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ON SOME PROPERTIES OF A DIFFERENTIAL OPERATOR ON THE POLYDISK

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ABSTRACT. We study the action and properties of a differential operator in the polydisk, extending some classical results from the unit disk. Using so called dyadic decomposition of the polydisk we find precise connections between quazinorms of holomorphic function in the polydisk with quazinorms on the subframe and the unit disk. All our results were previously well-known in the unit disk.

1. INTRODUCTION AND PRELIMINARIES

Let $n \in \mathbb{N}$ and $\mathbb{C}^n = \{z = (z_1, ..., z_n) \,|z_k \in \mathbb{C}, 1 \leq k \leq n\}$ be the n-dimensional space of complex coordinates. We denote the unit polydisk by

$$U^n = \{z \in \mathbb{C}^n : \ |z_k| < 1, 1 \leq k \leq n\}$$

and the distinguished boundary of U^n by

$$T^n = \{z \in \mathbb{C}^n : \ |z_k| = 1, 1 \leq k \leq n\}.$$

We use m_{2n} to denote the volume measure on U^n and m_n to denote the normalized Lebesgue measure on T^n. Let $H(U^n)$ be the space of all holomorphic functions on U^n. When $n = 1$, we simply denote U^1 by U, T^1 by T, m_{2n} by m_2, m_n by m. We refer to [15] for further details.

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The Hardy spaces, denoted by $H^p(U^n)(0<p\le\infty)$, are defined by

$$H^p(U^n)=\{f\in H(U^n):\sup_{0\le r<1}M_p(f,r)<\infty\},$$

where

$$M_p^p(f,r)=\int_{T^n}|f(r\xi)|^p dm_n(\xi),\ M_\infty(f,r)=\max_{\xi\in T^n}|f(r\xi)|,\ r\in(0,1),\ f\in H(U^n).$$

For $\alpha_j>-1, j=1,\cdots,n, 0<p<\infty$, recall that the weighted Bergman space $A^p_{\vec{\alpha}}(U^n)$ consists of all holomorphic functions on the polydisk satisfying the condition

$$\|f\|^p_{A^p_{\vec{\alpha}}}=\int_{U^n}|f(z)|^p\prod_{i=1}^n(1-|z_i|^2)^{\alpha_i}dm_{2n}(z)<\infty.$$

Throughout this paper, constants are denoted by C, C_α, or $C(\alpha)$, they are positive and may differ from one occurrence to other. The notation $A\asymp B$ means that there is a positive constant C such that $B/C\le A\le CB$.

Let $z=(z_1,\dots,z_n)\in U^n$, $f_j(z)\in H(U^n)$, $j=1,\cdots,n$. It is easy to see that if

$$f_j(z)=\sum_{k_1,\cdots,k_n\ge0}a^{(j)}_{k_1,\dots,k_n}z_1^{k_1}\cdots z_n^{k_n},\ j=1,\cdots n$$

is a usual Taylor expansion in U^n of f_j, then

$$S(f_1,\cdots,f_n)=f_1+\cdots f_n=\sum_{k_1,\dots,k_n\ge0}\Big(\sum_{j=1}^n a^{(j)}_{k_1,\dots,k_n}\Big)z_1^{k_1}\cdots z_n^{k_n}.$$

We consider a very particular case when

$$a^{(j)}_{k_1,\dots,k_n}=k_ja_{k_1,\dots,k_n},\ j=1,\cdot\cdot\cdot,n,$$

where $a_{k_1,\dots,k_n}$ is a certain sequence. We have

$$S(f_1,\cdots,f_n)=\sum_{k_1,\dots,k_n\ge0}(k_1+\cdots k_n)a_{k_1,\dots,k_n}z_1^{k_1}\cdots z_n^{k_n}.$$

Motivated by the above expression we define a operator in the polydisk as follows

$$\mathcal{R}f=\sum_{k_1,\dots,k_n\ge0}(k_1+\cdots+k_n+1)a_{k_1,\dots,k_n}z_1^{k_1}\cdots z_n^{k_n}$$

or more general form

$$\mathcal{R}^sf=\sum_{k_1,\dots,k_n\ge0}(k_1+\cdots+k_n+1)^sa_{k_1,\dots,k_n}z_1^{k_1}\cdots z_n^{k_n},\ s\in\mathbb{R},$$

where

$$f(z)=\sum_{k_1,\cdots,k_n\ge0}a_{k_1,\dots,k_n}z_1^{k_1}\cdots z_n^{k_n}\in H(U^n).$$

It is easy to see that

$$\mathcal{R}f=\mathcal{R}^1f=f+\sum_{j=1}^n z_j\frac{\partial f}{\partial z_j}. \tag{1.1}$$

In the case of the unit ball an analogue of $\mathcal{R}^s$ operator is a well known radial derivative which is well studied (see [17]). We note that in polydisk the following fractional derivative is well studied (see [4, 11]),

$$(\mathcal{D}^\alpha f)(z) = \sum_{k_1,\dots,k_n\geq 0} (k_1+1)^\alpha \cdots (k_n+1)^\alpha a_{k_1,\dots,k_n} z_1^{k_1} z_n^{k_n},$$

where $\alpha \in \mathbb{R}, f \in H(U^n)$ and $\mathcal{D}^\alpha : H(U^n) \to H(U^n)$. We also note that in polydisk the following derivative was studied in [16],

$$\mathcal{D} = \prod_{k=1}^{n} \Big(2 + z_k \frac{\partial}{\partial z_k}\Big).$$

Apparently the $\mathcal{R}^s$ operator was studied in [6] for the first time. Then in [12], the first author studied some properties of this operator. The aim of this paper continue to study the $\mathcal{R}^s$ operator.

We need the following vital formula which can be checked by easy calculation

$$f(\tau\xi_1,\cdot\cdot\cdot,\tau\xi_n) = C_s \int_0^1 \mathcal{R}^s f(\tau\xi_1\rho,\cdot\cdot\cdot,\tau\xi_n\rho)(\log\frac{1}{\rho})^{s-1} d\rho, \tag{1.2}$$

where $s > 0, \tau \in (0,1), C_s > 0, \xi_j \in T, j = 1,\dots,n$. The integral representation of functions via these operators will allow us to consider them in U^n in close connection with functional spaces on subframe

$$\widetilde{U^n} = \{z \in U^n, |z_j| = r, r \in (0,1], j = 1,\dots,n\}.$$

The following dyadic decomposition of subframe and polydisk were introduced in [4] and will be used by us.

$$\widetilde{U}_{k,l_1,\cdots,l_n} = \widetilde{U}_{k,l_1} \times \cdots \widetilde{U}_{k,l_n} = \Big\{(\tau\xi_1,\dots,\tau\xi_n) : \tau \in (1-\frac{1}{2^k}, 1-\frac{1}{2^{k+1}}],$$
$$k = 0,1,2,\cdots;\ \frac{\pi l_j}{2^k} < \xi_j \leq \frac{\pi(l_j+1)}{2^k},\ l_j = -2^k,\cdots,2^k-1,\ j = 1,\cdots,n\Big\},$$

$$m(\widetilde{I}_{k,l_j}) = m(\xi \in T : \frac{\pi l_j}{2^k} < \xi_j \leq \frac{\pi(l_j+1)}{2^k}) \asymp 2^{-k},\ m_{2n}(\widetilde{U}_{k,l_1,\cdots,l_n}) \asymp 2^{-2kn},$$

$$\begin{aligned} U_{k_1,\cdots,k_n,l_1,\cdots,l_n} &= U_{k_1,l_1} \times \cdots U_{k_n,l_n} = \Big\{(\tau_1\xi_1,\cdots,\tau_n\xi_n), \\ &\tau_j \in (1-\frac{1}{2^{k_j}}, 1-\frac{1}{2^{k_j+1}}],\ k_j = 0,1,\cdots,\ j = 1,2,\cdots,n, \\ &\xi_j \in (\frac{\pi l_j}{2^{k_j}}, \frac{\pi(l_j+1)}{2^{k_j}}],\ l_j = -2^{k_j},\cdots,2^{k_j}-1,\ j = 1,\cdots,n,\Big\}. \end{aligned}$$

