

# Some Structural Considerations on the Theory of Gravitational Field

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## Abstract

When the gravitational field is regarded as the time-sequence of space-slices (i.e., the evolution space), the gravitational field itself is treated by means of the differential geometry of total space of the vector bundle whose base manifold is the one-dimensional time-axis and fibre at each time is the three-dimensional space. From this vector bundle-like standpoint, new field equations and new conservation laws are proposed.

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**Key words:** space-time, bundle-like structures, field equations, conservation laws

## 1 Introduction

In this paper, the gravitational field is regarded as the ensemble of space-time events and then, as the time-sequence of space-slices. That is to say, the four-dimensional gravitational field itself is decomposed into the one-dimensional time-part and the three-dimensional space-part and the three-dimensional space evolves along the time-axis. This idea arises from the concept of evolution space in the theory of dynamical systems (cf. [1]). Therefore, from the vector bundle-like standpoint, the gravitational field can be adapted to the total space of the vector bundle whose base manifold is the one-dimensional time-axis ( $x^0 = t$ :time) and the fibre at each time is the three-dimensional space spanned by points  $\{x^i\}$  ( $i = 1, 2, 3$ ).

Therefore, the gravitational field can be treated by means of the differential geometry of total space of the vector bundle [2], [3]. From this standpoint, the metrical and connection structures will be introduced and then, new field equations and new conservation laws will be proposed in the following.

## 2 On the vector bundle-like structures - I

Now, in the total space mentioned above, the so-called adapted frame is set as follows:

$$(2.1) \quad \begin{aligned} dX^A &\equiv (dx^0, \delta x^i = dx^i + N_0^i dx^0), \\ \frac{\partial}{\partial X^A} &\equiv \left( \frac{\delta}{\delta x^0} = \frac{\partial}{\partial x^0} - N_0^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i} \right), \end{aligned}$$

where  $X^A = (x^0, x^i)$  ( $A = (0, i)$ ;  $i = 1, 2, 3$ ) and  $x^0 = t$  (time). The quantity  $N_0^i$  denotes the nonlinear connection playing physically the role of velocity.

On the basis of (2.1), the connection structures is introduced by

$$(2.2) \quad \begin{aligned} \nabla \frac{\partial}{\partial X^C} \frac{\partial}{\partial X^B} &= \Gamma_{BC}^A \frac{\partial}{\partial X^A}; \\ \Gamma_{BC}^A &\equiv (L_{00}^0, L_{j0}^i, C_{0k}^0, C_{jk}^i), \end{aligned}$$

where

$$\nabla \frac{\delta}{\delta x^0} \frac{\delta}{\delta x^0} = L_{00}^0 \frac{\delta}{\delta x^0}, \quad \nabla \frac{\delta}{\delta x^0} \frac{\partial}{\partial x^j} = C_{jk}^i \frac{\partial}{\partial x^i}, \text{ etc.}$$

Namely, the following four kinds of covariant derivatives can be defined, for an arbitrary vector  $V^A = (V^0, V^i)$

$$(2.3) \quad \left\{ \begin{array}{l} V_{|0}^0 = \frac{\delta V^0}{\delta x^0} + L_{00}^0 V^0, \\ V^0|_k = \frac{\partial V^0}{\partial x^k} + C_{0k}^0 V^0, \\ V_{|0}^i = \frac{\delta V^i}{\delta x^0} + L_{j0}^i V^j, \\ V^i|_k = \frac{\partial V^i}{\partial x^k} + C_{jk}^i V^j. \end{array} \right.$$

On the other hand, the metrical structure is introduced by

$$(2.4) \quad G \equiv G_{AB} dX^A dX^B = g_{00} dx^0 \otimes dx^0 + g_{ij} \delta x^i \otimes \delta x^j,$$

where  $g_{00}$  and  $g_{ij}$  are metric tensors depending on  $(x^0, x^i)$ . The connection (2.2) can be made metrical by imposing the metrical conditions such as  $g_{00|0} = 0$ ,  $g_{00|k} = 0$ ,  $g_{ij|0} = 0$  and  $g_{ij|k} = 0$ . In the metrical case, the canonical connection coefficients can be determined as follows [2], [3]

$$(2.5) \quad \left\{ \begin{array}{l} L_{00}^0 = \frac{1}{2} g^{00} \frac{\delta g_{00}}{\delta x^0}, \\ L_{j0}^i = \frac{1}{2} g^{il} \frac{\delta g_{lj}}{\delta x^0}, \\ C_{0k}^0 = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^k}, \\ C_{jk}^i = \frac{1}{2} g^{il} \left( \frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right), \end{array} \right.$$

where  $g^{00}$  and  $g^{il}$  are the inverse of  $g_{00}$  and  $g_{li}$  respectively and the torsion tensor  $S_{jk}^i (= C_{jk}^i - C_{kj}^i)$  is assumed to vanish (see (3.5)).  $C_{jk}^i$  is nothing but the three-dimensional Christoffel symbol formed with  $g_{ij}$ . And also, in (2.5), if  $g_{00} = \text{constant}$ , then  $L_{00}^0 = 0$  and  $C_{0k}^0 = 0$ .

