# Gauge Transformations on Holomorphic Bundles

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**Abstract**

In holomorphic tangent bundle $T'M$ we define a generalization of classical gauge transformation, called complex gauge transformation, and related to it we shall study the invariant geometric objects: $d$-gauge tensors, nonlinear gauge connection, gauge complex derivatives.

The problem of global invariance concerning a complex Lagrangian is treated in the section related to Einstein-Yang-Mills complex equations. Finally, we shall discuss a few applications regarding infinitesimal complex gauge transformations.

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**Key words**: gauge transformation, connection, Lagrangian, holomorphic bundle.

## 1 Introduction. The holomorphic bundle $T'M$

In many physical theories related to relativistic quantum, such as spontaneous symmetry-breaking, the Higgs theory, etc., the gauge Lagrangians which are used are complex scalar fields. Consequently, a geometrical approach of such problems requires to create a complex model for gauge geometric fields and their derivatives.

For the real case the geometric methods of fibre bundles with structural group are used for a long period of time in gauge theories ([3]). A generalization for the vectorial bundles of gauge transformation was made in [1], [7].

In a previous paper ([8]) we studied the Lagrange spaces having as base the holomorphic tangent bundle $T'M$, endowed with a nonlinear complex connection. Following such ideas, in the present paper we shall deal with complex gauge transformations on $T'M$.

Briefly, we shall introduce the basic results from [8].

Let us consider $M$ a complex manifold, $\dim_C M = n$, and $(U, (z^i))$ the complex coordinates in a local map. The complexification $T_CM$ of the tangent bundle $TM$ in each $z \in U$ is decomposed in the $(1,0)$-vectors and their conjugates of $(0,1)$-type ([5], [6]),

$$T_CM = T'M \oplus T''M.$$

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The bundle $\pi_T : T'M \longrightarrow M$ is holomorphic and $\dim_C T'M = 2n$. With $V(T'M) = \{\xi \in T'(T'M)' \ \pi_{T*}(\xi) = 0\}$ is denoted the vertical sub-bundle, which is also holomorphic, a local basis in $V(T'M)$ being $\left\{\dfrac{\partial}{\partial \eta^i}\right\}_{i=\overline{1,n}}$.

A nonlinear complex connection, $(nl.c.c.)$ for short, is a distribution $N$ in $T'(T'M)$, $N : u = (z^i, \eta^i) \to H_u(T'M)$ where $H_u(T'M)$ is a supplementary sub-bundle of the vertical bundle in $T'(T'M)$. A local basis in $H_u(T'M)$ is denoted by

$$\left\{\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N_i^j \frac{\partial}{\partial \eta^j}\right\}_{i,j=\overline{1,n}},$$

where $N_i^j$ are the $(nl.c.c.)$-coefficients and satisfying the following rules of transformation at the change of local map

$$N'^i_k \frac{\partial z'^k}{\partial z^j} = \frac{\partial z'^i}{\partial z^k} N_j^k - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^k. \tag{1.1}$$

From (1.1) we can easily verify that the adapted basis $\left\{\dfrac{\delta}{\delta z^i}\right\}$ is changed after the rule

$$\frac{\delta}{\delta z^i} = \frac{\partial z'^j}{\partial z^i} \frac{\delta}{\delta z'^j}. \tag{1.2}$$

Taking the conjugate bundles $\overline{H(T'M)}$ and $\overline{V(T'M)}$ with the corresponding local bases $\left\{\dfrac{\delta}{\delta \overline{z}^i}\right\}$ and $\left\{\dfrac{\partial}{\partial \overline{\eta}^i}\right\}$, we obtain the following decomposition for the complexified bundle:

$$T_C M = H(T'M) \oplus V(T'M) \oplus \overline{H(T'M)} \oplus \overline{V(T'M)}, \tag{1.3}$$

with the corresponding projectors denoted by $h, v, \overline{h}, \overline{v}$.

Furthermore, we shall use the following abbreviate notations: $\{\delta_i, \partial_i, \delta_{\overline{i}}, \partial_{\overline{i}}\}$ for the adapted bases in (1.3) decomposition.

Let us consider $D$ a derivative law on $T_C(T'M)$. For a given $(nl.c.c.)$ $N$, the derivative $D$ is called *$N$-linear complex connection,* shortly: $N-(l.c.c.)$, if $D$ preserves the four distributions from (1.3). In the adapted basis $\{\delta_i, \partial_i, \delta_{\overline{i}}, \partial_{\overline{i}}\}$ the $N-(.c.c.)D$ has the following local expression:

$$\begin{aligned}
&D_{\delta_k}\delta_j = \overset{1}{L}{}^i_{jk}\,\delta_i; \quad D_{\partial_k}\delta_j = \overset{1}{C}{}^i_{jk}\,\delta_i; \quad D_{\delta_{\overline{k}}}\delta_j = \overset{3}{L}{}^i_{j\overline{k}}\,\delta_i; \quad D_{\partial_{\overline{k}}}\delta_j = \overset{3}{C}{}^i_{j\overline{k}}\,\delta_i;\\
&D_{\delta_k}\partial_j = \overset{2}{L}{}^i_{jk}\,\partial_i; \quad D_{\partial_k}\partial_j = \overset{2}{C}{}^i_{jk}\,\partial_i; \quad D_{\delta_{\overline{k}}}\partial_j = \overset{4}{L}{}^i_{j\overline{k}}\,\partial_i; \quad D_{\partial_{\overline{k}}}\partial_j = \overset{4}{C}{}^i_{j\overline{k}}\,\partial_i;\\
&D_{\delta_k}\delta_{\overline{j}} = \overset{3}{\overline{L}}{}^{\overline{i}}_{\overline{j}k}\,\delta_{\overline{i}}; \quad D_{\partial_k}\delta_{\overline{j}} = \overset{3}{\overline{C}}{}^{\overline{i}}_{\overline{j}k}\,\delta_{\overline{i}}; \quad D_{\delta_{\overline{k}}}\delta_{\overline{j}} = \overset{1}{\overline{L}}{}^{\overline{i}}_{\overline{j}\overline{k}}\,\delta_{\overline{i}}; \quad D_{\partial_{\overline{k}}}\delta_{\overline{j}} = \overset{1}{\overline{C}}{}^{\overline{i}}_{\overline{j}\overline{k}}\,\delta_{\overline{i}};\\
&D_{\delta_k}\partial_{\overline{j}} = \overset{4}{\overline{L}}{}^{\overline{i}}_{\overline{j}k}\,\partial_{\overline{i}}; \quad D_{\partial_k}\partial_{\overline{j}} = \overset{4}{\overline{C}}{}^{\overline{i}}_{\overline{j}k}\,\partial_{\overline{i}}; \quad D_{\delta_{\overline{k}}}\partial_{\overline{j}} = \overset{2}{\overline{L}}{}^{\overline{i}}_{\overline{j}\overline{k}}\,\partial_{\overline{i}}; \quad D_{\partial_{\overline{k}}}\partial_{\overline{j}} = \overset{2}{\overline{C}}{}^{\overline{i}}_{\overline{j}\overline{k}}\,\partial_{\overline{i}},
\end{aligned}$$

with $\overline{D_X Y} = D_{\overline{X}}\overline{Y}$. If

$$\overset{1}{L}{}^{i}_{jk}=\overset{2}{L}{}^{i}_{jk}=L^{i}_{jk};\ \overset{3}{L}{}^{\bar{i}}_{\bar{j}k}=\overset{4}{L}{}^{\bar{i}}_{\bar{j}k}=L^{\bar{i}}_{\bar{j}k};\ \overset{1}{C}{}^{i}_{jk}=\overset{2}{C}{}^{i}_{jk}=C^{i}_{jk};\ \overset{3}{C}{}^{\bar{i}}_{\bar{j}k}=\overset{4}{C}{}^{\bar{i}}_{\bar{j}k}=C^{\bar{i}}_{\bar{j}k}$$

(and their conjugates coincide too), then the $N-l.c.c.$ $D$ is said to be *normal*, and will be denoted by $M-(l.c.c)$.

The components of the torsion and the curvature tensors of one $M-(l.c.c.)$, $D$ have been calculated in [8].

A $N-(l.c.c.)$ is decomposed in : $D = D' + D''$, where $D'' = \overline{D'}$ and their turn for each of them: $D' = D'^h + D'^v$, $D'' = D''^h + D''^v$, with $D'^h = D_{\delta_k}$, $D'^v = D_{\partial_k}$, $D''^h = D_{\delta_{\bar{k}}}$, $D''^v = D_{\partial_{\bar{k}}}$. A system of functions on $T'M$, $w^{i_1\dots i_p\bar{i}_1\dots\bar{i}_r}_{j_1\dots j_q\bar{j}_1\dots\bar{j}_s}(z,\eta)$ is called a *d−complex tensor field* of $\begin{pmatrix} p & \bar{r} \\ q & \bar{s}\end{pmatrix}$-type if at change of local maps on $M$ it well be modified after the rule:

$$w'^{i_1\dots i_p\bar{i}_1\dots\bar{i}_r}_{j_1\dots j_q\bar{j}_1\dots\bar{j}_s}(z',\eta') = \frac{\partial z'^{i_1}}{\partial z^{k_1}}\cdots\frac{\partial z'^{i_p}}{\partial z^{k_p}}\cdot\frac{\partial z^{h_1}}{\partial z'^{j_1}}\cdots\frac{\partial z^{h_q}}{\partial z'^{j_q}}\cdot$$
$$\cdot\frac{\partial \bar{z}'^{\bar{i}_1}}{\partial \bar{z}^{\bar{k}_1}}\cdots\frac{\partial \bar{z}'^{\bar{i}_r}}{\partial \bar{z}^{\bar{k}_r}}\cdot\frac{\partial \bar{z}^{\bar{h}_1}}{\partial \bar{z}'^{\bar{j}_1}}\cdots\frac{\partial \bar{z}^{\bar{h}_r}}{\partial \bar{z}'^{\bar{j}_r}}w^{k_1\dots k_p\bar{k}_1\dots\bar{k}_r}_{h_1\dots h_p\bar{h}_1\dots\bar{h}_s}(z,\eta).$$

The derivations of the $d-$tensor $w$ will be expressed by "$|$" and by "$||$" for $h-$and respectively $v-$covariant derivative $D'$, and by "$\bar{|}$", "$\bar{||}$" respectively for $\bar{h}-$and $\bar{v}-$covariant derivative $D''$.