The goal of this paper is to extend some known assertions connected with fractional derivative in the unit disk to the polydisk and use the diadic decomposition of subframe and polydisk to study the action and properties of $\mathcal{R}^s$ operator and quasinorm connected with them on subframe. In section 2 we give preliminaries, several useful inequalities for the study of $\mathcal{R}^s$ operators, and show connections between $\mathcal{R}^s$ and $\mathcal{D}^s$ operators in the polydisk. In section 3, we using the $\mathcal{R}^s$ operator establish some embedding theorems extending some known embeddings for Hardy classes and weighted Bergman classes in the unit disk. In section 3 we also establish connections between holomorphic spaces with quasinorms in the subframe and polydisk.

2. Some observations concerning $\mathcal{R}^s$ and $\mathcal{D}^s$ differential operators and proofs of preliminaries

When we look at $\mathcal{R}^s$ operators then we have the following natural problem: Is it possible to reduce the study of $\mathcal{R}^s$ operators to the study of $\mathcal{D}^s$ operators? which was studied by many authors (see for example [4] and references there). Then we will be able to use known properties of $\mathcal{D}^s$ to get new results for $\mathcal{R}^s$ operators. The differential operator $\mathcal{D}^s$ is much more convenient at least because of the following property of g function. Let

$$g(z) = \frac{1}{1-z} = \sum_{k_1,\cdots,k_n \geq 0} z_1^{k_1} \cdots z_n^{k_n}.$$

We have

$$\mathcal{R}^s g(z) = \sum_{k_1,\cdots,k_n \geq 0} (k_1 + \cdots + k_n + 1)^s z_1^{k_1} \cdots z_n^{k_n}$$

and

$$\mathcal{D}^s g(z) = \sum_{k_1 \geq 0} (k_1+1)^s z_1^{k_1} \cdots \sum_{k_n \geq 0} (k_n+1)^s z_n^{k_n}.$$

For $\mathcal{D}^s$ we reduce things to one dimensional differential operators for one function in the unit disk, that why we will find ways to reduce the study of $\mathcal{R}^s$ to $\mathcal{D}^s$.

Let

$$\widetilde{\mathcal{R}}^s f(z) = \sum_{k_1,\cdots,k_n \geq 0} (k_1 + \cdots + k_n)^s a_{k_1,\cdots,k_n} z_1^{k_1} \cdots z_n^{k_n}.$$

The following lemma is playing an important role in the study of $\mathcal{R}^s$ operator. We use some known facts in the proof of Lemma 2.1 about action of one dimensional $\mathcal{D}^\alpha$ operator, for example the following estimate (see [2])

$$\int_T |\mathcal{D}^\alpha g(\tau\xi)|^p d\xi \;\leq\; C \int_T |\mathcal{D}^\beta g(\tau\xi)|^p d\xi \qquad (2.1)$$
$$(\tau \in (0,1), 0 < p < \infty, \alpha \leq \beta, g \in H(U))$$

which can be transferred by induction to polydisk.

Lemma 2.1. *Let $w = |w|\xi$, $w, z, \in U^n$, $1 - w\overline{z} = \prod_{k=1}^n (1 - w_k \overline{z_k})$, $s \in \{0\} \cup \mathbb{N}$, $\beta > 0, p \in (0,\infty)$. Then we have*

$$\int_{T^n} \left| \mathcal{R}^s \frac{1}{(1-\xi|w|z)^\beta} \right|^p dm_n(\xi)$$
$$\leq \; C \sum_{\substack{\alpha_j \geq 0 \\ \sum \alpha_j = s}} \Big(\prod_{k=1}^n \frac{1}{(1-|w_k||z_k|)^{p(\alpha_k+\beta)-1}} \Big), \; p > \frac{1}{\min_k \alpha_k + \beta}.$$

Proof. Let first $\beta = 1$. Since

$$(k_1 + \cdots k_n)^s = \sum_{\substack{\alpha_j \geq 0 \\ \sum \alpha_j = s}} C_\alpha k_1^{\alpha_1} \cdots k_n^{\alpha_n}, \ k_i \in \mathbb{N}, \ i = 1, \cdots, n,$$

$$k_i^{\alpha_i} = (k_i + 1 - 1)^{\alpha_i} = \sum_{l=0}^{\alpha_i} C_{\alpha_i}^l (k_i + 1)^l (-1)^{\alpha_i - l},$$

and

$$\frac{1}{1-w} = \prod_{k=1}^n \frac{1}{1-w_k}, \ \frac{1}{1-w} = \sum_{\substack{k_i \geq 0 \\ i=1,\cdots,n}} w_1^{k_1} \cdots w_n^{k_n}, \ |w_j| < 1, \ k_j \in \mathbb{N}, \ j = 1, \cdots, n,$$

where $C_\alpha = C(\alpha_1, \cdots \alpha_n) = \frac{s!}{\alpha_1! \cdots \alpha_n!}$, we have

$$\widetilde{\mathcal{R}}^s \frac{1}{1-w} = \sum_{\substack{k_i \geq 0 \\ i=1,\cdots,n}} (k_1 + \cdots + k_n)^s w^k$$

$$= \sum_{\substack{k_i \geq 0 \\ i=1,\cdots,n}} (k_1 + \cdots + k_n)^s w_1^{k_1} \cdots w_n^{k_n}$$

$$= \sum_{\substack{k_i \geq 0 \\ i=1,\cdots,n}} \sum_{\substack{\alpha_j \geq 0 \\ \sum \alpha_j = s}} C_\alpha k_1^{\alpha_1} \cdots k_n^{\alpha_n} w_1^{k_1} \cdots w_n^{k_n}$$

$$= \sum_{\substack{k_i \geq 0 \\ i=1,\cdots,n}} \sum_{\substack{\alpha_j \geq 0 \\ \sum \alpha_j = s}} C_\alpha \prod_{t=1}^n \sum_{l=0}^{\alpha_t} C_{\alpha_t}^l (k_t + 1)^l (-1)^{\alpha_t - l} w_1^{k_1} \cdots w_n^{k_n}, \ |w_j| < 1, \ j = 1, \cdots, n.$$