### 3 On the vector bundle-like structures - II

Now, the torsion and curvature tensors are defined by [2], [3], respectively,

$$(3.1) \quad T_{BC}^A = \Gamma_{BC}^A - \Gamma_{CB}^A + \omega_{BC}^A,$$

$$(3.2) \quad R_{BCD}^A = \frac{\partial}{\partial X^D} \Gamma_{BC}^A - \frac{\partial}{\partial X^C} \Gamma_{BD}^A + \Gamma_{BC}^E \Gamma_{ED}^A - \Gamma_{BD}^E \Gamma_{EC}^A + \Gamma_{DE}^A \omega_{CD}^E,$$

where  $\omega_{BC}^A$  means the non-holonomic object defined by

$$(3.3) \quad \left[ \frac{\partial}{\partial X^B}, \frac{\partial}{\partial X^C} \right] = \omega_{BC}^A \frac{\partial}{\partial X^A}.$$

In our case, only one component  $\omega_{0j}^i = -\omega_{j0}^i = \frac{\partial N_0^i}{\partial x^j}$  appears (because  $\omega_{00}^i = R_{00}^i = 0$ , see below).

Five components of the torsion tensor

$$T_{BC}^A \equiv (T_{00}^0 = 0, R_{00}^i = 0, C_{0i}^0, P_{0j}^i, S_{jk}^i)$$

and six components of the curvature tensor

$$\mathcal{R}_{BCD}^A \equiv (R_{000}^0 = 0, R_{j00}^i = 0, P_{k0j}^i, P_{00i}^0, S_{0ij}^0, S_{jkl}^i)$$

appear explicitly in the following Ricci-identities:

$$(3.4) \quad \left\{ \begin{array}{l} V^0|_{0|0} - V^0|_{0|0} = R_{000}^0 V^0 - T_{00}^0 V^0|_0 - R_{00}^i V^0|_i (= 0), \\ V^0|_0|_i - V^0|_{i|0} = P_{00i}^0 V^0 - C_{0i}^0 V^0|_0 - P_{0i}^j V^0|_j, \\ V^0|_i|_j - V^0|_j|_i = S_{0ij}^0 V^0 - S_{ij}^k V^0|_k, \\ V^i|_{0|0} - V^i|_{0|0} = R_{j00}^i V^j - T_{00}^0 V^i|_0 - R_{00}^j V^i|_j (= 0), \\ V^i|_0|_j - V^i|_{j|0} = P_{k0j}^i V^k - C_{0j}^0 V^i|_0 - P_{0j}^k V^i|_k, \\ V^i|_j|_k - V^i|_k|_j = S_{jkl}^i V^l - S_{jk}^l V^i|_l. \end{array} \right.$$

The concrete definitions of those components are given as follows

$$(3.5) \quad \left\{ \begin{array}{l} T_{00}^0 = L_{00}^0 - L_{00}^0 = 0, \\ R_{00}^i = \frac{\delta N_0^i}{\delta x^0} - \frac{\delta N_0^i}{\delta x^0} = 0, \\ P_{0j}^i = \frac{\partial N_0^i}{\partial x^j} - L_{j0}^i, \\ S_{jk}^i = C_{jk}^i - C_{kj}^i. \end{array} \right.$$

$$(3.6) \quad \left\{ \begin{array}{l} R_{000}^0 = \frac{\delta L_{00}^0}{\delta x^0} - \frac{\delta L_{00}^0}{\delta x^0} + L_{00}^0 L_{00}^0 - L_{00}^0 L_{00}^0 + C_{0i}^0 R_{00}^i = 0, \\ R_{j00}^i = \frac{\delta L_{j0}^i}{\delta x^0} - \frac{\delta L_{j0}^i}{\delta x^0} + L_{j0}^i L_{i0}^i - L_{j0}^i L_{i0}^i + C_{jl}^i R_{00}^l = 0, \\ P_{j0k}^i = \frac{\partial L_{j0}^i}{\partial x^k} - C_{jk|0}^i + C_{jl}^i P_{0k}^l, \\ P_{00k}^0 = \frac{\partial L_{00}^0}{\partial x^k} - C_{0k|0}^0 + C_{0l}^0 P_{0k}^l, \\ S_{0kh}^0 = \frac{\partial C_{0k}^0}{\partial x^h} - \frac{\partial C_{0h}^0}{\partial x^k} + C_{0k}^0 C_{0h}^0 - C_{0h}^0 C_{0k}^0, \\ S_{jkh}^i = \frac{\partial C_{jk}^i}{\partial x^h} - \frac{\partial C_{jh}^i}{\partial x^k} + C_{jk}^l C_{lh}^i - C_{jh}^l C_{lk}^i. \end{array} \right.$$

In the canonical case of (2.5), the tensor  $S_{jkh}^i$  is just the three-dimensional Riemann-Christoffel curvature tensor and also, if  $g_{00} = \text{constant}$ , then  $P_{00k}^0 = 0$  and  $S_{0kh}^0 = 0$  in (3.6).

