Gauge complex field theory

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Abstract

On the total space of a $G-$complex vector bundle we study the gauge transformations. A gauge complex nonlinear connection determined by a gauge invariant Lagrangian plays a special role in our approach. The associated Chern-Lagrange and complex Bott connections are gauge invariant too.

The complex field equations are studied with respect to a gauge invariant vertical connection, particularly with respect to the complex Bott connection.

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1 Gauge transformations on a G-complex vector bundle

From a geometrical point of view, gauge theory is the study of principal bundles, their connection space, and the curvature of these connections.

A principal $G-$bundle P over a manifold M is a manifold with a smooth $G-$action and its orbit space is $P/G = M$. In our investigation, we will work with an associated vector bundle of the principal bundle (P, M, G).

Let M be a complex manifold, $(z^k)_{k=\overline{1,n}}$ complex coordinates in a local chart $(U_\alpha, \varphi_\alpha)$, $\pi : E \to M$ a complex vector bundle of $\mathbb{C}^m$ fibre, $\eta = \eta^a s_a$, local section on E, $a = \overline{1,m}$. Consider G a closed $m-$dimensional Lie group of complex matrices, whose elements are holomorphic functions over M.

Definition 1.1. *A structure of $G-$complex vector bundle of E is a fibration with transition functions taking values in G.*

This means that if $z'^i = z'^i(z)$ is a local change of charts on M, then the section η changes by the rule:

$$z'^i = z'^i(z) \ \ ; \ \ \eta'^a = M^a_b(z)\eta^b \tag{1.1}$$

where $M^a_b(z) \in G$ and $\partial M^a_b(z)/\partial \bar{z}^k = 0$ for any $a, b = \overline{1,m}$ and $k = \overline{1,n}$.

E has a natural structure of $(n+m)-$complex manifold, a point of E is designed by $u = (z^k, \eta^a)$.

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The geometry of E manifold (the total space), endowed with a Hermitian metric $g_{a\bar{b}} = \partial^2 L/\partial\eta^a\partial\bar{\eta}^b$ derived from a homogeneous Lagrangian $L : E \to \mathbb{R}^+$, was intensively studied by T. Aikou ([1, 2, 3, 13]). Particularly, if $E = T'M$, the holomorphic tangent bundle of M, then is obtaine a structure of $GL(n,\mathbb{C})-$complex vector bundle. Let us consider $VE = \ker \pi^* \subset T'E$ the vertical bundle; a local base for its sections is $\{\dot{\partial}_a := \frac{\partial}{\partial\eta^a}\}_{a=\overline{1,m}}$. The vertical distribution V_uE is isomorphic with the sections module of E in u.

A supplementary subbundle in $T'E$ of VE, i.e. $T'E = VE \oplus HE$, is called a *complex nonlinear connection*, in brief (*c.n.c.*). A local base for the horizontal distribution H_uE, called *adapted* for the (*c.n.c.*), is $\{\delta_k := \frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^a \frac{\partial}{\partial\eta^a}\}_{k=\overline{1,n}}$, where $N_k^a(z,\eta)$ are the coefficients of the (*c.n.c.*). Locally $\{\delta_k\}$ defines an isomorphism of $\pi^*(T'M)$ with HE if and only if they are changed under de rules

$$\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'^j \;\; ; \;\; \dot{\partial}_b = M_b^a \dot{\partial}_a \tag{1.2}$$

and consequently we have for its coefficients (see (7.1.9) in [13]):

$$\frac{\partial z'^k}{\partial z^j} N_k'^a = M_b^a N_j^b - \frac{\partial M_b^a}{\partial z^j}\eta^k. \tag{1.3}$$

The existence of a (*c.n.c.*) is an important ingredient in the "linearizing" of this geometry. The adapted basis, denoted $\{\delta_{\bar{k}} := \frac{\delta}{\delta\bar{z}^k}\}$ and $\{\dot{\partial}_{\bar{a}} := \frac{\partial}{\partial\bar{\eta}^a}\}$, for $\overline{HE}$ and $\overline{VE}$ distributions are obtained respectively by conjugation everywhere. The conjugate coefficients of the (*c.n.c.*) are denoted by $N_{\bar{k}}^{\bar{a}} = \overline{N_k^a}$.