Therefore using polydisk version of (2.1) we have

$$J = \int_{T^n} \left| \widetilde{\mathcal{R}}^s \frac{1}{(1 - \xi |w| z)} \right|^p dm_n(\xi)$$

$$\leq C \sum_{\substack{\alpha_j \geq 0 \\ \sum \alpha_j = s}} C_\alpha \int_{T^n} \left| \sum_{\substack{k_i \geq 0 \\ i=1,\cdots,n}} (k_1 + 1)^{\alpha_1} \cdots (k_n + 1)^{\alpha_n} \widetilde{w_1}^{k_1} \cdots \widetilde{w_n}^{k_n} \right|^p dm_n(\xi)$$

$$\leq C \sum_{\substack{\alpha_j \geq 0 \\ \sum \alpha_j = s}} C_\alpha \int_T \left| \sum_{k_1 \geq 0} (k_1 + 1)^{\alpha_1} \widetilde{w_1}^{k_1} \right|^p dm(\xi_1) \cdots$$

$$\times \int_T \left| \sum_{k_n \geq 0} (k_n + 1)^{\alpha_n} \widetilde{w_n}^{k_n} \right|^p dm(\xi_n),$$

where $\widetilde{w_j} = \xi_j |w_j||z_j|, \ j = 1, \cdots, n$. Hence by the estimate

$$\int_T \frac{dm(\eta)}{|1 - \tau\eta|^\gamma} \leq \frac{C(\gamma)}{(1-\tau)^{\gamma - 1}}, \ \gamma > 1, \ \tau \in (0,1), \tag{2.2}$$

we finally have

$$\begin{aligned} J &\le C \sum_{\substack{\alpha_j\ge 0\\ \sum \alpha_j = s}} C_\alpha \int_T \left|\mathcal{D}^{\alpha_1} g_1(|w_1||z_1|\xi_1)\right|^p d\xi_1 \cdots \times \int_T \left|\mathcal{D}^{\alpha_n} g_n(|w_n||z_n|\xi_n)\right|^p d\xi_n \\ &\le C \sum_{\substack{\alpha_j\ge 0\\ \sum \alpha_j = s}} \left(\prod_{k=1}^n \int_T \frac{dm(\xi_k)}{|1-\widetilde{w_k}|^{p(\alpha_k+1)}} \right) \\ &\le C \sum_{\substack{\alpha_j\ge 0\\ \sum \alpha_j = s}} \prod_{k=1}^n \frac{1}{(1-|w_k||z_k|)^{p(\alpha_k+1)-1}}, \end{aligned}$$

where $p > 1/(\min_k \alpha_k + 1)$, $|w_k| \in (0,1)$, $|z_k| \in (0,1)$, $k = 1, \cdots, n$, and

$$\mathcal{D}^{\alpha_s} g_s(z) = \sum_{k\ge 0} (k+1)^{\alpha_s} z^k, \ z \in U, \ s = 1, \cdots, n.$$

The case of $\beta \in (0,\infty)$ needs small modification since

$$\frac{1}{(1-z)^\beta} = \sum_{k\ge 0} C_k^\beta z^k, \ C_k^\beta \asymp (k+1)^{\beta-1}, \ \beta > 0.$$

Lemma 2.1 is proved since

$$\int_{T^n} |\mathcal{R}^s f(\tau\xi)|^p dm_n(\xi) \le C \int_{T^n} |\widetilde{\mathcal{R}}^s f(\tau\xi)|^p dm_n(\xi),$$

for $f(z) = \prod_{k=1}^n \frac{1}{(1-z_k)^\alpha}$, $\alpha > 0$, $s \ge 0, p \in (0,\infty)$, which follows from equality

$$(k_1 + \cdots + k_n + 1)^s = \sum_{j=0}^s C_s^j (k_1 + \cdots + k_n)^j$$

and some calculations similar to those that we used above. □

Corollary 2.2. *Let $0 < p < \infty$, $s \in \mathbb{N} \cup \{0\}$, $l \in (0,\infty), \gamma > 1/p + l, w \in U^n$. Then*

$$\int_{U^n} |\mathcal{R}^s \frac{1}{(1-w\overline{z})^\gamma}|^p (1-|z|)^{pl-1} dm_{2n}(z) \le \sum_{\alpha_j \ge 0, \sum \alpha_j = s} \prod_{k=1}^n \frac{C}{(1-|w_k|)^{(\alpha_k+\gamma)p-pl-1}}.$$

Proof. The result follows directly from Lemma 2.1 and the following estimate (see [13]),

$$\int_0^1 (1-\rho\tau)^{-\lambda}(1-\tau)^\alpha d\tau \le C(1-\rho)^{-\lambda+\alpha+1}, \ \lambda > \alpha+1, \alpha > -1, \rho \in (0,1). \quad (2.3)$$

□

Remark 2.3. Our estimates in Lemma 2.1 and Corollary 2.2 coincide with well known estimates in the unit disk for $n = 1$, they also known in ball, see for example [8, 17].

For the proof of our main results some additional lemmas will be needed.

Lemma 2.4. *Let* $1 \le p \le q \le \infty$, $p' = \frac{p}{p-1}$, $u(t), v(t), \varphi(t)$ *are positive functions on* $(0,1)$. *Then*

$$\Big(\int_0^1 u(t)\Big(\int_0^t \varphi(r)dr\Big)^q dt\Big)^{1/q} \le K_1\Big(\int_0^1 \varphi^p(t)v(t)dt\Big)^{1/p}$$

for some constant K_1 *if and only if*

$$\sup_t\Big(\int_t^1 u(\tau)d\tau\Big)^{p/q}\Big(\int_0^t v(\tau)^{1-p'}d\tau\Big)^{p-1} < \infty.$$

Remark 2.5. These are well known so-called Hardy-type inequalities, see [9].

Lemma 2.6. *Let* $\gamma > 1$. *Then*

$$\int_{T^n} \frac{1}{|1-w\overline{z}|^\gamma} dm_n(\xi) \le \frac{C_\gamma}{(1-|w||z|)^{\gamma-1}},$$

where $w = \xi|w|$, $z, w \in U^n$, $1 - w\overline{z} = \prod_{k=1}^n (1 - w_k\overline{z_k})$.

Proof. The proof easily follows by (2.2), we omit the details. □

We want to get the analogue of Bergman representation formula for the $\mathcal{R}^s$ operator (see [4]). Using (1.2) we get

$$\begin{aligned} & f(\tau z_1, \cdots, \tau z_n) = f_\tau(z_1, \cdots, z_n) \\ = \ & C\int_{T^n} \frac{f_\tau(\xi_1, \cdots, \xi_n)}{(1-\overline{\xi}_1 z_1)\cdots(1-\overline{\xi}_n z_n)} dm_n(\xi) \\ = \ & C_s \int_{T^n}\int_0^1 \mathcal{R}^s f(\tau\xi_1\rho, \cdots, \tau\xi_n\rho)(\log\frac{1}{\rho})^{s-1} d\rho \frac{dm_n(\xi)}{(1-\overline{\xi}_1 z_1)\cdots(1-\overline{\xi}_n z_n)} \\ = \ & C_s \int_{T^n}\int_0^1 f(\tau\xi_1\rho, \cdots, \tau\xi_n\rho)(\log\frac{1}{\rho})^{s-1}\mathcal{R}^s\Big(\prod_{k=1}^n \frac{1}{(1-\overline{\xi}_k z_k)}\Big) d\rho\, dm_n(\xi), \end{aligned}$$

where $\tau \in (0,1)$. Let $\tau \to 1$. Then

$$f(z_1, \cdots, z_n) = C_s \int_{T^n}\int_0^1 f(\rho\xi)(\log\frac{1}{\rho})^{s-1}\mathcal{R}^s \frac{1}{(1-\overline{\xi}\varphi\widetilde{\rho})} d\rho\, dm_n(\xi), \qquad (2.4)$$

for $f \in H(U^n)$, $s > 0$, $z \in U^n$, $z = \widetilde{\rho}\varphi$.