From  $\mathcal{R}_{BCD}^A$ , the Ricci-tensor is given by  
(3.7)

$$\mathcal{R}_{BC} \equiv \mathcal{R}_{BCA}^A \equiv (R_{00} \equiv R_{000}^0 = 0, \overset{1}{P}_{j0} \equiv P_{j0i}^i, -\overset{2}{P}_{0k} \equiv -P_{0k0}^0 = P_{00k}^0, S_{ij} \equiv S_{ijk}^k).$$

Namely, three non-vanishing components appear. And the total scalar is given by

$$(3.8) \quad \mathcal{R} \equiv \mathcal{R}_{AB} G^{AB} = R_{00} g^{00} + S_{ij} g^{ij} = S_{ij} g^{ij} \equiv S.$$

(3.7) and (3.8) will be used in the next Section.

## 4 On the field equations and the conservation laws

As the field equation for the total space (i.e., the gravitational field), we shall put it in the form [2], [3], with use of the Ricci-tensor  $\mathcal{R}_{AB}$  (3.7) and the total scalar  $\mathcal{R}$  (3.8),

$$(4.1) \quad \mathcal{R}_{AB} - \frac{1}{2} \mathcal{R} G_{AB} = \tau_{AB},$$

where  $\tau_{AB}$  represents the energy-momentum tensor with four components  $\tau_{AB} \equiv (\tau_{00}, \tau_{j0}, \tau_{0i}, \tau_{ij})$ . Then, by use of the components of (3.7) and (3.8), we can obtain the following four kinds of field equations

$$(4.2) \quad \left\{ \begin{array}{l} -\frac{1}{2} S g_{00} = \tau_{00}, \\ \overset{1}{P}_{j0} = \tau_{j0}, \\ \overset{2}{P}_{0i} = -\tau_{0i}, \\ S_{ij} - \frac{1}{2} S g_{ij} = \tau_{ij}. \end{array} \right.$$

These are new equations, different from those obtained in [2], [3]. In the case of the canonical connection (2.5), the last equation is just the (three-dimensional) Einstein's field equation.

As to the conservative law, we can formulate it in the form [2], [3] (i.e., the divergence-zero of (4.1))

$$(4.3) \quad \frac{\nabla_\partial}{\partial X^A} \left( \mathcal{R}_B^A - \frac{1}{2} \mathcal{R} \delta_B^A \right) = 0,$$

where

$$(4.4) \quad \begin{aligned} \mathcal{R}_B^A &\equiv \mathcal{R}_{BC} G^{CA} \equiv (R_0^0 \equiv R_{00} g^{00} = 0, P_0^i \equiv P_{j0}^1 g^{ji}, \\ &\quad - P_j^0 \equiv - P_{0j}^2 g^{00}, S_j^i \equiv S_{jl} g^{li}). \end{aligned}$$

Therefore, we can obtain the following two kinds of conservation laws

$$(4.5) \quad \begin{cases} P_0^i |_i - \frac{1}{2} S_{|0} = 0, \\ (S_j^i - \frac{1}{2} S \delta_j^i) |_i - P_j^0 |_0 = 0. \end{cases}$$

These are new conservation laws, different from those obtained in [2], [3]. In some special cases where  $P_j^0 = 0$  or  $P_j^0 |_0 = 0$ , we can obtain the pure conservation law

$$\left( S_j^i - \frac{1}{2} S \delta_j^i \right) |_i = 0,$$

which is the same as the Einstein's one. In those special cases, the conditions such as

$$(g_{00} = g_{00}(x^0) \& g_{ij} = g_{ij}(x^i)) \text{ or } (g_{00} = g_{00}(x^i) \& g_{ij} = g_{ij}(x^i)),$$

etc. must be taken into account.

## 5 Conclusion

Thus, we can treat the gravitational field by means of the differential geometry of total space of the vector bundle whose base manifold is the time-axis and fibre at each time is the space-slice. And we can propose new field equations (4.2) and new conservation laws (4.5). (Some physical aspects of these subjects are referred to [4]).

## References

- [1] J.M. Souriau, *Structure de systemes dynamiques*, Dunod, Paris, 1970.
- [2] R.Miron and M.Anastasiei, *The geometry of Lagrange spaces: Theory and Applications*, Kluwer Academic Publishers, Dordrecht, 1994.

- [3] R.Miron and M.Anastasiei, *Vector bundles and Lagrange spaces with applications to relativity*, Geometry Balkan Press, Bucharest, 1997.
- [4] S.Ikeda, *Some physical aspects underlying the Lagrangian theory of relativity*, Chap. 12 of [3], 183-198.

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