Definition 1.2. *A gauge complex transformation on $G-$complex vector bundle E, is a pair $\Upsilon = (F_0, F_1)$, where locally $F_1 : E \to E$ is an F_0-holomorphic isomorphism which satisfies*

$$\pi_* \circ F_1 = F_0 \circ \pi_*. \tag{1.4}$$

The local character concerns the open sets of local charts. This notion generalizes that considered in [12] for the holomorphic bundle $T'M$. When Υ is globally defined, the complex structure of E is preserved by Υ and

Proposition 1.1. *A gauge complex transformation $\Upsilon : u \to \widetilde{u}$ is locally given by a system of analytic functions :*

$$\widetilde{z}^i = X^i(z)\,; \qquad \widetilde{\eta}^i = Y^i(z,\eta) \tag{1.5}$$

with the regularity condition: $\det\left(\frac{\partial X^i}{\partial z^j}\right)\cdot\det\left(\frac{\partial Y^i}{\partial\eta^j}\right) \neq 0$

We denote by $X_j^i := \frac{\partial X^i}{\partial z^j}$ and $Y_b^a := \frac{\partial Y^a}{\partial\eta^b}$ and by $X_{\bar{j}}^{\bar{i}}$, $Y_{\bar{b}}^{\bar{a}}$ their conjugates.

Obviously, from the holomorphy requirements we have $X_{\bar{j}}^i = \frac{\partial X^i}{\partial\bar{z}^j} = 0$ and $Y_{\bar{j}}^a = \frac{\partial Y^a}{\partial\bar{z}^j} = 0, Y_{\bar{b}}^a = \frac{\partial Y^a}{\partial\bar{\eta}^b} = 0.$

Some ideas from [12] can be easily generalized here. For instance a $d-$complex gauge tensor is a set of functions on E, $w_{j_1\ldots\bar{j}_1\ldots b_1\ldots\bar{b}_1\ldots}^{i_1\ldots\bar{i}_1\ldots a_1\ldots\bar{a}_1\ldots}(z,\eta)$ which transform under the (1.4) changes with the matrices $X^i_k, X^{\bar{i}}_{\bar{k}}, Y^a_c, Y^{\bar{a}}_{\bar{c}}$ for the upper indices and with their inverses $\overset{*}{X}{}^i_k, \overset{*}{X}{}^{\bar{i}}_{\bar{k}}, \overset{*}{Y}{}^a_c, \overset{*}{Y}{}^{\bar{a}}_{\bar{c}}$ for the lower indices. In addition we require for these functions to be $d-$tensors (see [13]).

A (*c.n.c.*) is said to be gauge, (*g.c.n.c*), if it transforms the adapted frames into $d-$complex gauge fields, i.e. in addition to (1.2) we have

$$\delta_j = X^i_j \delta_{\tilde{i}} \;\; ; \;\; \dot{\partial}_b = Y^a_b \dot{\partial}_{\tilde{b}} \; , \tag{1.6}$$

where $\delta_{\tilde{i}} = \frac{\delta}{\delta \tilde{z}^k}$ and $\dot{\partial}_{\tilde{b}} = \frac{\partial}{\partial \tilde{\eta}^b}$.

