A Recurrent Curvature-Like Tensor on Semi-Definite Kähler Manifolds

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Abstract

The purpose of this paper is to investigate properties of semi-definite Kähler manifolds with the second curvature-like tensor H. In paticular, we study a semi-definite H-recurrent Kähler manifold.

Mathematics Subject Classification: 53C50, 53C55
Key words: Semi-definite Kähler manifold, curvature-like tensor, Ricci-like tensor, recurrent curvature tensor, complex space form

1 Introduction

As is well known, there exist some special tensors on the Kähler manifold which extend the Riemannian curvature tensor on the Riemannian manifold, for example, the concircular, projective or conformal curvature tensor. In [3], Bochner introduced three kinds of curvature-like tensors on the Kähler manifold which are closely related to the projective curvature tensor and the conformal curvature tensor, and calculated Betti numbers on a compact Kähler manifold under suitable conditions for these three curvature-like tensors. We can see detailed calculations with respect to these three kinds of curvature-like tensors in Yano and Bochner [8]. The first tensor W is called the Weyl curvature tensor and the third tensor B is known as the Bochner curvature tensor. In [4], Matsumoto and Tanno treated Kähler manifolds with parallel Bochner curvature tensor.

In this note, we investigate properties of semi-definite Kähler manifolds with the second curvature-like tensor H of the above three curvature-like tensors.

Let (M, g) be an n-dimensional semi-definite Kähler manifold $n \geq 2$ with the Kähler connection ∇. Let R or S be the Riemannian curvature tensor or the Ricci tensor with components $K_{\bar{i}jk\bar{l}}$ or $S_{i\bar{j}}$ and let H be the second curvature-like tensor on M with components $H_{\bar{i}jk\bar{l}}$ defined by

$$H_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}\{\epsilon_j(\delta_{ji}S_{k\bar{l}} + \delta_{jl}S_{k\bar{i}}) + \epsilon_k(\delta_{ki}S_{j\bar{l}} + \delta_{kl}S_{j\bar{i}})\}.$$

Balkan Journal of Geometry and Its Applications, Vol.6, No.2, 2001, pp. 89-98

The Riemannian curvature tensor R (resp. the second curvature-like tensor H) is said to be *recurrent*, if there exists an 1-form α such that $\nabla R = \alpha \otimes R$ (resp. $\nabla H = \alpha \otimes H$) and then M is said to be *R-recurrent* (resp. *H-recurrent*). The class of semi-definite H-recurrent Kähler manifolds contains locally symmetric semi-definite Kähler manifolds, semi-definite Kähler manifolds with parallel second curvature-like tensor and semi-definite Kähler manifolds with vanishing second curvature-like tensor. From these facts it is natural to consider properties of semi-definite R-recurrent or H-recurrent Kähler manifolds.

The purpose of this paper is to prove the following

Theorem. *Let M be an n-dimensional semi-definite Kähler manifold ($n \geq 2$) with non-null second curvature like tensor H. If it is H-recurrent, then it is R-recurrent.*

2 Semi-definite Kähler manifolds

This section is concerned with recalling basic formulas on semi-Kähler manifolds.

Let M be an n-dimensional semi-definite Kähler manifold ($n \geq 2$) with a semi-definite Kähler metric tensor g and almost complex structure J. From the semi-definite Kähler structure $\{g, J\}$, it follows that J is integrable and the index of g is even, say $2s (0 \leq s \leq n)$. In the case where the index $2s$ is contained in the range $0 < s < n$, the structure $\{g, J\}$ is said to be *indefinite Kähler structure* and, in particular, in the case where $s = 0$ or n, it is said to be *definite Kähler structure.*

In this section, we shall consider M an n-dimensional connected semi-definite Kähler manifold ($n \geq 2$) of index $2s$, $0 \leq s \leq n$. Then a local unitary frame field $\{U_j\} = \{U_1, \dots, U_n\}$ on a neighborhood of M can be chosen. This is a complex linear frame which is orthonormal with respect to the semi-definite Kähler metric g of M, that is, $g(U_j, U_k) = \epsilon_j \delta_{jk}$, where

$$\epsilon_j = -1 \text{ or } 1 \text{ according as } 0 \leq j \leq s \text{ or } s+1 \leq j \leq n.$$