A metric Hermitian structure $G$ on $T_C(T'M)$ is defined by a $d-$complex tensor $g_{i\bar{j}}(z,\eta)$ of $\begin{pmatrix} 0 & \bar{0} \\ 1 & \bar{1}\end{pmatrix}$-type, nondegenerate, so that:

$$G = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + g_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j, \tag{1.6}$$

where $(dz^i, \delta\eta^i, d\bar{z}^j, \delta\bar{\eta}^j)$ is the dual adapted basis.

## 2 Gauge complex transformations

Let us consider the holomorphic bundle $\pi_T : T'M \to M$.

**Definition 2.1** A *gauge transformation* on complex manifold $M$ is a pair of analytic isomorphisms $\Upsilon = (F^0, F^1)$ on the manifolds $F^0 : M \to M$ and $F^1 : T'M \to T'M$, satisfying the property

$$\pi_T \circ F^1 = F^0 \circ \pi_T. \tag{2.1}$$

We can see that a gauge transformation $\Upsilon$ preserves the geometric structures of the manifold and the whole set determine a group structure with respect to the composition of mappings.

It is useful to obtain a local representation of complex gauge transformation $\Upsilon$.

Supposing that $\Upsilon = (F^0, F^1)$ applies the point $u = (z^i, \eta^i) \in \pi_T^{-1}(U_\alpha)$ in $\widetilde{u} = (\widetilde{z}^i, \widetilde{\eta}^i) \in \pi_T^{-1}(U_\beta)$ and by adding the condition (2.1), it results

**Proposition 2.1** *A gauge complex transformation $\Upsilon : u \to \widetilde{u}$ is locally given by a system of analytic functions:*

$$\widetilde{z}^i = X^i(z), \qquad \widetilde{\eta}^i = Y^i(z,\eta), \tag{2.2}$$

*with the regularity condition* $\det\left(\dfrac{\partial X^i}{\partial z^j}\right)\cdot\det\left(\dfrac{\partial Y^i}{\partial \eta^j}\right)\neq 0$.

At the changes of local coordinates at $u$ and $\widetilde{u}$ it is necessary to require the conditions of global existence of the transformation $\Upsilon$. If at $\widetilde{u}=(\widetilde{z}^i,\widetilde{\eta}^i)$ we consider the local changes of the coordinates $\widetilde{z}'^i=\widetilde{z}'^i(z)$; $\widetilde{\eta}'^i=\dfrac{\partial \widetilde{z}'^i}{\partial \widetilde{z}^j}\widetilde{\eta}^j$, then according to (2.2) we have

$$\widetilde{z}'^i = X'^i(z); \qquad \widetilde{\eta}'^i = Y'^i(z,\eta), \tag{2.3}$$

and the inverses $X'^i(z)=\widetilde{z}'^i(X(z))$, $\quad Y'^i(z,\eta)=\frac{\partial \widetilde{z}'^i}{\partial \widetilde{z}^j}Y^j(z,\eta)$. The global existence of $\Upsilon$ transformation implies:

$$X'^i(z'(z)) = X^i(z)\ ;\ Y'^i(z'(z),\eta'(z,\eta)) = Y^i(z,\eta), \tag{2.4}$$

with the regularity conditions of transformation.

Next, it is convenient to denote the following derivatives by $X^i_j=\dfrac{\partial X^i}{\partial z^j}$, $Y^i_j=\dfrac{\partial Y^i}{\partial \eta^j}$, their inverses by $\overset{*}{X}{}^i_j$, $\overset{*}{Y}{}^i_j$, and the conjugates by $\overline{X^i_j}$, etc.

From (2.4) we infer

**Proposition 2.2** *At the change of local maps, the following derivatives changed by the rules*

$$X'^i_k=\frac{\partial \widetilde{z}'^i}{\partial \widetilde{z}^j}X^j_k,\quad Y'^i_k=\frac{\partial \widetilde{z}'^i}{\partial \widetilde{z}^j}Y^j_k,\quad X^i_k=\frac{\partial z^i}{\partial z'^j}X'^j_k,\quad Y^i_k=\frac{\partial z^i}{\partial z'^j}Y'^j_k.$$

**Definition 2.2** *A $d-$gauge complex tensor* of $\begin{pmatrix} p & \overline{r} \\ q & \overline{s}\end{pmatrix}$ type is a system of functions $w^{i_1\dots i_p\overline{i}_1\dots\overline{i}_r}_{j_1\dots j_q\overline{j}_1\dots\overline{j}_s}(z,\eta)$, which satisfies in addition to (1.5) the following rules of change

$$\begin{aligned}\widetilde{w}^{i_1\dots i_p\overline{i}_1\dots\overline{i}_r}_{j_1\dots j_q\overline{j}_1\dots\overline{j}_s}(\widetilde{z},\widetilde{\eta}) &= X^{i_1}_{h_1}\cdots X^{i_p}_{h_p}\cdot \overset{*}{X}{}^{k_1}_{j_1}\cdot\ldots\cdot \overset{*}{X}{}^{k_q}_{j_q}\cdot\overline{X}^{\overline{i}_1}_{\overline{h}_1}\cdot\ldots\cdot \\ &\cdot\ \overline{X}^{\overline{i}_r}_{\overline{h}_r}\cdot\overset{*}{\overline{X}}{}^{\overline{k}_1}_{\overline{j}_1}\cdot\ldots\cdot\overset{*}{\overline{X}}{}^{\overline{k}_s}_{\overline{j}_s}\cdot\cdot w^{h_1\dots h_p\overline{h}_1\dots\overline{h}_r}_{k_1\dots k_q\overline{k}_1\dots\overline{k}_s}(z,\eta).\end{aligned} \tag{2.5}$$

Denote by $J$ the natural almost tangent structure on $T'M$, which is defined by $J\left(\dfrac{\partial}{\partial z^i}\right)=\dfrac{\partial}{\partial \eta^i}$, $J\left(\frac{\partial}{\partial \eta^i}\right)=0$, and is globally defined. The kernel of $J$ is just the vertical distribution, and if $N$ is a $(nl.c.c.)$ then $J(\dfrac{\delta}{\delta z^i})=\dfrac{\partial}{\partial \eta^i}$. This means that the image of the horizontal distribution is the vertical one.