Indeed, this implies that in addition to (1.3) we have

$$X^k_j \tilde{N}^a_k = Y^a_b N^b_j - \frac{\partial Y^a}{\partial z^j}. \tag{1.7}$$

Proposition 1.2. *Let $L_0(z,\eta)$ be a gauge invariant Lagrangian on E, i.e. $L_0(z,\eta) = L_0(\tilde{z},\tilde{\eta})$, and $g_{a\bar{b}} = \partial^2 L_0 / \partial\eta^a \partial\bar{\eta}^b$ its metric tensor. Then*

$$N^a_k = g^{\bar{b}a} \frac{\partial^2 L_0}{\partial z^k \partial \bar{\eta}^k} \tag{1.8}$$

is a (g.c.n.c.).

Proof. From holomorphy conditions $\partial M^a_b / \partial \bar{z}^k = 0$, $X^i_{\bar{k}} = 0$, $Y^a_{\bar{k}} = 0$, $Y^a_{\bar{b}} = 0$ and taking into account (1.2), (1.6), and the gauge invariance of the Lagrangian, it results that $g_{a\bar{b}}$ is a $d-$complex gauge tensor, which means $g_{a\bar{b}} = M^c_a \overset{*}{M}{}^{\bar{d}}_{\bar{b}} \, g'_{c\bar{d}}$ and $g_{a\bar{b}} = Y^c_a \overset{*}{Y}{}^{\bar{d}}_{\bar{b}} \, \tilde{g}_{c\bar{d}}$. Now we can easily check the (1.3) and (1.7) rules for the gauge changes of N^a_k. Therefore (1.8) defines the coefficients of a (*g.c.n.c.*). ■

Other fundamental notion in our study of the geometry of E is that of $d-$complex linear connection. We proved in [13] that generally this notion has too much connection coefficients, except the particular case $E = T'M$ when we can discuss on $N - (c.l.c.)$.

In [2, 3, 13] it is proved that the calculus is simplified enough if we consider a Hermitian connection in the vertical bundle VE. According to (7.2.4) from [13], the connection form of a vertical connection is written in adapted frames as follows:

$$\omega^a_b = L^a_{bk} dz^k + L^a_{b\bar{k}} d\bar{z}^k + C^a_{bc} \delta\eta^c + C^a_{b\bar{c}} \delta\bar{\eta}^c \; , \tag{1.9}$$

where

$$\begin{aligned} L^a_{bk} &= g^{\bar{d}a} \delta_k g_{b\bar{d}} \;\; ; \;\; L^a_{b\bar{k}} = g^{\bar{d}a} \delta_{\bar{k}} g_{b\bar{d}} \\ C^a_{bc} &= g^{\bar{d}a} \dot{\partial}_c g_{b\bar{d}} \;\; ; \;\; C^a_{b\bar{c}} = g^{\bar{d}a} \dot{\partial}_{\bar{c}} g_{b\bar{d}}. \end{aligned}$$

Such a complex vertical connection still behaves some difficulties in calculus. A simplification presents the *Chern-Lagrange complex connection*, which can be introduced by the same technique as we proceed on $T'M$ bundle (Corollary 5.1.1, [13]):

$$\overset{CL}{L}{}^a_{bk} = g^{\bar{d}a} \delta_k g_{b\bar{d}} \;\; ; \;\; \overset{CL}{L}{}^a_{b\bar{k}} = 0 \; ; \; \overset{CL}{C}{}^a_{bc} = g^{\bar{d}a} \dot{\partial}_c g_{b\bar{d}} \;\; ; \;\; \overset{CL}{C}{}^a_{b\bar{c}} = 0 \tag{1.10}$$

Certainly, it is of $(1,0)-$type and we easy check that it is metrical with respect to $\mathcal{G}= g_{a\bar{b}}d\eta^a \otimes d\bar{\eta}^b$ on E.

In [2, 3], a *partial complex connection* $D = D' + D'' : VE \to (HE \oplus \overline{HE}^*) \otimes VE$ is studied, whose connection form reduces to $\omega_b^a = L_{bk}^a dz^k + L_{b\bar{k}}^a d\bar{z}^k$.