Its dual frame field $\{\omega_j\} = \{\omega_1, \dots, \omega_n\}$ consists of complex valued 1-forms of $(1,0)$ on M such that $\omega_j(U_k) = \epsilon_j \delta_{jk}$ and $\omega_1, \dots, \omega_n, \bar{\omega}_1, \dots, \bar{\omega}_n$ are linearly independent. Thus the natural extension g^c of the semi-definite Kähler metric g of M can be expressed as $g^c = 2\sum_j \epsilon_j \omega_j \otimes \bar{\omega}_j$. Associated with the frame field $\{U_j\}$, there exist complex valued forms ω_{ik}, where the indices i and k run over the range $1, \dots, n$. They are usually called *connection forms* on M such that they satisfy the structure equations of M :

$$d\omega_i + \sum_j \epsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0,$$

$$d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \tag{2.1}$$

$$\Omega_{ij} = \sum_{k,l} \epsilon_{kl} K_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l,$$

where $\epsilon_{k \dots l} = \epsilon_k \dots \epsilon_l$ and $\Omega = (\Omega_{ij})$ (*resp.* $K_{\bar{i}jk\bar{l}}$) denotes the curvature form (*resp.* components of the semi-definite Riemannian curvature tensor R) of M in Kobayashi and Nomizu [6]. The second formula of (2.1) means the skew-Hermitian symmetricity of $\Omega_{ij,}$ which is equivalent to the symmetric condition

$$K_{\bar{i}jk\bar{l}} = \bar{K}_{\bar{j}il\bar{k}}. \tag{2.2}$$

Moreover, substitution the third equation of (2.1) into the exterior differential of the first equation of (2.1), the first Bianchi identity

$$\sum_j \epsilon_j \Omega_{ij} \wedge \omega_j = 0 \tag{2.3}$$

is given. It implies further symmetric relations

$$K_{\bar{i}jk\bar{l}} = K_{\bar{i}kj\bar{l}} = K_{\bar{l}kj\bar{i}} = K_{\bar{l}jk\bar{i}}. \tag{2.4}$$

Now, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows :

$$S = \sum_{i,j} \epsilon_{ij}(S_{i\bar{j}}\omega_i \otimes \bar{\omega}_j + S_{\bar{i}j}\bar{\omega}_i \otimes \omega_j), \tag{2.5}$$

where $S_{i\bar{j}} = \sum_k \epsilon_k K_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \bar{S}_{\bar{i}j.}$ The scalar curvature r of M is also given by

$$r = 2\sum_j \epsilon_j S_{j\bar{j}}. \tag{2.6}$$

An n-dimensional semi-definite Kähler manifold M is said to be *Einstein*, if the Ricci tensor S is given by

$$S_{i\bar{j}} = \frac{r}{2n}\epsilon_i\delta_{ij}, \quad S = \frac{r}{2n}g. \tag{2.7}$$

The components $K_{\bar{i}jk\bar{l}m}$ and $K_{\bar{i}jk\bar{l}\bar{m}}$ (*resp.* $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative ∇R (resp. ∇S) of the Riemannian curvature tensor R (*resp.* the Ricci tensor S) are obtained by

$$\begin{aligned} &\sum_m \epsilon_m(K_{\bar{i}jk\bar{l}m}\omega_m + K_{\bar{i}jk\bar{l}\bar{m}}\bar{\omega}_m) = dK_{\bar{i}jk\bar{l}} - \\ &-\sum_m \epsilon_m(K_{\bar{m}jk\bar{l}}\bar{\omega}_{mi} + K_{\bar{i}mk\bar{l}}\omega_{mj} + K_{\bar{i}jm\bar{l}}\omega_{mk} + K_{\bar{i}jk\bar{m}}\bar{\omega}_{ml}), \end{aligned} \tag{2.8}$$

and

$$\sum_k \epsilon_k(S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k \epsilon_k(S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}). \tag{2.9}$$

By differentiating the third equation in (2.1) exteriorly, we have

$$K_{\bar{i}jk\bar{l}m} = K_{\bar{i}jm\bar{l}k}, \tag{2.10}$$

and hence we have

$$S_{i\bar{j}k} = S_{k\bar{j}i} = \sum_l \epsilon_l K_{\bar{j}ik\bar{l}l}. \tag{2.11}$$

On the other hand, the exterior differential dr of the scalar curvature r on M is given by

$$dr = \sum_m \epsilon_m (r_m \omega_m + r_{\bar{m}} \bar{\omega}_m). \tag{2.12}$$