**Definition 2.3** A nonlinear complex gauge connection (shortly $(nl.c.g.c)$) is a $(nl.c.c)$ $N$ which is preserved by the tangent map of the transformation, i.e., $\Upsilon_*:T'_u(T'M)\to T'_{\widetilde{u}}(T'M)$ preserves the distributions.

If $\left\{ \frac{\delta}{\delta \widetilde{z}^i} = \frac{\partial}{\partial \widetilde{z}^i} - \widetilde{N}_i^j \frac{\partial}{\partial \eta^j}; \ \frac{\partial}{\partial \widetilde{\eta}^i} \right\}$ is the adapted basis in $T'_{\widetilde{u}}(T'M)$ then for $\Upsilon_*$ to preserve the distributions it is necessary that $J(\frac{\delta}{\delta \widetilde{z}^i}) = \frac{\partial}{\partial \widetilde{\eta}^i}$. For this reason $N$ is a $(nl.c.g.c.)$ if

$$\frac{\delta}{\delta z^i} = X_i^j \frac{\delta}{\delta \widetilde{z}^j} \quad \text{and} \quad \frac{\partial}{\partial \eta^i} = X_i^j \frac{\partial}{\partial \widetilde{\eta}^j}, \tag{2.6}$$

this means that the elements of the adapted bases are $d-$gauge complex tensors.
**Proposition 2.3.** *The coefficients $N_i^j(z,\eta)$ of one (nl.c.g.c.) satisfy in addition to the condition (1.1) the next transformation law* $\widetilde{N}_k^j X_j^k = X_k^i N_j^k - \frac{\partial Y^i}{\partial z^k}$. The proof results from the first relation (2.6)

Considering the conjugate $\widetilde{N}_{\bar{i}}^{\bar{j}}$ one of the $(nl.c.g.c.)$ and the corresponding adapted bases on $T''(T'M)$, we obtain that the extension $\Upsilon_*$ of the tangent gauge transformation to the whole complexification $T_C(T'M)$ preserves the four distributions.

# 3 Complex gauge derivatives

For a given $(nl.c.g.c.)N$, let us consider the components $D^h$, $D^v$, $D^{\widetilde{h}}$, $D^{\widetilde{v}}$ of one $N-(l.c.c.)$ $D$ on $T_C(T'M)$ with the local coefficients written in formula (1.4), and which at the change of local maps on $M$ are transformed by the rules

$$\overset{\alpha}{L}{}'^{i}_{jk} \frac{\partial z'^j}{\partial z^h} \frac{\partial z'^k}{\partial z^m} = \overset{\alpha}{L}{}^{l}_{hm} \frac{\partial z'^i}{\partial z^l} - \frac{\partial^2 z'^i}{\partial z^h \partial z^m} \qquad \overset{\alpha}{C}{}'^{i}_{jk} \frac{\partial z'^j}{\partial z^h} \frac{\partial z'^k}{\partial z^m} = \overset{\alpha}{C}{}^{l}_{hm} \frac{\partial z'^i}{\partial z^l} \ ; \ \alpha = 1,2. \tag{3.1}$$

For the conjugate indices the rules are obtained from (3.1) deriving with respect to the conjugate basis.

The covariant derivatives $D^h$, $D^v$, $D^{\widetilde{h}}$, $D^{\widetilde{v}}$ act on one complex $d-$tensor field $w$ of $\begin{pmatrix} p & \bar{r} \\ q & \bar{s} \end{pmatrix}-$type, determining the following $d-$tensor fields

$$w^{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_r}_{j_1 \ldots j_q \bar{j}_1 \ldots \bar{j}_s \,|m} \ ; \ w^{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_r}_{j_1 \ldots j_q \bar{j}_1 \ldots \bar{j}_s \,\|m}, \quad w^{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_r}_{j_1 \ldots j_q \bar{j}_1 \ldots \bar{j}_s \,|\bar{m}}, \quad w^{i_1 \ldots i_p \bar{i}_1 \ldots \bar{i}_r}_{j_1 \ldots j_q \bar{j}_1 \ldots \bar{j}_s \,\|\bar{m}}.$$

**Proposition 3.1** *The covariant derivatives $D^h$, $D^v$, $D^{\widetilde{h}}$, $D^{\widetilde{v}}$ of one complex $d-$gauge tensor field are $d-$complex gauge tensor fields if and only if the local coefficients of $D$ derivative satisfy in addition to (3.1) the following rules*

$$\overset{\alpha}{\widetilde{L}}{}^{i}_{jk} = \overset{*}{X}{}^h_j \overset{*}{X}{}^m_k X^i_l \overset{\alpha}{L}{}^l_{hm} - \overset{*}{X}{}^h_j \overset{*}{X}{}^m_k \frac{\delta X^i_m}{\delta z^h} \qquad \overset{\alpha}{\widetilde{C}}{}^{i}_{jk} = \overset{*}{X}{}^h_j \overset{*}{X}{}^m_k X^i_l \overset{\alpha}{C}{}^l_{hm}, \quad \alpha = 1,2. \tag{3.2}$$

*and analogous conditions for the conjugates indices, the $\overset{*}{X}{}^i_j$ is replaced by its conjugate $\overline{\overset{*}{X}{}^{i}_{j}}$.*

**Proof.** For instance, starting with the first relation (1.4) $D_{\delta_k} \delta_j = \overset{1}{L}{}^i_{jk} \, \delta_i$, and considering (2.6) and their conjugates it results directly the relations (3.2).