Finally, there exist a partial complex connection of $(1,0)-$type on VE, called the *complex Bott connection*, which is not metrical but has a very simple expression

$$D_X Y = v\,[X,Y] \ , \ \forall X \in HE \ , \ Y \in VE. \tag{1.11}$$

¿From the calculus of the Lie brackets, see (7.1.10) in [13], it results that the connection form of the complex Bott connection is:

$$\omega_b^a = \overset{B}{L}{}_{bk}^a \, dz^k \ , \text{ where } \overset{B}{L}{}_{bk}^a = \frac{\partial N_k^a}{\partial \eta^b}. \tag{1.12}$$

Like for (5.1.15) from [13], we can prove that $\frac{\partial N_k^a}{\partial \eta^b} = g^{\bar{d}a}\delta_k g_{b\bar{d}}$, where the adapted frames δ_k is with respect to (1.8) (*c.n.c.*).

Because $\delta_k, \dot{\partial}_a$ and $g_{a\bar{b}}$ are gauge $d-$fields, respectively $d-$tensor, it follows

Proposition 1.3. *If L_0 is a gauge invariant Lagrangian, then the complex Chern-Lagrange and Bott connections are gauge invariant.*

Clearly, we can treat the complex Bott connection as a vertical one with $L_{b\bar{k}}^a = C_{bc}^a = C_{b\bar{c}}^a = 0$, and generally this is an appropriate connection for our approach.

A complex vertical connection determines the following derivative laws in VE :

$$\begin{aligned} \overset{h}{D}_{\delta_k} \dot{\partial}_b &= L_{bk}^a \dot{\partial}_a \ ; \ \overset{\bar{h}}{D}_{\delta_{\bar{k}}} \dot{\partial}_b = L_{b\bar{k}}^a \dot{\partial}_a \ ; \\ \overset{v}{D}_{\dot{\partial}_c} \dot{\partial}_b &= C_{bc}^a \dot{\partial}_a \ ; \ \overset{\bar{v}}{D}_{\dot{\partial}_{\bar{c}}} \dot{\partial}_b = C_{b\bar{c}}^a \dot{\partial}_a \ . \end{aligned}$$

The covariant derivatives of a vertical field $\Phi = \Phi^a \frac{\partial}{\partial \eta^a}$ will be denoted with $\Phi^a{}_{\shortmid k}$, $\Phi^a{}_{\shortmid \bar{k}}$ and $\Phi^a{}_{| b}$, $\Phi^a{}_{| \bar{b}}$ where

$$\begin{aligned} \Phi^a{}_{\shortmid k} &= \delta_k \Phi^a + L_{bk}^a \Phi^b \ ; \ \Phi^a{}_{\shortmid \bar{k}} = \delta_{\bar{k}} \Phi^a + L_{b\bar{k}}^a \Phi^b \ ; \\ \Phi^a{}_{| c} &= \dot{\partial}_c \Phi^a + C_{bc}^a \Phi^b \ ; \ \Phi^a{}_{| \bar{c}} = \dot{\partial}_{\bar{c}} \Phi^a + C_{b\bar{c}}^a \Phi^b. \end{aligned}$$

If D is a gauge invariant connection, because $\delta_k, \dot{\partial}_c$ and $\delta_{\bar{k}}, \dot{\partial}_{\bar{c}}$ are gauge invariant, we may conclude that these covariant derivatives of Φ are gauge invariant as long as Φ is gauge invariant.

2 Complex field equations

Let E be a $G-$complex vector bundle over M. ¿From the physical point of view a section in E is treated as a field particle. The field particle dynamics assumes to consider the variation of a Lagrangian particle $L_p : E \to R$, which is a first order differential operator over the sections of E. This is $L_p = L_p(j_1\Phi)$, where $\Phi = \Phi^a s_a$ is a section and $j_1\Phi$ its first jet. Enlarge this is,

$$\hat{L}_p(\Phi) = L_p(\Phi^a, \partial_i \Phi^a, \partial_{\bar{i}} \Phi^a, \dot{\partial}_i \Phi^a, \dot{\partial}_{\bar{i}} \Phi^a) \tag{2.1}$$

where $\partial_i = \frac{\partial}{\partial z^i}$, $\dot{\partial}_i = \frac{\partial}{\partial \eta^i}$.