Putting $i = j$ in (2.11) and summing up with respect to i, we have

$$\sum_r \epsilon_r S_{r\bar{r}k} = \sum_{r,s} \epsilon_{rs} K_{\bar{r}rk\bar{s}s} = \sum_{r,s} \epsilon_{rs} K_{\bar{r}rs\bar{s}k} = \frac{r_k}{2}. \tag{2.13}$$

Now, a semi-definite Kähler manifold M of constant holomorphic sectional curvature is called a *semi-definite complex space form*. An n-dimensional semi-definite complex space form of constant holomorphic sectional curvature c and of index $2s$ $0 \le s \le n$, is denoted by $M_s^n(c)$. The standard models of semi-definite complex space forms are the following three kinds : the semi-definite complex Euclidean space $\mathbf{C}_s^n$, the semi-definite complex projective space $\mathbf{CP}_s^n(c)$ or the semi-definite complex hyperbolic space $\mathbf{CH}_s^n(c)$, according as $c = 0$, $c > 0$ or $c < 0$. For any integer s $(0 \le s \le n)$, it is seen that they are only complete and simply connected semi-definite complex space forms of dimension n and of index $2s$. The Riemannian curvature tensor $K_{\bar{i}jk\bar{l}}$ of $M_s^n(c)$ is given by

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}\epsilon_{jk}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{2.14}$$

3 The second curvature-like tensor H

This section is concerned with the second curvature-like tensor H on a semi-definite Kähler manifold M.

Under the notation prepared in §2, the *second curvature-like tensor H* with components $H_{\bar{i}jk\bar{l}}$ is defined by

$$H_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{1}{2(n+1)}\{\epsilon_j(\delta_{ji}S_{k\bar{l}} + \delta_{jl}S_{k\bar{i}}) + \epsilon_k(\delta_{ki}S_{j\bar{l}} + \delta_{kl}S_{j\bar{i}})\}. \tag{3.1}$$

As is easily seen (cf. Yano and Bocher [8] in the definite case), the semi-definite Kähler manifold M with vanishing second curvature-like tensor H is of constant holomorphic sectional curvature.

On the other hand, let Z be a tensor with components $Z_{\bar{i}jk\bar{l}}$ such that

$$Z_{\bar{i}jk\bar{l}} = K_{\bar{i}jk\bar{l}} - \frac{r}{2n(n+1)}\epsilon_{jk}(\delta_{ji}\delta_{kl} + \delta_{jl}\delta_{ki}). \tag{3.2}$$

In the case where the semi-definite Kähler manifold M is Einstein, the tensor Z is equivalent to the second curvature-like tensor H. It is trivial that the semi-definite Kähler manifold M with $Z = 0$ is of constant holomorphic sectional curvature. The tensor Z is the formal analogue to the concircular curvature tensor on the Riemannian manifold [8].

Lemma 3.1([5]). *The second curvature-like tensor H with components $H_{\bar{i}jk\bar{l}}$ satisfies*

$$H_{\bar{i}jk\bar{l}} = H_{\bar{i}kj\bar{l}} = H_{\bar{l}jk\bar{i}} = H_{\bar{l}kj\bar{i}}, \tag{3.3}$$

$$H_{i\bar{j}} = H_{\bar{j}i} = \bar{H}_{j\bar{i}} = \frac{1}{4(n+1)}\{2nS_{i\bar{j}} - r\epsilon_i\delta_{ij}\}. \tag{3.4}$$

where $H_{i\bar{j}}$ is the components of the Ricci-like tensor of H defined by $H_{i\bar{j}} = \sum_r \epsilon_r H_{\bar{r}ri\bar{j}}$. Moreover, $\sum_i H_{i\bar{i}} = 0$.

Theorem 3.2. *Let M be an n-dimensional semi-(definite) Kähler manifold $(n \geq 2)$. If the Ricci tensor S is parallel, then the second fundamental tensor H is parallel to the curvature tensor R.*

Proof. By the definition of H, we have

$$\begin{aligned} H_{\bar{i}jk\bar{l}m} &= K_{\bar{i}jk\bar{l}m} - \\ &- \frac{1}{2(n+1)}\{\epsilon_j(\delta_{ji}S_{k\bar{l}m} + \delta_{jl}S_{k\bar{i}m}) + \epsilon_k(\delta_{kl}S_{j\bar{i}m} + \delta_{ki}S_{j\bar{l}m})\}. \end{aligned} \tag{3.5}$$