**Definition 3.1** A $N-$linear complex connection $D$ satisfying the conditions from *Prop.3.1* is named a *$N-$linear complex gauge connection* $(N-(l.c.g.c.))$.

The corresponding gauge complex derivatives will be specified by $\mathcal{D}^h$, $\mathcal{D}^v$, $\widetilde{\mathcal{D}^h}$, $\widetilde{\mathcal{D}^v}$, replacing the short, long and conjugate bar of derivation of a $d-$complex gauge tensor.

**Definition 3.2.** A $N-(l.c.g.c.), D$ is said to be *metrical* if

$$\mathcal{D}^h g_{i\bar{j}} = 0 \ ; \ \mathcal{D}^v g_{i\bar{j}} = 0, \quad \mathcal{D}^{\bar{h}} g_{i\bar{j}} = 0, \quad \mathcal{D}^{\bar{v}} g_{i\bar{j}} = 0, \tag{3.3}$$

where $g_{i\bar{j}}$ defines the Hermitian metric structure (1.6).

If we use [8] we obtain a $N-(l.c.c.)$, denoted by $\overset{c}{D}$ and called *canonical connection*, which is metrical and $M-$connection, too:

$$\begin{aligned} \overset{c}{L}{}^{i}_{jk} &= \frac{1}{2} g^{\bar{l}i} \left( \frac{\delta g_{j\bar{l}}}{\delta z^k} + \frac{\delta g_{k\bar{l}}}{\delta z^j} \right), & \overset{c}{C}{}^{i}_{jk} &= \frac{1}{2} g^{\bar{l}i} \left( \frac{\partial g_{j\bar{l}}}{\partial \eta^k} + \frac{\partial g_{k\bar{l}}}{\partial \eta^j} \right), \\ \overset{c}{L}{}^{\bar{i}}_{\bar{j}k} &= \frac{1}{2} g^{\bar{i}l} \left( \frac{\delta g_{l\bar{j}}}{\delta z^k} - \frac{\delta g_{k\bar{j}}}{\delta z^l} \right), & \overset{c}{C}{}^{\bar{i}}_{\bar{j}k} &= \frac{1}{2} g^{\bar{i}l} \left( \frac{\partial g_{l\bar{j}}}{\partial \eta^k} - \frac{\partial g_{k\bar{j}}}{\partial \eta^l} \right), \end{aligned} \tag{3.4}$$

and the conjugates.

Reiterating the calculus in Proposition 3.1 for the canonical connection $\overset{c}{D}$, using the conditions (2.6) of $(nl.c.g.c.)$, we infer

**Theorem 3.2.** *The canonical connection $\overset{c}{D}$ is a $N-(l.c.g.c.)$ with the $T^i_{jk}$ and $S^i_{jk}$ vanishing torsions.*

The connection (3.4) will be called the *Miron canonical complex connection.*

# 4 Complex Einstein-Yang-Mills equations

Let us consider $N$ a $(nl.c.g.c.)$ and $L_0 : T'M \to R$ a complex Lagrangian, i.e.,

$$g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \overline{\eta}^J} \tag{4.1}$$

is a nondegenerated $d-$tensor.

From (4.1) we remark that $g_{i\bar{j}}$ is a gauge complex tensor. The $(nl.c.g.c.), N$ can be determined from $L_0$ as is shown in [8].

In calculus $L_0$ depends on the point $u = (z, \eta)$ via the wave functions $\Phi^A$, $A = \overline{1,p}$, which are gauge fields, and on their derivatives which are supposed to be with respect to the adapted basis

$$L_0(z, \eta) = L \left( \Phi^A, \frac{\delta \Phi^A}{\delta z^i}, \frac{\delta \Phi^A}{\delta \overline{z}^i}, \frac{\partial \Phi^A}{\partial \eta^i}, \frac{\partial \Phi^A}{\partial \overline{\eta}^i} \right). \tag{4.2}$$

The Euler equation gives the extremes of action of the $L_0$ Lagrangian. But this action depends on the local maps. To remove this drawback it is useful to consider the following modified Lagrangian

$$\mathcal{L}(z, \eta) = L_0(z, \eta) \cdot |g|^2, \tag{4.3}$$

where $|g| = |\det(g_{i\bar{j}})|$.