Note for $n = 1$, we get the well known representation formula (see [4, 17]) after small modifications. The last estimate is equivalent to

$$(\mathcal{R}^{-s}f)(z) = C_s \int_{T^n}\int_0^1 f(\rho\xi)(\log\frac{1}{\rho})^{s-1} \frac{1}{(1-\overline{\xi}z)} d\rho\, dm_n(\xi), \qquad (2.5)$$

where $\mathcal{R}^{-s}\mathcal{R}^s f = f$, $s > 0$, $z \in U^n$, $f \in H(U^n)$.

The following theorem connect $\mathcal{D}^s$ and $\mathcal{R}^s$ operators.

Theorem 2.7. *Let $0 < p < \infty$, $\alpha > -1$, $s \in \mathbb{N}$, $f \in H(U^n)$. If $\gamma > \frac{\alpha+2}{p} - 1$ for $p \le 1$ and $\gamma > \frac{\alpha+1}{p} + \frac{1}{n}(1 - \frac{1}{p})$ for $p > 1$, $v = sp + \alpha n - \gamma pn + n - 1$, then*

$$\begin{aligned}&\int_{U^n} |\mathcal{D}^\gamma f(z)|^p (1-|z|^2)^\alpha dm_{2n}(z)\\ \le\ & C\int_0^1 \int_{T^n} |\mathcal{R}^s f(w)|^p (1-|w|^2)^v dm_n(\xi) d|w|\,,\end{aligned}$$

where $w = |w|\xi$.

Proof. Let $f \in H(U^n)$, $p \le 1$. Then (see [14])

$$M_1(f,\tau^2) \le C(1-\tau)^{n(1-1/p)} M_p(f,\tau),\ \tau \in (0,1). \tag{2.6}$$

Using (2.6) and the fact that $M_p(f,r)$ is increasing as a function of r (see [5]) we easily get

$$\begin{aligned}&\left(\int_{T^n}\int_0^1 |f(w)|(1-|w|)^t dm_n(\xi)d|w|\right)^p\\ &\le C\int_{T^n}\int_0^1 |f(w)|^p(1-|w|)^{tp+(n+1)(p-1)} dm_n(\xi)d|w|,\end{aligned} \tag{2.7}$$

where $t > \frac{n(1-p)}{p} - 1$, $f \in H(U^n)$, $p \le 1$. By the Cauchy formula,

$$|\mathcal{D}^\gamma f_\rho(z_1,\cdots,z_n)|^p \le C\left|\int_{T^n} \frac{f_\rho(\xi_1,\cdots,\xi_n)dm_n(\xi)}{(1-\overline{\xi}z)^{\gamma+1}}\right|^p,\ \rho \in (0,1).$$

Using (1.2) or (2.4), (2.5) for large s and (2.7) we obtain

$$\begin{aligned}&\left|\int_{T^n} \frac{f_\rho(\xi_1,\cdots,\xi_n)dm_n(\xi)}{\prod_{k=1}^n(1-\overline{\xi_k}z_k)^{\gamma+1}}\right|^p\\ \le\ & C\left(\int_0^1\int_{T^n} \frac{|\mathcal{R}^s f_\rho(w)|(1-|w|)^{s-1}dm_n(\xi)d|w|}{|1-\overline{w}z|^{\gamma+1}}\right)^p\\ \le\ & C\int_0^1\int_{T^n} \frac{|\mathcal{R}^s f_\rho(w)|^p(1-|w|)^{(s-1)p}(1-|w|)^{(n+1)(p-1)}}{|1-\overline{w}z|^{(\gamma+1)p}} dm_n(\xi)d|w|.\end{aligned} \tag{2.8}$$

Therefore, from (2.6)-(2.8) and using Fubini's theorem we have

$$\begin{aligned}&\int_{U^n} |\mathcal{D}^\gamma f_\rho(z)|^p(1-|z|^2)^\alpha dm_{2n}(z)\\ \le\ & C\int_{U^n}\int_0^1\int_{T^n} \frac{|\mathcal{R}^s f_\rho(w)|^p(1-|w|)^{sp+n(p-1)-1}dm_n(\xi)d|w|}{|1-\overline{w}z|^{(\gamma+1)p}}(1-|z|^2)^\alpha dm_{2n}(z)\\ \le\ & C\int_0^1\int_{T^n}\int_{U^n} \frac{(1-|z|^2)^\alpha dm_{2n}(z)}{|1-\overline{w}z|^{(\gamma+1)p}} |\mathcal{R}^s f_\rho(w)|^p(1-|w|)^{sp+n(p-1)-1} dm_n(\xi)d|w|.\end{aligned}$$

Using the following estimate

$$\int_{U^n} \frac{(1-|z|)^t dm_{2n}(z)}{\prod_{k=1}^n |1-\overline{z_k}w_k|^{t_1}} \le \frac{C}{\prod_{k=1}^n(1-|w_k|)^{t_1-t-2}},\ t > -1,\ t_1 > t+2,$$

we get

$$\int_{U^n} \frac{(1-|z|^2)^\alpha dm_{2n}(z)}{|1-\overline{w}z|^{(\gamma+1)p}} \le C(1-|w|)^{(2+\alpha-p\gamma-p)n}$$
$$\left(\gamma > \frac{\alpha+2}{p} - 1, \alpha > -1, |w| \in (0,1)\right).$$

Using limit argument we can get the desired result.

Consider now the case $p > 1$. The arguments are partially the same as in the case $p \le 1$. Let ϵ be positive and small enough. Using (1.2), Cauchy formula, Hölder inequality we have

$$|\mathcal{D}^\gamma f(z)|^p \le \left(\int_0^1 \int_{T^n} \frac{|\mathcal{R}^s f(w)|(1-|w|)^{s-1} dm_n(\xi) d|w|}{|1-\overline{w}z|^{\gamma+1}}\right)^p$$
$$\le C \int_0^1 \int_{T^n} \frac{|\mathcal{R}^s f(w)|^p (1-|w|)^{(s-1)p} dm_n(w) d|w|}{|1-\overline{w}z|^{(\gamma-1)p+2+p\epsilon}} \left(\int_{T^n} \int_0^1 \frac{dm_n(\xi) d|w|}{|1-\overline{w}z|^{2-p'\epsilon}}\right)^{p/p'}$$
$$= CM_1 M_2^{p/p'}.$$

Furthermore by Lemma 2.6,

$$M_2 \le \int_0^1 \frac{dR}{\prod_{k=1}^n (1-R|z_k|)^{1-\epsilon p'}} \le \left(\int_0^{R_0} + \int_{R_0}^1\right) \frac{dR}{\prod_{k=1}^n (1-R|z_k|)^{1-\epsilon p'}}, \quad (2.9)$$

where $R_0 = \max_{1\le k\le n} |z_k|$. In addition,

$$\int_{R_0}^1 \frac{dR}{\prod_{k=1}^n (1-R|z_k|)^{1-\epsilon p'}} \le \frac{(1-R_0)^{1/n+\cdots 1/n}}{\prod_{k=1}^n (1-|z_k|)^{1-\epsilon p'}}$$
$$\le \frac{C}{\prod_{k=1}^n (1-|z_k|)^{1-\epsilon p' - \frac{1}{n}}} = \widetilde{M}. \quad (2.10)$$

Using (2.3),

$$\int_0^{R_0} \frac{dR}{\prod_{k=1}^n (1-R|z_k|)^{1-\epsilon p'}}$$
$$= \int_0^{R_0} \frac{(1-R)^{\frac{1}{n}-1} \times (1-R)^{\frac{n-1}{n}} dR}{\prod_{k=1}^n (1-R|z_k|)^{1-\epsilon p'}}$$
$$\le C \int_0^{R_0} \frac{(1-R|z_1|)^{1/n} \cdots (1-R|z_{n-1}|)^{1/n}}{(1-R|z_1|)^{1-\epsilon p'} \cdots (1-R|z_n|)^{1-\epsilon p'}} (1-R)^{1/n-1} dR$$
$$\le \frac{C}{\prod_{k=1}^{n-1} (1-|z_k|)^{1-\epsilon p' - \frac{1}{n}}} \int_0^{R_0} \frac{(1-R)^{1/n-1} dR}{(1-R|z_n|)^{1-\epsilon p'}} \le C\widetilde{M}. \quad (2.11)$$

Combing (2.9) with (2.10), (2.11) and repeating the arguments that we used for $p \le 1$, we can get the desired result at the case of $p > 1$. □

Remark 2.8. As we see from Theorem 2.7, the $\mathcal{R}^s$ operator are closely connected with quasinorms defined on subframe. Theorem 2.7 is an extension of result on the action of fractional derivatives on weighted Bergman spaces in the unit disk, see [2].