Putting $i = m$ in (3.5) and summing up with respect to i, we obtain

$$\sum_r \epsilon_r H_{\bar{r}jk\bar{l}r} = S_{j\bar{l}k} - \frac{1}{4(n+1)}(4S_{j\bar{l}k} + r_j\epsilon_k\delta_{kl} + r_k\epsilon_j\delta_{jl}) = 0,$$

where we have used (2.10), (2.11), (2.13) and the relation

$$H_{i\bar{j}k} = \frac{1}{4(n+1)}(2nS_{i\bar{j}k} - \epsilon_i\delta_{ij}r_k).$$

From this it follows that

$$4nS_{j\bar{l}k} = r_j\epsilon_k\delta_{kl} + r_k\epsilon_j\delta_{jl}. \tag{3.6}$$

Putting $k = l$ in (3.6) and summing up with respect to k, we have $(n-1)r_j = 0$. Since $n \geq 2$, we get $r_j = 0$. It means that the scalar curvature r is constant, and hence by the above equation (3.6), we have $S_{j\bar{l}k} = 0$, from which together with (3.5) we have $H_{\bar{i}jk\bar{l}m} = K_{\bar{i}jk\bar{l}m}$. Thus we obtain $\nabla H = \nabla R$.

It completes the proof. □

In [5], the authors of the present paper proved the following :

Let M be an n-dimensional semi-definite Kähler manifold $(n \geq 2)$. Then its second curvature-like tensor H is parallel if and only if its Riemannian curvature tensor R is parallel.

4 Semi-definite H-recurrent Kähler manifolds

Now, we assume that the curvature-like tensor H is recurrent, namely, there exists an 1-form α such that $\nabla H = \alpha \otimes H$. Then M is said to be *H-recurrent.* In terms of components, we assume that

$$H_{\bar{i}jk\bar{l}m} = \alpha_m H_{\bar{i}jk\bar{l}}. \tag{4.1}$$

Substituting (3.1) and (3.5) into (4.1), we get

$$\begin{aligned} K_{\bar{i}jk\bar{l}m} &= \alpha_m K_{\bar{i}jk\bar{l}} + \\ &+ \frac{1}{2(n+1)}[\epsilon_j\{\delta_{ji}(S_{k\bar{l}m} - \alpha_m S_{k\bar{l}}) + \delta_{jl}(S_{k\bar{i}m} - \alpha_m S_{k\bar{i}})\} \\ &+ \epsilon_k\{\delta_{kl}(S_{j\bar{i}m} - \alpha_m S_{j\bar{i}}) + \delta_{ki}(S_{j\bar{l}m} - \alpha_m S_{j\bar{l}})\}]. \end{aligned} \tag{4.2}$$

Putting $k = l$ in (4.2) and summing up with respect to k, we have

$$S_{j\bar{i}m} - \alpha_m S_{j\bar{i}} = \frac{1}{2n}\{(r_m - r\alpha_m)\epsilon_j \delta_{ji}\}, \tag{4.3}$$

where we have used (2.5) and (2.11).

Theorem 4.1. *Let M be an $n(\geq 2)$-dimensional semi-definite Kähler manifold. Put $M' = \{x \in M| < H,\ H >= 0,\ H \neq 0$ at $x\}$, and assume that the subset $M - M'$ is dense in M. If it is H-recurrent, then it is R-recurrent.*

Proof. If $H = 0$ on an open subset U in M, then the holomorphic sectional curvature is constant and hence $\nabla H = \nabla R = 0$ on U. So it is sufficient to consider on an open set $V = \{x \in M|\langle H,\ H\rangle \neq 0$ at $x\}$. The assumption $\nabla H = \alpha \otimes H$ implies that $\alpha_{m\bar{p}} = \alpha_{\bar{p}m}$ on V. Thus

$$H_{\bar{i}jk\bar{l}m\bar{p}} = (\alpha_{m\bar{p}} + \alpha_m \bar{\alpha}_p) H_{\bar{i}jk\bar{l}} = H_{\bar{i}jk\bar{l}\bar{p}m}. \tag{4.4}$$