The field equations imply to find the particle Φ from the variational principle $\delta\mathcal{A} = \frac{d}{dt} \mid_{t=0} \mathcal{A}(\Phi + t\delta\Phi)$, where $\mathcal{A}(\Phi) = \int \hat{L}_p(\Phi)$ is the action integral. Actually, the action integral is defined on a compact subset $\theta \subset E$ and, for the independence of the integral at the changes of local charts, inside of $\hat{L}_p(\Phi)$ we consider the Lagrangian density $\mathcal{L}_p(\Phi) = \hat{L}_p(\Phi) \mid g \mid^2$, where $\mid g \mid = \mid \det g_{a\bar{b}} \mid$ and $g_{a\bar{b}} = \partial^2 L_p / \partial\eta^a \partial\bar{\eta}^b$ (L_p depends on (z, η) by means of Φ).

This problem of solutions for the field equations is one difficult, first because the chosen Lagrangian needs to be one gauge invariant (by means of Φ and its derivatives). Then the derivations in field equations are with respect to the natural frames ∂_i, $\partial_{\bar{i}}$ which, for a gauge invariant expression of the field equations, need to be replaced with the adapted frames of one $(g.c.n.c.)$, $\partial_i = \delta_i + N_i^a \ \dot{\partial}_a$. Such a way was followed in [12] in order to obtain the gauge invariant field equations. In the present paper we propose a somehow different approach of this problem, based on the "minimal replacement" principle ([14]), which is nothing but a generalization of Einstein's covariance principle.

The minimal replacement principle consists in replacement in $L_p(\Phi^a, \partial_i \Phi^a, \partial_{\bar{i}} \Phi^a, \dot{\partial}_i \Phi^a, \dot{\partial}_{\bar{i}} \Phi^a)$ the partial derivatives with the covariant derivatives of a gauge invariant vertical connection, possible the complex Bott connection. At the first glance this seems to be a notational process, but it is a more subtle idea. The connection becomes a dynamical variable which joints mechanics with the geometry of the space. Thus we will study the variation of the action for the Lagrangian $L_p(\Phi, D\Phi)$. But for the beginning let us introduce, as in standard theory ([14]), the (complex) currents on E :

$$J(\Phi, D\Phi) \wedge \delta\omega := \frac{d}{dt} \mid_{t=0} \mathcal{L}(\Phi, D\Phi + t\delta\omega) \tag{2.2}$$

where $\delta\omega$ is a variation for the connection form of D connection.

Direct calculus in (2.2) yields the following complex currents:

$$\overset{h}{J}{}^i_a = \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}} \ ; \ \overset{\bar{h}}{J}{}^{\bar{i}}_a = \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid \bar{i}}} \ ; \ \overset{v}{J}{}^b_a = \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\mid b}} \ ; \ \overset{\bar{v}}{J}{}^{\bar{b}}_a = \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\mid \bar{b}}} \tag{2.3}$$

which implicitly contain the following components

$$\overset{h}{J}{}^{ib}_a = \frac{\partial \mathcal{L}}{\partial L^a_{bi}} \ ; \ \overset{\bar{h}}{J}{}^{\bar{i}b}_a = \frac{\partial \mathcal{L}}{\partial L^a_{b\bar{i}}} \ ; \ \overset{v}{J}{}^{cb}_a = \frac{\partial \mathcal{L}}{\partial C^a_{bc}} \ ; \ \overset{\bar{v}}{J}{}^{\bar{c}b}_a = \frac{\partial \mathcal{L}}{\partial C^a_{b\bar{c}}} \ .$$