3. Embedding theorems for some analytic spaces connected with $\mathcal{R}^s$ operator

In this section we state some new embedding theorems for various quasinorms where the $\mathcal{R}^s$ operator is participating, note that practically all results are well known or obvious in the unit disk. Proofs of our results are heavily based on definitions and preliminaries from previous sections.

Theorem 3.1. *i) Let $0 < q < \infty$, $\alpha \in [0,\infty)$, $s \in \mathbb{N}$, $1 < p < \infty$, $f \in H(U^n)$. Then*

$$\int_{T^n}\Big(\int_0^1 (1-|z|)^\alpha |f(z)|^p |z|^{sp} d|z|\Big)^{q/p} dm_n(\xi)$$
$$\leq C\int_{T^n}\Big(\int_0^1 |\mathcal{R}^s f(u\xi)|^p (1-u)^{sp+\alpha} du\Big)^{q/p} dm_n(\xi).$$

ii) Let $0 < q < \infty$, $\alpha \in [0,\infty)$, $s \in \mathbb{N}$, $1 < p < \infty$, $\gamma \in (-1/p, 1/p')$, $1/p' + 1/p = 1$, $f \in H(U^n)$. Then

$$\int_{T^n}\Big(\int_0^1 (1-r)^\gamma |f(r\xi)|^p r^p dr\Big)^{q/p} dm_n(\xi)$$
$$\leq C\int_{T^n}\Big(\int_0^1 |\mathcal{R}^s f(r\xi)|^p (1-r)^{sp+\gamma} dr\Big)^{q/p} dm_n(\xi)\,.$$

Proof. Using (1.2) we have

$$\begin{aligned} M &= \Big(\int_0^1 (1-|z|)^\alpha |f(z)|^p |z|^{sp} d|z|\Big)^{1/p} \\ &\leq C\Big(\int_0^1 \Big(\int_0^1 |\mathcal{R}^s f(\rho z)|(1-\rho)^{s-1} d\rho\Big)^p (1-|z|)^\alpha |z|^{sp} d|z|\Big)^{1/p} \\ &= C\int_0^1\int_0^1 |\mathcal{R}^s f(\rho z)|(1-\rho)^{s-1}(1-|z|)^{\alpha/p}\psi(|z|)|z|^s d|z| d\rho\,, \end{aligned}$$

where $\psi(|z|) \in L^{p'}(d|z|)$, $1/p' + 1/p = 1$.

Changing the variables we have

$$\int_0^1 |\mathcal{R}^s f(\rho z)|(1-\rho)^{s-1} d\rho \leq \int_0^{|z|} |\mathcal{R}^s f(u\xi)|(1-u)^{s-1}\frac{du}{|z|^s}.$$

Using the last inequality and Hölder inequality, we get

$$\begin{aligned} M &\leq C\int_0^1\int_0^{|z|} |\mathcal{R}^s f(u\xi)|(1-u)^{s-1}(1-|z|)^{\alpha/p}\psi(|z|) d|z| du \\ &\leq C\int_0^1 \frac{|\mathcal{R}^s f(v\xi)|(1-v)^s}{1-v}\int_v^1 (1-|z|)^{\alpha/p}\psi(|z|) d|z| dv \\ &\leq C \Big(\int_0^1 \Big(\frac{1}{1-v}\int_v^1 \psi(|z|) d|z|\Big)^{p'} dv\Big)^{1/p'}\Big(\int_0^1 |\mathcal{R}^s f(u\xi)|^p (1-u)^{\alpha+ps} du\Big)^{1/p}. \end{aligned}$$

From the last inequality and Lemma 2.4 we easily get the first part of the theorem.

Now we prove the second part. From (1.2) and changing the variables we have

$$\begin{aligned} f(\tau\xi) &= f(\tau\xi_1,\cdots,\tau\xi_n) = \int_0^1 \mathcal{R}^s f(\tau\xi_1\rho,\cdots,\tau\xi_n\rho)(\log\frac{1}{\rho})^{s-1}d\rho \\ &= \int_0^\tau \mathcal{R}^s f(u\xi_1,\cdots,u\xi_n)(\log\frac{\tau}{u})^{s-1}\frac{du}{\tau} \\ &\leq C\int_0^\tau \mathcal{R}^s f(u\xi_1,\cdots,u\xi_n)\times\frac{(1-u\tau)^{-1}}{\tau}(1-u^2)^s d\tau,\ s\geq 1\,. \end{aligned}$$

Using duality argument, (2.3) and Hölder inequality we have

$$\begin{aligned} &\Big(\int_0^1 f(\tau\xi)^p\tau^p(1-\tau)^\gamma d\tau\Big)^{1/p} \\ \leq\ & C\Big(\int_0^1\Big(\int_0^1|\mathcal{R}^s f(u\xi)|(1-u)^s(1-\tau u)^{-1}du\Big)^p(1-\tau)^\gamma d\tau\Big)^{1/p} \\ =\ & C\int_0^1\int_0^1|\mathcal{R}^s f(u\xi)|(1-u)^s(1-\tau u)^{-1}(1-\tau)^\gamma\psi(\tau)(\frac{1-u}{1-\tau})^{\frac{-1}{pp'}}(\frac{1-u}{1-\tau})^{\frac{1}{pp'}}dud\tau \\ \leq\ & C\Big(\int_0^1\int_0^1\psi^{p'}(\tau)(1-\tau u)^{-1}\Big(\frac{1-u}{1-\tau}\Big)^{-\frac{1}{p}}(1-\tau)^\gamma dud\tau\Big)^{1/p'} \\ & \times\Big(\int_0^1\int_0^1(1-u)^{sp}|\mathcal{R}^s f(u\xi)|^p\Big(\frac{1-u}{1-\tau}\Big)^{\frac{1}{p'}}(1-\tau)^\gamma(1-\tau u)^{-1}dud\tau\Big)^{1/p} \\ \leq\ & C\Big(\int_0^1\psi^{p'}(\tau)(1-\tau)^\gamma d\tau\Big)^{1/p'}\Big(\int_0^1(1-u)^{sp+\gamma}|\mathcal{R}^s f(u\xi)|^p du\Big)^{1/p}, \end{aligned}$$

where $\psi\in L^{p'}((1-\tau)^\gamma d\tau), 1/p+1/p'=1$. We can get the desired inequality by integrating both sides on T^n. The proof is completed. □

Remark 3.2. Our Theorem 3.1 extends the well-known Hardy-Littlwood theorems(case $n=1$ and $p=q$) with fractional derivative (see [2, 5]) to the case of mixed norm spaces with $\mathcal{R}^s$ operators.