It follows from $d^2\ H_{\bar{k}lm\bar{p}} = 0$ combined with (4.6) that

$$\sum_r \epsilon_r(-K_{\bar{i}jr\bar{k}} H_{\bar{r}lm\bar{p}} + K_{\bar{i}jl\bar{r}} H_{\bar{k}rm\bar{p}} + K_{\bar{i}jm\bar{r}} H_{\bar{k}lr\bar{p}} - K_{\bar{i}jr\bar{p}} H_{\bar{k}lm\bar{r}}) = 0. \tag{4.5}$$

Putting $i = j$ in (4.5) and summing up with respect to i, we have

$$\sum_r \epsilon_r(-S_{r\bar{k}} H_{\bar{r}lm\bar{p}} + S_{l\bar{r}} H_{\bar{k}rm\bar{p}} + S_{m\bar{r}} H_{\bar{k}lr\bar{p}} - S_{r\bar{p}} H_{\bar{k}lm\bar{r}}) = 0. \tag{4.6}$$

Moreover, differentiating (4.5) covariantly, we get

$$\sum_r \epsilon_r(-K_{\bar{i}jr\bar{k}q} H_{\bar{r}lm\bar{p}} + K_{\bar{i}jl\bar{r}q} H_{\bar{k}rm\bar{p}} + K_{\bar{i}jm\bar{r}q} H_{\bar{k}lr\bar{p}} - K_{\bar{i}jr\bar{p}q} H_{\bar{k}lm\bar{r}}) = 0, \tag{4.7}$$

where the equality is derived from (4.1) and (4.5).

Differentiating (4,6) covariantly and then, from (4.1) and (4.8), we get

$$\sum_r \epsilon_r(-S_{r\bar{k}q} H_{\bar{r}lm\bar{p}} + S_{l\bar{r}q} H_{\bar{k}rm\bar{p}} + S_{m\bar{r}q} H_{\bar{k}lr\bar{p}} - S_{r\bar{p}q} H_{\bar{k}lm\bar{r}}) = 0. \tag{4.8}$$

Subtraction α_q times (4.6) from (4.8), we have

$$\begin{aligned}(4.9)\qquad &\sum_r \epsilon_r\{-(S_{r\bar{k}q}-\alpha_q S_{r\bar{k}})H_{\bar{r}lm\bar{p}}+(S_{l\bar{r}q}-\alpha_q S_{l\bar{r}})H_{\bar{k}rm\bar{p}}+\\ &+(S_{m\bar{r}q}-\alpha_q S_{m\bar{r}})H_{\bar{k}lr\bar{p}}-(S_{r\bar{p}q}-\alpha_q S_{r\bar{p}})H_{\bar{k}lm\bar{r}}\}=0.\end{aligned}$$

Substituting (4.2) into (4.7), we have

$$\begin{aligned}&\sum_r \epsilon_{rj}[-\{\delta_{ji}(S_{r\bar{k}q}-\alpha_q S_{r\bar{k}})+\delta_{jk}(S_{r\bar{i}q}-\alpha_q S_{r\bar{i}})\}H_{\bar{r}lm\bar{p}}+\\ &+\{\delta_{ji}(S_{l\bar{r}q}-\alpha_q S_{l\bar{r}})+\delta_{jr}(S_{l\bar{i}q}-\alpha_q S_{l\bar{i}})\}H_{\bar{k}rm\bar{p}}+\\ &+\{\delta_{ji}(S_{m\bar{r}q}-\alpha_q S_{m\bar{r}})+\delta_{jr}(S_{m\bar{i}q}-\alpha_q S_{m\bar{i}})\}H_{\bar{k}lr\bar{p}}-\\ &-\{\delta_{ji}(S_{r\bar{p}q}-\alpha_q S_{r\bar{p}})+\delta_{jp}(S_{r\bar{i}q}-\alpha_q S_{r\bar{i}})\}H_{\bar{k}lm\bar{r}}\ ]+\\ &+\sum_r \epsilon_r[-\epsilon_r\{\delta_{rk}(S_{j\bar{i}q}-\alpha_q S_{j\bar{i}})+\delta_{ri}(S_{j\bar{k}q}-\alpha_q S_{j\bar{k}})\}H_{\bar{r}lm\bar{p}}+\\ &+\epsilon_l\{\delta_{lr}((S_{j\bar{i}q}-\alpha_q S_{j\bar{i}})+\delta_{li}(S_{j\bar{r}q}-\alpha_q S_{j\bar{r}})\}H_{\bar{k}rm\bar{p}}+\\ &+\epsilon_m\{\delta_{mr}(S_{j\bar{i}q}-\alpha_q S_{j\bar{i}})+\delta_{mi}(S_{j\bar{r}q}-\alpha_q S_{j\bar{r}})\}H_{\bar{k}lr\bar{p}}-\\ &-\epsilon_r\{\delta_{rp}(S_{j\bar{i}q}-\alpha_q S_{j\bar{i}})+\delta_{ri}(S_{j\bar{p}q}-\alpha_q S_{j\bar{p}})\}H_{\bar{k}lm\bar{r}}\ ]=0.\end{aligned}$$