Now, let us come back to the variation of the action for the Lagrangian, $\delta\mathcal{A}(\Phi) = \frac{d}{dt} \mid_{t=0} \int_\theta \mathcal{L}(\Phi, D\Phi + t\delta\omega) = 0$. This implies

$$\int_\theta \{ \tfrac{\partial \mathcal{L}}{\partial \Phi^a} \delta\Phi^a + \tfrac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}} \delta(\Phi^a_{\,\shortmid i}) + \tfrac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid \bar{i}}} \delta(\Phi^a_{\,\shortmid \bar{i}}) + \tfrac{\partial \mathcal{L}}{\partial \Phi^a_{\,\mid b}} \delta(\Phi^a_{\,\mid b}) + \tfrac{\partial \mathcal{L}}{\partial \Phi^a_{\,\mid \bar{b}}} \delta(\Phi^a_{\,\mid \bar{b}}) \} = 0.$$

Further, for instance the calculus of the second term involves

$$\tfrac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}} \delta(\Phi^a_{\,\shortmid i}) = \tfrac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}} \tfrac{\partial}{\partial z^i} (\delta\Phi^a) + \tfrac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}} \delta(L^a_{bi} \Phi^b) =$$

$$= \tfrac{\partial}{\partial z^i} \left(\tfrac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}} \delta(\Phi^a) \right) - \tfrac{\partial}{\partial z^i} \left(\tfrac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}} \right) (\delta\Phi^a) + \tfrac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}} \delta(L^a_{bi} \Phi^b)$$

and analogously for the other terms. If we assume a variation with $\delta\Phi^a = 0$ on the boundary of θ, then finally for the variation of the integral action we obtain

$$\frac{\partial\mathcal{L}}{\partial\Phi^a} = \frac{\partial}{\partial z^i}\left(\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,\shortmid i}}\right) + \frac{\partial}{\partial \bar{z}^i}\left(\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,\shortmid \bar{i}}}\right) + \frac{\partial}{\partial \eta^b}\left(\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,| b}}\right) + \frac{\partial}{\partial \bar{\eta}^b}\left(\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,| \bar{b}}}\right) - < J, \delta\omega >$$

where,

$$< J, \delta\omega > = \int_\theta \{ \overset{h}{J}{}^i_a\, \delta(L^a_{bi}\Phi^b) + \overset{\bar{h}}{J}{}^i_a\, \delta(L^a_{b\bar{i}}\Phi^b) + \overset{v}{J}{}^c_a\, \delta(C^a_{bc}\Phi^b) + \overset{\bar{v}}{J}{}^{\bar{c}}_a\, \delta(L^a_{b\bar{c}}\Phi^b)\}.$$

Taking into account the (2.3) expressions of the complex currents, in adapted frames of the (1.8) (*c.n.c.*) the previous field equations are written

$$\frac{\partial\mathcal{L}}{\partial\Phi^a} = \delta_i\, \overset{h}{J}{}^i_a + \delta_{i}\, \overset{\bar{h}}{J}{}^{\bar{i}}_a + \dot{\partial}_b\, \overset{v}{J}{}^b_a + \dot{\partial}_{\bar{b}}\, \overset{\bar{v}}{J}{}^{\bar{b}}_a + N^b_i \dot{\partial}_b\, \overset{h}{J}{}^i_a + N^{\bar{b}}_{\bar{i}} \dot{\partial}_{\bar{b}}\, \overset{\bar{h}}{J}{}^{\bar{i}}_a - < J, \delta\omega > . \tag{2.4}$$

The (2.4) equations, for $a = \overline{1, m}$, will be called the *complex field equations* of the particle Φ.

The gauge invariance of the Lagrangian L_p, with respect to particle Φ and their covariant derivatives, implies the gauge invariance of the complex currents and consequently the gauge invariance of (2.4) complex field equations. Certainly, everywhere we take in discussion a gauge invariant vertical connection D.