Then the action $I = \int \mathcal{L}(z,\eta)d\omega$ does not depend on the local coordinates, and if $\Phi$ is one of the generic field $\Phi^A$, then the direct calculus gives the following

**Proposition 4.1.** *The extremum of the action, $\delta I(\Phi) = 0$, determines the Euler-Lagrange complex equation*

(4.4)
$$\frac{\partial\mathcal{L}}{\partial\Phi} - \frac{\partial}{\partial z^i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Phi}{\partial z^i}\right)}\right) - \frac{\partial}{\partial \bar{z}^i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Phi}{\partial \bar{z}^i}\right)}\right) - \frac{\partial}{\partial \eta^i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Phi}{\partial \eta^i}\right)}\right) - \frac{\partial}{\partial \bar{\eta}^i}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Phi}{\partial \bar{\eta}^i}\right)}\right) = 0.$$

The complex $(E-L)$ equation can be rewritten in an adapted basis $\{\delta_i, \partial_i, \delta_{\bar{i}}, \partial_{\bar{i}}\}$ in the equivalent form

$$|g|^2\left\{\frac{\partial L}{\partial\Phi} - \frac{\delta}{\delta z^i}\left(\frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta z^i}\right)}\right) - \frac{\delta}{\delta \bar{z}^i}\left(\frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta \bar{z}^i}\right)}\right) - \frac{\partial}{\partial \eta^i}\left(\frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial \eta^i}\right)}\right)_{|c_1} - \right.$$

$$(4.5)\qquad \left. - \frac{\partial}{\partial \bar{\eta}^i}\left(\frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial \bar{\eta}^i}\right)}\right)_{|c_2} + \frac{\partial N^i_j}{\partial\eta^i}\frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta z^j}\right)} + \frac{\partial \overline{N}^i_j}{\partial\bar{\eta}^i}\frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta \bar{z}^j}\right)}\right\} =$$

$$= \frac{\delta|g|^2}{\delta z^i}\cdot\frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta z^i}\right)} + \frac{\delta|g|^2}{\delta \bar{z}^i}\cdot\frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta \bar{z}^i}\right)} + \frac{\partial|g|^2}{\partial \eta^i}\cdot\frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial \eta^i}\right)} + \frac{\partial|g|^2}{\partial \bar{\eta}^i}\cdot\frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial \bar{\eta}^i}\right)},$$

with $c_1 = \dfrac{\delta\Phi}{\delta z^i}$; $c_2 = \dfrac{\delta\Phi}{\delta \bar{z}^i}$ taking constant values.

The formula (4.5) shows that the $(E.L)$ equation is invariant to the local change of coordinates.

Next, we shall use the notations

$$\overset{h}{\Phi}{}^i_A = \frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta z^i}\right)},\quad \overset{\bar{h}}{\Phi}{}^{\bar{i}}_A = \frac{\partial L}{\partial\left(\frac{\delta\Phi}{\delta \bar{z}^i}\right)},\quad \overset{v}{\Phi}{}^i_A = \frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial \eta^i}\right)},\quad \overset{\bar{v}}{\Phi}{}^{\bar{i}}_A = \frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial \bar{\eta}^i}\right)},$$

in order to abbreviate the written form of (4.5).

Also, for a given $N-(l.c.g.c.)D$ (for instance the (3.4) canonical connection), let us consider the $h-, v-, \bar{h}-, \bar{v}-$derivations of one vector $X = (X^i)$,

$$(4.6)\qquad \begin{aligned} X^i_{|k} &= \frac{\delta X^i}{\delta z^k} + L^i_{jk}X^j, & X^i_{\|k} &= \frac{\partial X^i}{\partial \eta^k} + C^i_{jk}X^j,\\ X^i_{|\bar{k}} &= \frac{\delta X^i}{\delta \bar{z}^k} + L^i_{j\bar{k}}X^j, & X^i_{\|\bar{k}} &= \frac{\partial X^i}{\partial \bar{\eta}^k} + C^i_{j\bar{k}}X^j, \end{aligned}$$

and their conjugates. So, the formula (4.5) is written in the equivalent form

$$(4.7)\qquad \frac{\partial L}{\partial\Phi^A} - \overset{h}{\Phi}{}^i_{A|i} - \overset{\bar{h}}{\Phi}{}^{\bar{i}}_{A|\bar{i}} - \overset{v}{\Phi}{}^i_{A\|i} - \overset{\bar{v}}{\Phi}{}^{\bar{i}}_{A\|\bar{i}} = E_A,$$

where

$$E_A = \frac{1}{|g|^2}\left\{\frac{\delta |g|^2}{\delta z^i}\overset{h}{\Phi^i_A} + \frac{\delta |g|^2}{\delta \overline{z}^i}\overset{\overline{h}}{\Phi^{\overline{i}}_A} + \frac{\partial |g|^2}{\partial \eta^i}\overset{v}{\Phi^i_A} + \frac{\partial |g|^2}{\partial \overline{\eta}^i}\overset{\overline{v}}{\Phi^{\overline{i}}_A}\right\} - \tag{4.8}$$
$$- \left(L^i_{jk} + \frac{\partial N^i_j}{\partial \eta^i}\right)\overset{h}{\Phi^i_A} - \left(L^{\overline{i}}_{\overline{ji}} + \frac{\partial \overline{N}^{\overline{i}}_{\overline{j}}}{\partial \overline{\eta}^{\overline{i}}}\right)\overset{\overline{h}}{\Phi^{\overline{i}}_A} - C^i_{ji}\overset{v}{\Phi^i_A} - C^{\overline{i}}_{\overline{ji}}\overset{\overline{v}}{\Phi^{\overline{i}}_A}$$

are gauge scalars if $D$ is a $N-(l.c.g.c.)$.