Combining (4.9) with the above equation, we can obtain

$$\begin{aligned}&\sum_r \epsilon_{rj}\{-\delta_{jk}(S_{r\bar{i}q}-\alpha_p S_{r\bar{i}})H_{\bar{r}lm\bar{p}}+\delta_{jr}(S_{l\bar{i}q}-\alpha_q S_{l\bar{i}})H_{\bar{k}rm\bar{p}}+\\ (4.10)\qquad &+\delta_{jr}(S_{m\bar{i}q}-\alpha_q S_{m\bar{i}})H_{\bar{k}lr\bar{p}}-\delta_{jp}(S_{r\bar{i}q}-\alpha_q S_{r\bar{i}})H_{\bar{k}lm\bar{r}}\}-\\ &-(S_{j\bar{k}q}-\alpha_q S_{j\bar{k}})H_{\bar{i}lm\bar{p}}-(S_{j\bar{p}q}-\alpha_q S_{j\bar{p}})H_{\bar{k}lm\bar{i}}+\\ &+\sum_r \epsilon_r\{\epsilon_l\delta_{li}(S_{j\bar{r}q}-\alpha_q S_{j\bar{r}})H_{\bar{k}rm\bar{p}}+\epsilon_m\delta_{mi}(S_{j\bar{r}q}-\alpha_q S_{j\bar{r}})H_{\bar{k}lr\bar{p}}\}=0.\end{aligned}$$

Putting $j=k$ in (4.10) and summing up with respect to j, we have

$$\begin{aligned}&(n+1)\sum_r \epsilon_r(S_{r\bar{i}q}-\alpha_q S_{r\bar{i}})H_{\bar{r}lm\bar{p}}-(S_{l\bar{i}q}-\alpha_q S_{l\bar{i}})H_{m\bar{p}}-\\ &-(S_{m\bar{i}q}-\alpha_q S_{m\bar{i}})H_{l\bar{p}}+(r_q-r\alpha_q)H_{\bar{i}lm\bar{p}}+\textstyle\sum_r \epsilon_r(S_{r\bar{p}q}-\alpha_q S_{r\bar{p}})H_{\bar{r}lm\bar{i}}-\\ &-\sum_{r,s}\epsilon_{rs}\{\epsilon_l\delta_{li}(S_{r\bar{s}q}-\alpha_q S_{r\bar{s}})H_{\bar{r}sm\bar{p}}+\epsilon_m\delta_{mi}(S_{r\bar{s}q}-\alpha_q S_{r\bar{s}})H_{\bar{r}sl\bar{p}}\}=0.\end{aligned}$$

Taking account of (4.3), the last equation turns out to be

$$(4.11)\qquad (r_q-r\alpha_q)\{(2n+1)H_{\bar{i}lm\bar{p}}-2\epsilon_m\delta_{mi}H_{l\bar{p}}-2\epsilon_l\delta_{li}H_{m\bar{p}}\}=0.$$

Putting $m=p$ in (4.11) and summing up with respect to the index p, we have

$$3n(r_q-r\alpha_q)H_{l\bar{i}}=0$$

for any indices $q,\ l$ and i. Hence (4.11) implies that

$$(r_q - r\,\alpha_q)H_{\bar{i}lm\bar{p}} = 0 \tag{4.12}$$

for any indices $q,\ i,\ l,\ m$ and p.

Since $H \neq 0$ on V, (4.12) implies that $\nabla r = r\alpha$ on V. So, from (4.5), we obtain that $\nabla S = \alpha \otimes S$.

Moreover, it follows from (4.4) that $\nabla R = \alpha \otimes R$.

It completes the proof. □

Remark 4.1. In the last section, an example of an $n(\geq 2)$-dimensional indefinite H-recurrent Kähler manifolds which satisfies $H \neq 0$ and $\|H\| = 0$ is given.