The existence of a such gauge invariant particle Lagrangian is a somewhat fastidious problem since, as a rule, in general relativity L_p must be Lorentz invariant. One can be of the Klein-Gordon type, which can be constructed by help of a fixed Hermitian Lorentz metric $\gamma_{i\bar{j}}(z)$ on the complex manifold M (eventually the classical complex inner product). Moreover, we can add to $\gamma_{i\bar{j}}(z)$ a gauge invariant Hermitian metric $\gamma_{a\bar{b}}(z, \eta)$ on E which comes from a Finsler type Lagrangian ([2]). The Hermitian metric $\gamma_{a\bar{b}}$ determines the (1.8) (*c.n.c.*) and its adapted frames. Then, with respect to complex Bott connection (for instance), such a gauge invariant Lagrangian can be

$$L_p(\Phi, D\Phi) = \frac{1}{2}\sum_a \{\gamma^{\bar{j}i} D_{\delta_i}\Phi^a D_{\delta_{\bar{j}}}\bar{\Phi}^a + \gamma^{\bar{b}c} D_{\dot{\partial}_c}\Phi^a D_{\dot{\partial}_{\bar{b}}}\bar{\Phi}^a\} + V(\Phi). \tag{2.5}$$

As we already know from the classical field theory, this particle Lagrangian $L_p(\Phi, D\Phi)$ is not able, quite so in a generalized form, to offer a solid physical theory because it does not contain enough the geometrical aspects of the space (curvature, etc.). For this purpose, in the generalized Maxwell equations, the total Lagrangian of electrodynamics is taken in the form:

$$L_e(\Phi, D\Phi) = L_p(\Phi, D\Phi) + L_{YM}(D), \tag{2.6}$$

where

$$L_{YM}(D) = -\frac{1}{2}\Omega \wedge {}^*\Omega, \tag{2.7}$$

is a connection Lagrangian, Ω being the curvature form of D and ${}^*\Omega$ is its dual.

For the complex Bott connection in (7.2.8) from [13] is showed that

$$\Omega^a_b = R^a_{bi\bar{j}} dz^i \wedge d\bar{z}^j \ \text{ with } R^a_{bi\bar{j}} = -\delta_{\bar{j}} \overset{B}{L^a_{bi}} \ . \tag{2.8}$$

Thus we have,

$$L_{YM}(D) = -\frac{1}{2}\sum_{a,b} \gamma^{\bar{j}i}\gamma^{\bar{k}l} R^a_{bi\bar{j}} R^a_{bl\bar{k}} \ .$$

The connection form of Chern-Lagrange connection is a bit complicate:

$$\Omega^a_b = R^a_{bi\bar{j}} dz^i \wedge d\bar{z}^j \ + R^a_{bi\bar{d}} dz^i \wedge \delta\bar{\eta}^d + R^a_{bc\bar{j}} \delta\eta^c \wedge d\bar{z}^j + R^a_{bc\bar{d}} \delta\eta^c \wedge \delta\bar{\eta}^d,$$

where the curvature components are not so simply likes for Bott connection. The computations for $L_{YM}(D)$ become very laborious and hence now we renounce to do it.

Since, $\delta_D \mathcal{A}_e(\Phi, D\Phi) = \delta_D \mathcal{A}_p(\Phi, D\Phi) + \delta_D \mathcal{A}_{YM}(D)$, and $\delta_D \mathcal{A}_p(\Phi, D\Phi) = -< J, \delta\omega >= -< \delta\omega,^* J >$ ($^*J$ is the dual form current), a computation like in [14], yields for the complex Bott connection that $\delta_D \mathcal{A}_{YM}(D) =< \delta\omega,^* D^*\Omega >$. Hence, for the complex Bott connection we have that

$$D^*\Omega =^* J \ , \tag{2.9}$$

this generally being called the *complex Yang-Mills equation* on E.