Next we give an application of Lemma 2.1. For measurable function f in the disk we define

$$A_p(f)(\xi)=\Big(\int_{\Gamma_\delta(\xi)}\frac{|f(z)|^p}{(1-|z|)^2}dm_2(z)\Big)^{1/p},\ p<\infty;$$

$$A_\infty(f)(\xi)=\sup\{|f(z)|:z\in\Gamma_\delta(\xi)\},\ \xi\in T,\ \delta>1;$$

$$C_p(f)(\xi)=\sup_{\xi\in I}\Big(\frac{1}{|I|}\int_{S(I)}\frac{|f(z)|^p}{1-|z|}dm_2(z)\Big)^{1/p},\ p<\infty,\ \xi\in T,$$

where $S(I)=\{z\in U,\frac{z}{|z|}\in I, 1-|z|\leq\frac{1}{2\pi}|I|\}$, $I\subset T$,

$$\Gamma_\delta(\xi)=\{z\in U:|1-\xi\overline{z}|<\delta(1-|z|),\ \delta>1\}.$$

The following result can be found in [1] or [10]:

For any functions $f(z)$ and $g(z)$ measurable in the unit disk

$$\int_U \frac{|f(z)||g(z)|}{1-|z|} dm_2(z) \le C \int_T A_p(f)(\xi) C_{p'}(g)(\xi) dm(\xi), 1 < p \le \infty, \frac{1}{p} + \frac{1}{p'} = 1 \quad (3.1)$$

The above result can be easily by iteration extended to polydisk.

Using the polydisk version of (3.1), by Cauchy formula in polydisk, Littlewood-Paley inequality (see [1]) and 2.1 we have for any holomorphic function f in the polydisk,

$$\begin{aligned} |\mathcal{R}^s f_\rho(\varphi\tau^2)| &= C \Big| \int_{T^n} f(\tau\xi) \mathcal{R}^s \frac{1}{1-\tau\rho\bar{\xi}\varphi} dm_n(\xi) \Big| \\ &\le C \int_{U^n} |\mathcal{D}^\alpha \mathcal{R}^s \frac{1}{1-w\varphi\rho} |f(w)|(1-|w|)^{\alpha-1} dm_{2n}(w) \\ &\le C \int_{T^n} \sup_{w\in\Gamma_t(\xi)} |\mathcal{D}^\alpha \mathcal{R}^s \frac{1}{1-w\varphi\rho} (1-|w|)^\alpha dm_n(\xi) \times \|C_1(f)\|_{L^\infty} \\ &\le C \int_{T^n} \sup_{w\in\Gamma_t(\xi)} |\mathcal{R}^s \frac{1}{1-w\varphi\rho}| dm_n(\xi) \times \|C_1(f)\|_{L^\infty}. \end{aligned}$$

Here $\alpha > 0$, $\varphi, \xi \in T^n, \rho \in [0,1]^n$, $\varphi\tau^2 = (\varphi_1\tau_1^2, \cdots, \varphi_n\tau_n^2)$, $\tau_j \in (1/2, 1)$, $j = 1, \cdots, n$. We used in the last inequality a maximal theorem for $\mathcal{D}^s$ operators, see [7]. Therefore

$$|\mathcal{R}^s f(\rho\varphi)| \times \Big(\sum_{\substack{\alpha_j \ge 0 \\ \sum \alpha_j = s}} \prod_{k=1}^n (1-\rho_k)^{-\alpha_k} \Big)^{-1} \le C \|C_1(f)\|_{L^\infty}$$

and

$$\begin{aligned} &\sup_{\rho\in[0,1]^n} M_\infty(\mathcal{D}^{-1}\mathcal{R}^s f, \rho)| \times \Big(\sum_{\substack{\alpha_j \ge 0 \\ \sum \alpha_j = s}} \prod_{k=1}^n (1-\rho_k)^{-\alpha_k} \Big)^{-1} \\ &\le C \|C_1(f \times (1-|z|))\|_{L^\infty}, \end{aligned} \quad (3.2)$$

where

$$C_1(f \times (1-|z|))(\xi_1, \cdots, \xi_n) = \sup_{\xi_1 \in I} \frac{1}{|I|} \int_{S(I)} \cdots \sup_{\xi_n \in I} \frac{1}{|I|} \int_{S(I)} |f(z)| dm_{2n}(z).$$

Remark 3.3. Estimates (3.2) extends known one dimensional inequality which can be found in [10]. We just give one application of Lemma 2.1. Various other generalizations can be obtained with the help of Lemma 2.1. Estimate (3.2) is just an example.

We see that the $\mathcal{R}^s$ operator are connected with the quasinorms on subframe via integral representations (2.4) and (2.5). So it is natural to find new estimates for quasinorms on subframe. In the polydisk we consider the following three expressions

$$\int_0^1 \int_{T^n} |f(r\xi_1, \cdots, r\xi_n)|^p (1-r)^\alpha r dr dm_n(\xi), \ 0 < p < \infty, \ \alpha > -1;$$

$$\int_{T^n}\int_{[0,1]^n}|f(\tau_1\xi_1,\cdots,\tau_n\xi_n)|^p\prod_{k=1}^{n}(1-\tau_k)^{\alpha_k}\tau_1\cdots\tau_n d\tau_1\cdots d\tau_n dm_n(\xi);$$

and

$$\int_{T}\int_{[0,1]^n}|f(\tau_1\xi,\cdots,\tau_n\xi)|^p(1-\tau_1)^{\alpha_1}\cdots(1-\tau_n)^{\alpha_n}\tau_1\cdots\tau_n d\tau_1\cdots d\tau_n dm(\xi),$$

where $0<p<\infty,\alpha_j>-1,j=1,\cdots,n,f\in H(U^n)$. Our intension is to show some connection between these quasinorms using so-called dyadic decomposition of the polydisk and subframe, see [4]. First, we have the following result.

Theorem 3.4. *Let $f\in H(U^n)$, $0<p<\infty$, $\alpha>-1$, $n\in\mathbb{N}$. Then*

$$\begin{aligned}&\int_0^1\int_{T^n}|f(z)|^p(1-|z|^2)^{\alpha}dm_n(\xi)d|z|\\ \le\ & C\int_{U^n}|f(z)|^p(1-|z_1|^2)^{\frac{\alpha}{n}-1+\frac{1}{n}}\cdots(1-|z_n|^2)^{\frac{\alpha}{n}-1+\frac{1}{n}}dm_{2n}(z)\,.\end{aligned}$$