In particular, we consider the case that M is positive definite. Then the condition $\|H\| = 0$ is equivalent to $H = 0$. Thus we have the following

Corollary 4.2. *Let M be an $n(\geq 2)$-dimensional Kähler manifold. If it is H-recurrent, then it is R-recurrent.*

5 Examples

This section is devoted to some examples of H-recurrent indefinite Kähler manifold. For any integer $p(\geq 3)$ an indefinite complex hypersurface $M(p,c)$ of a $(2n+1)$-dimensional indefinite complex Euclidean space $\mathbf{C}_n^{2n+1}$ of index $2n$ defined as follows ([1], [6]) :

Let

$$\{z^A\} = \{z^j,\ z^{j^*},\ z^{2n+1}\} = \{z^1,\ \ldots,\ z^{2n+1}\}$$

be a complex coordinate of $\mathbf{C}_n^{2n+1}$ and let c be a complex number such that $|c| = 1$. Then $M(p,c)$ is an indefinite complete complex hypersurface of index $2n$ defined by

$$z^{2n+1} = \sum_j h_j(z^j + cz^{j^*}), \quad j^* = n+j, \quad h_j(z) = z^p.$$

Then the components h_{AB} and the components h_{ABC} of the second fundamental form and its covariant derivatives of M we have

$$h_{ij} = p(p-1)\delta_{ij}z^{p-2}, h_{i^*j} = p(p-1)c\delta_{ij}z^{p-2}, h_{i^*j^*} = p(p-1)c^2\delta_{ij}z^{p-2},$$
$$h_{ijk} = p(p-1)(p-2)\delta_{ij}\delta_{ik}z^{p-3}, \quad h_{i^*jk} = p(p-1)(p-2)c\delta_{ij}\delta_{ik}z^{p-3},$$
$$h_{i^*j^*k} = p(p-1)(p-2)c^2\delta_{ij}\delta_{ik}z^{p-3}, \quad h_{i^*j^*k^*} = p(p-1)(p-2)c^3\delta_{ij}\delta_{ik}z^{p-3}.$$

Thus under the assumption of p the second fundamental form is not parallel.

Let $K_{\bar{A}BC\bar{D}}$ (resp. $K_{\bar{A}BC\bar{D}E}$) the components of the Riemannian curvature tensor R (resp. the covariant derivative ∇R) of the indefinite complex hypersurface $M(p,c)$. In the theory of indefinite complex hypersurfaces in the indefinite complex space form the components are given by

$$K_{\bar{A}BC\bar{D}} = -h_{BC}\bar{h}_{AD},$$

$$K_{\bar{A}BC\bar{D}E} = -h_{BCE}\bar{h}_{AD},$$

which yield that $R \neq 0$ and $\nabla R \neq 0$. Namely, M is not flat and not locally symmetric.

Now, by the property of the second fundamental form, they give to us

$$K_{\bar{A}BC\bar{D}E} = -\frac{\delta_{AE}(p-2)K_{\bar{A}BC\bar{D}}}{z}.$$

Thus we see that M is R-recurrent. Furthermore, in the theory of indefinite submanifolds of an indefinite complex space form, we have

$$\begin{aligned} S_{i\bar{j}} &= -\sum_r h_{ir}\bar{h}_{rj} + \sum_r h_{ir^*}\bar{h}_{r^*j} = \\ &= p^2(p-1)^2(-1+|c|^2)\delta_{ij}z^{2(p-2)} = 0. \end{aligned}$$

Similarly we have

$$S_{A\bar{B}} = -\sum_r h_{Ar}\bar{h}_{rB} + \sum_r h_{Ar^*}\bar{h}_{r^*B} = 0,$$

and from this we see that the Ricci tensor S is flat and hence the scalar curvature r vanishes. Hence this implies

$$H_{\bar{A}BC\bar{D}} = K_{\bar{A}BC\bar{D}}.$$

Furthermore it is easily seen that

$$\begin{aligned} \sum_R \epsilon_R K_{\bar{A}BC\bar{R}}\bar{K}_{\bar{E}FG\bar{R}} &= -\sum_r K_{\bar{A}BC\bar{r}}\bar{K}_{\bar{E}FG\bar{r}} + \\ &+ \sum_r \epsilon_r K_{\bar{A}BC\bar{r}^*}\bar{K}_{\bar{E}FG\bar{r}^*} = 0, \end{aligned}$$

and hence we obtain

$$\|H\| = \|R\| = 0.$$

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