Also we can check that $D^*J = 0$ (the same calculus like for formulae (6.7) from [8]) and therefore the complex currents are conservative.

In a expansive writing form, the complex Y-M equation is

$$\delta_k \Omega^a_b + L^a_{ck}\Omega^c_b - L^c_{bk}\Omega^a_c =^* \overset{h}{J^a_{kb}} \ .$$

We note that in this complex Y-M equation the curvature form of Bott connection contains implicitly the Hermitian metric tensor $g_{a\bar{b}} = \partial^2 L_p / \partial\eta^a \partial\bar{\eta}^b$ of the particle Lagrangian.

Finally, for coupling with gravity we again consider the Lorentz hermitian metric $\gamma_{i\bar{j}}(z)$ on M, which now we assume it derives from a gravitational potential, and $G = \gamma_{i\bar{j}} dz^i \wedge d\bar{z}^j + g_{a\bar{b}} \delta\eta^a \wedge \delta\bar{\eta}^b$ a metric structure on $T_C E$.

By $S_{i\bar{j}} = \sum S^k_{ki\bar{j}}$ and by $\rho(\gamma) = \gamma^{\bar{j}i} S_{i\bar{j}}$ we denote the Ricci curvature and scalar, respectively, with respect to L-C connection of $\gamma_{i\bar{j}}$ metric. Also by $R_{i\bar{j}} = \sum R^a_{ai\bar{j}}$ and $\rho(g) = \gamma^{\bar{j}i} R_{i\bar{j}}$ we have the Ricci curvature and scalar, respectively, with respect to Bott connection of g metric. The sum $\rho = \rho(\gamma) + \rho(g)$ generates an Hilbert type Lagrangian $L_G = -\frac{1}{\chi}\rho$, where χ is the universal constant.

The complex Einstein equations on E will be

$$\begin{aligned} S_{i\bar{j}} - \frac{1}{2}\rho(\gamma)\gamma_{i\bar{j}} &= \chi T_{i\bar{j}} \\ R_{i\bar{j}} - \frac{1}{2}\rho(g)\gamma_{i\bar{j}} &= \chi T_{i\bar{j}} \end{aligned} \tag{2.10}$$

where, $T_{i\bar{j}}$ is the stress-energy tensor of the potential gravity $\gamma_{i\bar{j}}(z)$ on M.

The total Lagrangian for coupling gravity with electodinamics (complex nonhomogen Maxwell equations) is

$$L_t(\Phi, D\Phi) = L_p(\Phi, D\Phi) + L_{YM}(D) + L_G. \tag{2.11}$$

In [12] we considered for exact symmetry the (2.5) particle Lagrangian, with $a = 1$ (the term $\gamma^{\bar{b}c} D_{\dot{\partial}_c} \Phi D_{\dot{\partial}_{\bar{b}}} \bar{\Phi}$ coincides on $E = T'M$ with the first part of the Lagrangian) and $V(\Phi) = -m^2 \Phi\bar{\Phi} - \frac{1}{4} f(\Phi\bar{\Phi})^2$. Similarly, for the broken symmetry Lagrangian $V(\Phi)$ is $V(\Phi) = m^2 \Phi\bar{\Phi} - \frac{1}{4} f(\Phi\bar{\Phi})^2$. The gauge invariance of these Lagrangians is considered with respect to the transformation $\Phi \to \tilde{\Phi}(\tilde{z}, \tilde{\eta}) = e^{-ig\varepsilon} \Phi(z, \eta)$, where ε is the parameter of $U(1)$ group. Actually we consider (like in the classical theory) two particle field, $\Phi^1 = \Phi$ and $\Phi^2 = \bar{\Phi}$, its conjugate. The L_p Lagrangian from (2.5) is then gauge invariant at $U(1)$ group transformations and, for a chosen gauge invariant metric $\gamma_{i\bar{j}}(z)$ on M, the complex Bott connection is gauge invariant too.

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