Let us note that the invariance of the $(E-L)$ equation is assured if the Lagrangian $L$ is gauge invariant.

We shall analyze the gauge invariance of Lagrangian $L$ in the particular case of gauge infinitesimal transformations.

Let us consider $G$ a group of transformations, $\dim G = m$, that acts on $T'M$,

$$\widetilde{u} = \Upsilon(u, a), \qquad u = \Upsilon(u, 0), \tag{4.9}$$

with $u = (z^i, \eta^i)$ and $a = (a^1, a^2, \dots a^m) \in G$.

At an infinitesimal holomorphic of the group $G$, we have

$$\widetilde{z}^i = z^i + \overset{h}{\xi^i_\lambda}\,\varepsilon^\lambda, \quad \widetilde{\eta}^i = \eta^i + \overset{v}{\xi^i_\lambda}\,\varepsilon^\lambda. \tag{4.10}$$

Accordingly, the gauge fields $\Phi^A(u)$ will be transformed by the rules

$$\widetilde{\Phi}^A = \Phi^A + (X_\lambda \Phi^A)\varepsilon^\lambda, \tag{4.11}$$

where $X_\lambda = \overset{h}{\xi^i_\lambda}\,\frac{\partial}{\partial z^i} + \overset{v}{\xi^i_\lambda}\,\frac{\partial}{\partial \eta^i}$ are the generators of the group.

As in classical theory, let us consider a $p-$dimensional complex representation of the generators $X_\lambda \to [X_\lambda]^A_B$. Then the infinitesimal transformation of the gauge fields becomes

$$\widetilde{\Phi}^A = \Phi^A + ([X_\lambda]^A_B \Phi^B)\varepsilon^\lambda. \tag{4.12}$$

The gauge infinitesimal invariance condition of the Lagrangian $L$ is that of vanishing its variation,

$$\frac{\partial L}{\partial \Phi^A}\delta\Phi^A + \overset{h}{\Phi^i_A}\,\delta\left(\frac{\delta \Phi^A}{\delta z^i}\right) + \overset{\overline{h}}{\Phi^{\overline{i}}_A}\,\delta\left(\frac{\delta \Phi^A}{\delta \overline{z}^i}\right) + \overset{v}{\Phi^i_A}\,\delta\left(\frac{\partial \Phi^A}{\partial \eta^i}\right) + \overset{\overline{v}}{\Phi^{\overline{i}}_A}\,\delta\left(\frac{\partial \Phi^A}{\partial \overline{\eta}^i}\right) = 0.$$

We propose now to discuss only the global invariance, meaning that $\varepsilon^\lambda$ are constants, and hence the results

$$\delta\left(\frac{\delta \Phi^A}{\delta z^i}\right) = \frac{\delta}{\delta z^i}(\delta\Phi^A) = \varepsilon^\lambda [X_\lambda]^A_B \frac{\delta \Phi^B}{\delta z^i}, \quad \delta\left(\frac{\partial \Phi^A}{\partial \eta^i}\right) = \varepsilon^\lambda [X_\lambda]^A_B \frac{\partial \Phi^B}{\partial \eta^i},$$
$$\delta\left(\frac{\delta \Phi^A}{\delta \overline{z}^i}\right) = \frac{\delta}{\delta \overline{z}^i}(\delta\Phi^A) = \varepsilon^\lambda [X_\lambda]^A_B \frac{\delta \Phi^B}{\delta \overline{z}^i}, \quad \delta\left(\frac{\partial \Phi^A}{\partial \overline{\eta}^i}\right) = \varepsilon^\lambda [X_\lambda]^A_B \frac{\partial \Phi^B}{\partial \overline{\eta}^i}.$$

Replacing these variations in the invariance condition of the Lagrangian, it results

$$\left\{\frac{\partial L}{\partial \Phi^A}\Phi^B + \Phi^i_A\frac{\delta \Phi^B}{\delta z^i} + \Phi^{\bar{i}}_A\frac{\delta \Phi^B}{\delta \overline{z}^i} + \Phi^i_A\frac{\partial \Phi^B}{\partial \eta^i} + \Phi^{\bar{i}}_A\frac{\partial \Phi^B}{\partial \overline{\eta}^i}\right\}[X_\lambda]^A_B = 0. \tag{4.13}$$

Now, combining the formulae (4.7) and (4.13) we obtain the global invariance law of complex Lagrangian $L$ at the gauge infinitesimal transformation (4.11)

$$\begin{aligned}&\Big\{E_A\Phi^B + \overset{h}{\Phi}{}^i_{A|i}\,\Phi^B + \overset{\overline{h}}{\Phi}{}^{\bar{i}}_{A|\bar{i}}\,\Phi^B + \overset{v}{\Phi}{}^i_{A\|i}\,\Phi^B + \overset{\overline{v}}{\Phi}{}^{\bar{i}}_{A\|\bar{i}}\,\Phi^B + \\ &+ \overset{h}{\Phi}{}^i_A\,\frac{\delta \Phi^A}{\delta z^i} + \overset{\overline{h}}{\Phi}{}^{\bar{i}}_A\,\frac{\delta \Phi^A}{\delta \overline{z}^i} + \overset{v}{\Phi}{}^i_A\,\frac{\partial \Phi^A}{\partial \eta^i} + \overset{\overline{v}}{\Phi}{}^{\bar{i}}_A\,\frac{\partial \Phi^A}{\partial \overline{\eta}^i}\Big\}[X_\lambda]^A_B = 0.\end{aligned} \tag{4.14}$$