Proof. Using diadic decomposition that were introduced in introduction and the subharmonicity of the $|f(z)|^p(0<p<\infty)$ function, we have

$$\begin{aligned}&\int_0^1\int_{T^n}|f(z)|^p(1-|z|^2)^{\alpha}dm_n(\xi)d|z|\\ =\ &\sum_{k\ge0}\int_{1-2^{-k}}^{1-2^{-k-1}}\sum_{l_1,\cdots,l_n}\int_{I_{k,l_1}}\cdots\int_{I_{k,l_n}}|f(z)|^p(1-|z|^2)^{\alpha}dm_n(\xi)d|z|\\ \le\ & C\sum_{k\ge0}\sum_{l_1,\cdots,l_n}\max_{\tilde{U}_{k,l_1,\cdots,l_n}}|f(z)|^p\cdot2^{-k}\cdot2^{-k\alpha}\cdot(2^{-k}\cdots2^{-k})\\ \le\ & C\sum_{k_1\ge0}\cdots\sum_{k_n\ge0}\sum_{l_1,\cdots,l_n}\max_{U_{k_1,\cdots,k_n,l_1,\cdots,l_n}}|f(z)|^p\cdot\prod_{i=1}^{n}2^{-\frac{k_i}{n}}\cdot\prod_{i=1}^{n}2^{-\frac{k_i\alpha}{n}}\prod_{i=1}^{n}2^{-k_i}\\ \le\ & C\int_{U^n}|f(z)|^p(1-|z_1|^2)^{\frac{\alpha}{n}-1+\frac{1}{n}}\cdots(1-|z_n|^2)^{\frac{\alpha}{n}-1+\frac{1}{n}}dm_{2n}(z).\end{aligned}$$

In the last inequality we used two facts. The first is that we used the following inequality

$$\max_{z\in U_{k_1,\cdots,k_n,l_1,\cdots,l_n}}|f(z)|^p\le C2^{2(k_1+\cdots+k_n)}\int_{U^*_{k_1,\cdots,k_n,l_1,\cdots,l_n}}|f(z)|^p dm_{2n}(z)\,,$$

which follows from the subharmonicity of $|f(z)|^p(0<p<\infty,f\in H(U^n)$ (see [4]). Here $U^*_{k_1,\cdots,k_n,l_1,\cdots,l_n}=U^*_{k_1,l_1}\times\cdots\times U^*_{k_n,l_n}$ is a family of enlarged dyadic cubes, see [4]. The second is that the family of enlarged dyadic cubes is a finite covering of polydisk (see [4])

$$\begin{aligned}&\Big(\sum_{k_1,\cdots,k_n\ge0}\Big)\Big(\sum_{l_1,\cdots,l_n\ge0}\Big)\int_{U^*_{k_1,\cdots,k_n,l_1,\cdots,l_n}}|f(z)|^p(1-|z|^2)^{\gamma}dm_{2n}(z)\\ \le\ & C\int_{U^n}|f(z)|^p(1-|z|^2)^{\gamma}dm_{2n}(z),\ 0<p<\infty,\ \gamma>-1.\end{aligned}$$

The proof of the theorem is completed. □

Remark 3.5. It is easy to see that the assertion of Theorem 3.4 is true for $n = 1$.

The following theorem can be obtained using the same ideas by small modification of methods that we used above.

Theorem 3.6. *Let $f \in H(U^n)$. Then the following assertions are true.*

i) Let $p \in (0,\infty)$, $\alpha_j > -1, j-1,\cdots,n$, $n \in \mathbb{N}$. Then

$$\int_U |f(z,\cdots,z)|^p(1-|z|^2)^{\sum_{j=1}^n \alpha_j+n-1}dm_2(z)$$
$$\leq C\int_T\int_{[0,1]^n}|f(|z_1|\xi,\cdots,|z_n|\xi)|^p\prod_{k=1}^n(1-|z_k|^2)^{\alpha_k}d|z_1|\cdots d|z_n|dm(\xi).$$

ii) Let $0<p<\infty$, $\alpha>-1$. Then we have

$$\int_U |f(z,\cdots,z)|^p(1-|z|^2)^{\alpha+n-1}dm_2(z)$$
$$\leq C\int_{T^n}\int_0^1|f(|z|\xi_1,\cdots,|z|\xi_n)|^p(1-|z|^2)^{\alpha}dm_n(\xi)d|z|.$$

iii) Let $p \in (0,\infty)$, $\alpha_j > -1, j-1,\cdots,n$, $n \in \mathbb{N}$. Then

$$\int_T\int_{[0,1]^n}|f(|z_1|\xi,\cdots,|z_n|\xi)|^p\prod_{k=1}^n(1-|z_k|)^{\alpha_k+\frac{n-1}{n}}d|z_1|\cdots d|z_n|dm(\xi)$$
$$\leq C\int_{U^n}|f(z)|^p\prod_{k=1}^n(1-|z_k|^2)^{\alpha_k}dm_{2n}(z).$$

Proof. We prove only the second inequality. We use diadic decomposition of U and subframe and the same ideas that we used in the proof of Theorem 3.4. We have

$$\begin{aligned}
&\int_U |f(z,\cdots,z)|^p(1-|z|^2)^{\alpha+n-1}dm_2(z)\\
=&\sum_{k\geq 0}\sum_j\int_{U_{j,k}}|f(z,\cdots,z)|^p(1-|z|^2)^{\alpha+n-1}dm_2(z)\\
\leq& C\sum_{k\geq 0}\sum_j\max_{U_{j,k}}|f(z,\cdots,z)|^p2^{-2k}2^{-k(\alpha+n-1)}\\
\leq& C\sum_{k\geq 0}\sum_{j_1,\cdots j_n}\max_{z\in U_{k,j_1,\cdots,j_n}}|f(z,\cdots,z)|^p2^{-k(\alpha+n+1)}\\
\leq& C\sum_{k\geq 0}\sum_{j_1,\cdots j_n}2^{2kn}M2^{-k(\alpha+n+1)},
\end{aligned}$$

where

$$M = \int_{\widetilde{U}^*_{k,j_1,\cdots,j_n}}|f(z_1,\cdots,z_n)|^p dm_{2n}(z),$$

$\widetilde{U}^*_{k,j_1,\cdots,j_n}$ is a enlarged version of dyadic cube on dyadic decomposition of subframe (see [4])

$$\widetilde{U}^*_{k,j_1,\cdots,j_n} = \widetilde{U}^*_{k,j_1} \times \cdots \times \widetilde{U}^*_{k,j_n} = \Big\{(\tau\xi_1,\cdots,\tau\xi_n), \tau \in \widetilde{I}_k = (1-\frac{1}{2^{k-1}}, 1-\frac{1}{2^{k+2}}],$$

$$\xi_i \in \widetilde{I}_{k,j_i} = [\frac{\pi(j_i - 1/2)}{2^k}, \frac{\pi(j_i+1/2)}{2^k})\Big\}.$$

We have

$$\begin{aligned} M &= \Big(\int_{1-2^{-k+1}}^{1-2^{-k-2}} \int_{\widetilde{I}_{k,j_1}}\Big)\cdots\Big(\int_{1-2^{-k+1}}^{1-2^{-k-2}} \int_{\widetilde{I}_{k,j_n}}\Big)|f(z_1,\cdots,z_n)|^p dm_{2n}(z) \\ &\le C2^{-k}\cdots 2^{-k}\int_{\widetilde{I}_{k,j_1}}\cdots\int_{\widetilde{I}_{k,j_n}} |f(\widetilde{\tau_1}\xi_1,\cdots,\widetilde{\tau_n}\xi_n)|^p dm_n(\xi), \end{aligned}$$

where $|f(\widetilde{\tau_1}\xi_1,\cdots,\widetilde{\tau_n}\xi_n)| = \max_{\tau_j\in\widetilde{I}_k, j=1,\cdots,n.} |f(\tau_1\xi_1,\cdots,\tau_n\xi_n)|$. Therefore,

$$\begin{aligned} &\int_U |f(z,\cdots,z)|^p(1-|z|^2)^{\alpha+n-1}dm_2(z) \\ \le\ & C\sum_{k\ge 0}2^{-k\alpha}\sum_{j_1,\cdots,j_n}\int_{\widetilde{I}_{k,j_1}}\cdots\int_{\widetilde{I}_{k,j_n}} |f(\widetilde{\tau_1}\xi_1,\cdots,\widetilde{\tau_n}\xi_n)|^p dm_n(\xi)\int_{1-2^{-k-2}}^{1-2^{-k-3}} d\tau. \end{aligned}$$