Using the complex currents

$$\begin{aligned}&\overset{h}{J}{}^i_A = -\overset{h}{\Phi}{}^i_A\,[X_\lambda]^A_B\Phi^B, \quad \overset{v}{J}{}^i_A = -\overset{v}{\Phi}{}^i_A\,[X_\lambda]^A_B\Phi^B, \\ &\overset{\overline{h}}{J}{}^{\bar{i}}_A = -\overset{\overline{h}}{\Phi}{}^{\bar{i}}_A\,[X_\lambda]^A_B\Phi^B, \quad \overset{\overline{v}}{J}{}^{\bar{i}}_A = -\overset{\overline{v}}{\Phi}{}^{\bar{i}}_A\,[X_\lambda]^A_B\Phi^B,\end{aligned} \tag{4.15}$$

the global conservative law (4.13) is written in the form

$$\overset{h}{J}{}^i_{A|i} + \overset{\overline{h}}{J}{}^{\bar{i}}_{A|\bar{i}} + \overset{v}{J}{}^i_{A\|i} + \overset{\overline{v}}{J}{}^{\bar{i}}_{A\|\bar{i}} = E_A[X_\lambda]^A_B\Phi^B. \tag{4.16}$$

# 5 Applications

Let us consider a metric Hermitian $d-$tensor $\gamma_{i\bar{j}}(z)$ on $M$, and $\Phi : T'M \to \mathbf{C}$ one scalar field ($A = 1$). For a nonlinear complex connection $N^i_j(z,\eta)$ we can consider the one determined by the Christoffel symbols $\Gamma^k_{ij}$ of the metric $\gamma_{i\bar{j}}$, i.e., $\overset{c}{N}{}^i_j = \Gamma^i_{jk}\eta^k$ (see [8]).

In the adapted basis $\{\delta_i, \partial_i, \delta_{\bar{i}}, \partial_{\bar{i}}\}$ we shall consider a generalization of the classical exact symmetry Lagrangian ([4]) for the complex gauge field $\Phi : T'M \to \mathbf{C}$,

$$L = \gamma^{\bar{i}j}(z)\frac{\delta\overline{\Phi}}{\delta\overline{z}^i}\frac{\delta\Phi}{\delta z^j} - m^2\overline{\Phi}\cdot\Phi - \frac{1}{4}f\cdot(\overline{\Phi}\cdot\Phi)^2, \tag{5.1}$$

where $m^2 > 0$ is the mass and $f > 0$ is the coupling constant.

The Lagrangian $L$ is invariant to the change of local maps on $T'M$ and is also gauge invariant with respect to the phase transformation of $U_1$ group,

$$\Phi \to \widetilde{\Phi}(z,\eta) = e^{-ig\varepsilon}\Phi(z,\eta) \;\; ; \;\; \overline{\Phi} \to \widetilde{\overline{\Phi}}(z,\eta) = e^{ig\varepsilon}\Phi(z,\eta), \tag{5.2}$$

determined by the infinitesimal variations

$$\delta z^i = -ig\varepsilon, \quad \delta\eta^i = ig\varepsilon. \tag{5.3}$$

The $U_1$ group has one parameter $\varepsilon^1 = \varepsilon$; the variations of gauge field are $\delta\Phi = -ig\varepsilon\Phi$, and the generator of the group is $X = -\,ig\left(\dfrac{\partial}{\partial z^i} - \dfrac{\partial}{\partial \eta^i}\right)$.

In classical theories, in order to study the gauge invariance, there are considered two gauge fields: $\Phi^1 = \Phi$ and $\Phi^2 = \overline{\Phi}$. This doubles the dimension of the representation, in comparison to the present approach, where we consider only one generator.

The energy of the system, $E = \gamma^{\bar{i}j}(z)\frac{\delta\overline{\Phi}}{\delta\overline{z}^i}\frac{\delta\Phi}{\delta z^j} + m^2\overline{\Phi}\cdot\Phi + \frac{1}{4}f\cdot(\overline{\Phi}\cdot\Phi)^2$ is minimal if $\frac{\partial E}{\partial\Phi} = \frac{\partial E}{\partial\overline{\Phi}} = 0$; it results that $\Phi = \overline{\Phi} = 0$, and hence the vacuum state is nondegenerate, preserving the exact symmetry.

Another generalization of classical case is that of the spontaneously broken symmetry, where it is considered the complex Lagrangian

$$L_1 = \gamma^{\bar{i}j}(z)\frac{\delta\overline{\Phi}}{\delta\overline{z}^i}\frac{\delta\Phi}{\delta z^j} + m^2\overline{\Phi}\cdot\Phi - \frac{1}{4}f\cdot(\overline{\Phi}\cdot\Phi)^2, \tag{5.4}$$

which is also gauge invariant at (5.2) infinitesimal transformation.

The energy of the system, $E_1 = \gamma^{\bar{i}j}(z)\frac{\delta\overline{\Phi}}{\delta\overline{z}^i}\frac{\delta\Phi}{\delta z^j} - m^2\overline{\Phi}\cdot\Phi + \frac{1}{4}f\cdot(\overline{\Phi}\cdot\Phi)^2$ is minimal if $|\Phi| = \sqrt{2}\frac{m}{\sqrt{f}}$ and hence, the vacuum states are degenerated, i.e., we have in the holomorphic bundle a spontaneously broken symmetry.

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