Since

$$\int_{T^n} |f(\widetilde{\tau_1}\xi_1,\cdots,\widetilde{\tau_n}\xi_n)|^p dm_n(\xi) \le \int_{T^n} |f(\tau\xi_1,\cdots,\tau\xi_n)|^p dm_n(\xi),$$

for $0<p<\infty$, $\tau_j \in [1-2^{-k-2}, 1-2^{-k-3})$, $j=1,\cdots,n$, we obtain

$$\begin{aligned} &\int_U |f(z,\cdots,z)|^p(1-|z|^2)^{\alpha+n-1}dm_2(z) \\ \le\ & C\sum_{k\ge 0}2^{-k\alpha}\int_{1-2^{-k-2}}^{1-2^{-k-3}}\int_{T^n}|f(\tau\xi_1,\cdots,\tau\xi_n)|^p dm_n(\xi)d\tau \\ \le\ & C\int_{T^n}\int_0^1 |f(|z|\xi_1,\cdots,|z|\xi_n)|^p(1-|z|^2)^{\alpha}dm_n(\xi)d|z|. \end{aligned}$$

The proof of this theorem is finished. □

Remark 3.7. The assertions of Theorem 3.6 are obvious for $n=1$.

From Theorem 3.6, we easily get the following result, which was proved in [4].

Corollary 3.8. *Let $p\in(0,\infty)$, $\alpha>-1$, $n\in\mathbb{N}$, $f\in H(U^n)$. Then*

$$\begin{aligned} &\int_U |f(z,\cdots,z)|^p(1-|z|)^{\alpha n+2n-2}dm_2(z) \\ \le\ & C\int_{U^n}|f(z_1,\cdots,z_n)|^p(1-|z_1|)^{\alpha}\cdots(1-|z_n|)^{\alpha}dm_{2n}(z). \end{aligned}$$

Corollary 3.9. *Let $u \in H(U^n)$, $q \in (0,\infty)$, $\alpha_j > 0$, $j = 1, \cdots, n$, $n \in \mathbb{N}$. Then*

$$\int_U |u(z,\cdots,z)|^q (1-|z|^2)^{\sum_{j=1}^n \alpha_j - 1} dm_2(z)$$
$$\leq \; C \int_T \int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} |u(z_1,\cdots,z_n)|^q \prod_{k=1}^n (1-|z_k|^2)^{\alpha_k - 2} dm_{2n}(z) dm(\xi).$$

Proof. Fix $\xi \in T$, then (see [3])

$$\int_0^1 \frac{|\widetilde{u}(\rho\xi)|^q}{1-\rho} d\rho \leq C \int_{\Gamma_t(\xi)} \frac{|\widetilde{u}(z)|^q dm_2(z)}{(1-|z|^2)^2}.$$

Choosing $\widetilde{u} = u(\rho\xi)(1-\rho)^{\alpha/q}$, $u \in H(U)$, $\alpha, q \in (0,\infty)$, we have

$$\int_0^1 |u(\rho\xi)|^q (1-\rho)^{\alpha-1} d\rho \leq C \int_{\Gamma_t(\xi)} |u(z)|^q (1-|z|^2)^{\alpha-2} dm_2(z).$$

Using the last inequality, by each variable we get the following

$$\int_0^1 \cdots \int_0^1 |u(\rho_1\xi,\cdots,\rho_n\xi)|^q \prod_{k=1}^n (1-\rho_k)^{\alpha_k - 1} d\rho_1 \cdots d\rho_n$$
$$\leq \; C \int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} |u(z)|^q \prod_{k=1}^n (1-|z_k|^2)^{\alpha_k - 2} dm_{2n}(z).$$

Integrating both sides by T and using *i)* of Theorem 3.6, we have

$$\int_U |u(z,\cdots,z)|^q (1-|z|^2)^{\sum_{j=1}^n \alpha_j - 1} dm_2(z)$$
$$\leq \; C \int_T \int_0^1 \cdots \int_0^1 |u(\rho_1\xi,\cdots,\rho_n\xi)|^q \prod_{k=1}^n (1-\rho_k)^{\alpha_k - 1} d\rho_1 \cdots d\rho_n dm(\xi)$$
$$\leq \; C \int_T \int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} |u(z_1,\cdots,z_n)|^q \prod_{k=1}^n (1-|z_k|^2)^{\alpha_k - 2} dm_{2n} dm(\xi).$$

This completes the proof of Corollary 3.9. □

Remark 3.10. For $n = 1$, the assertion of Corollary 3.9 is contained in [3, 18].

References

1. K. L. Avetisyan and R. F. Shamoyan, *Some generalization of Littlewood-Paley inequality in the polydisk*, Matematicki Vesnik, **58** (3-4) (2006), 97–110.
2. S. M. Buckley, P. Koskela and D. Vukotic, *Fractional integration, differentiation, and weighted Bergman spaces*, Math. Proc. Cambridge Phil. Soc., **126** (1999), 369–385.
3. W. Cohn, *Weighted Bergman projections and tangential area integrals*, Studia Math., **106** (1993), 59–76.
4. A. E. Djrbashian and F. A. Shamoian, *Topics in the Theory of A^p_α Spaces*, Leipzig, Teubner, 1988.
5. P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.
6. V. S. Guliev and P. I. Lizorkin, *Spaces of holomorphic functins and harmonic functions in the polydisk and their connections with boundary values*, Trudy Mat. Inst. Steklov RAN, **204** (1993), 137–159.

7. M. I. Gvaradze, *Multipliers of a class of analytic functions defined on a polydisc*, Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze, **66** (1980), 15–21.
8. H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces*, Graduate Texts in Mathematics, 199, Springer, New York, 2000.
9. V. G. Mazya, *Sobolev Spaces*, Springer-Verlag, Berlin, 1985.
10. J. M. Ortega and J. Fàbrega, *Hardy's inequality and embeddings in holomorphic Triebel-Lizorkin spaces*, Illinois J. Math., **43** (1999), 733–751.
11. R. F. Shamoyan, *On multipliers from spaces of Bergman type to Hardy spaces in polydisk*, Ukrainian Math. J., **52** (10)(2000), 1606–1617.
12. R. F. Shamoyan, *On quasinorms of functions from holomorphic Lizorkin Triebel type spaces on subframe and polydisk*, Sympozium Fourier series and application, Conference materials, Rostov Na Donu, in Russian, 2002, 54-55.
13. A. L. Shields and D. L. Williams, *Bounded projections, duality, and multipliers in spaces of analytic functions*, Trans. Amer. Math. Soc., **162** (1971), 287–302.
14. R. M. Trigub, *Multipliers in the Hardy spaces $H^p(D^m)$ with $p \in (0,1]$ and approxiamation properties of summability methods for power series*, Mat. Sb., **188** (4) (1997), 145–160.
15. W. Rudin, *Function Theory in the Polydisk*, Benjamin, New York, 1969.
16. K. Zhu, *Weighted Bergman projections on the polydisk*, Houston J. Math., **20** (2) (1994), 275–292.
17. K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, New York, 2005.
18. A. Zygmund, *Trigonometric Series, (2nd.ed.)*, Vol.2, Cambridge Univ. Press, Cambridge, 1959.

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