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A Measure of Asymmetry for Domains of Constant Width

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Abstract. We introduce for convex domains of constant width a measure of asymmetry and show that the most asymmetric domains are Reuleaux triangles.

Measures of (central) symmetry, or, as we prefer, asymmetry for convex bodies have been extensively investigated, especially with respect to determining the extremal bodies. A survey of results of this kind (up to 1963) has been published by Grünbaum [4]. In some of these investigations the definition of such measures is restricted to certain subsets of the class of all convex bodies. For example, in his paper [1], Besicovitch considers a measure of asymmetry for domains of constant width in the euclidean plane $\mathbf{R}^2$.

In this note, we present another natural measure of asymmetry for domains of constant width and show that the most symmetric specimens are circular discs and the most asymmetric ones are Reuleaux triangles.

Let K be a convex domain, that is, a closed bounded convex subset of $\mathbf{R}^2$, and let u be a direction (unit vector). By a *diameter* of K of direction u we mean a line segment of direction u in K of maximal length. If K is of constant width then for any u there is exactly one diameter $D(u)$ of K of direction u, and the two lines that pass through the endpoints of $D(u)$ and are orthogonal to u are support lines of K. The diameter $D(u)$ splits K into two convex domains, say $K^+(u)$ and $K^-(u)$, where $K^+(u)$ lies in the 'positive' half-plane with respect to the line of direction u containing $D(u)$. For references regarding the known results about convex bodies of constant width that are used here see [2].

From now on, K will always denote a convex domain of constant width. C denotes the unit circle in $\mathbf{R}^2$ and $A(\cdot)$ signifies the area. We define the *asymmetry function* of K by

$$\alpha(K) = \max\{A(K^+(u))/A(K^-(u)) : u \in C\}.$$

For example, if K is Reuleaux triangle and if we write $\alpha_o = \alpha(K)$, then an obvious calculation shows that

$$\alpha_o = \frac{4\pi - 3\sqrt{3}}{2\pi - 3\sqrt{3}} = 6.780\ldots.$$

The following theorem contains our principal result.

Theorem. *Let K be a convex domain of constant width. Then,*

$$1 \leq \alpha(K) \leq \alpha_o. \tag{1}$$

Equality holds on the left-hand side precisely when K is a circular disc, and on the right hand side precisely when K is a Reuleaux triangle.

For the proof of this theorem we assume that $\mathbf{R}^2$ is equipped with the standard cartesian coordinate system. Then u is determined by the usual polar angle θ, so that $u = u(\theta) = (\cos\theta, \sin\theta)$. Hence we may consider $D(u)$ as a function of θ and write $D(\theta)$ instead of $D(u)$. We also use the corresponding notation for the other pertinent functions of u. We first prove two lemmas. Lemma 1 relates $A(K^+(\theta))$ to the lengths of the boundary arc of $K^+(\theta)$ (excluding $D(\theta)$). This lemma is actually known (see [2, p.57]) but for the sake of completeness we present here a very simple proof.

Lemma 1. *Let $L(K^+(\theta))$ be the lengths of the arc $\partial K^+(\theta) \cap \partial K$. Then, there is a constant $c(K)$ such that for all $\theta \in [0, 2\pi]$*

$$A(K^+(\theta)) - \frac{w}{2} L(K^+(\theta)) = c(K),$$

where w denotes the width of K.

Proof. Since K can be approximated (in the Hausdorff metric) by Reuleaux polygons it suffices to prove the lemma under the assumption that K is such a domain. Excluding the trivial case that ∂K is a circle each diameter of K has at least one endpoint at a 'corner' of K. Let now $D(\theta_1)$ and $D(\theta_2)$ be two diameters with one endpoint at the same corner and, consequently, the other in the same circular arc. Then it is evident that

$$A(K^+(\theta_1)) - \frac{w}{2} L(K^+(\theta_1)) = A(K^+(\theta_2)) - \frac{w}{2} L(K^+(\theta_2)).$$

Hence, setting

$$f(\theta) = A(K^+(\theta)) - \frac{w}{2} L(K^+(\theta)),$$

we have $f(\theta_1) = f(\theta_2)$ and this shows that $f(\theta)$ is constant on each interval corresponding to an arc of ∂K. Since $f(\theta)$ is obviously continuous, it must be constant and this proves the lemma.

Replacing θ by $\theta + \pi$ we note that this lemma implies that

$$A(K^-(\theta)) - \frac{w}{2} L(K^-(\theta)) = c(K).$$

Our second lemma is of a purely analytic nature.

Lemma 2. *Let $F(\theta)$ be a measurable function on $[0,\pi]$ with $0 \le F(\theta) \le 1$. If*

$$\int_0^\pi F(\theta)\sin\theta\, d\theta = 1,$$

then

$$\int_0^\pi F(\theta) d\theta \le \frac{2\pi}{3}.$$

Equality holds exactly if $F(\theta) = 1$ a.e. on $[0,\pi/3] \cup [2\pi/3,\pi]$.

Proof. Since $\int_0^{\pi/3} F(\theta)\sin\theta\, d\theta \le \int_0^{\pi/3} \sin\theta\, d\theta = 1/2$ and $\int_0^{2\pi/3} F(\theta)\sin\theta\, d\theta = 1 - \int_{2\pi/3}^{\pi} F(\theta)\sin\theta\, d\theta \ge 1/2$, there exists a number $c \in [\pi/3, 2\pi/3]$ such that

$$\int_0^c F(\theta)\sin\theta\, d\theta = \frac{1}{2}.$$

Hence, observing also that $\frac{2}{\sqrt{3}}\sin\theta \le 1$ if $\theta \in [0,\pi/3]$, and $\frac{2}{\sqrt{3}}\sin\theta \ge 1$ if $\theta \in [\pi/3, c]$ we find

$$\begin{aligned}\int_{\pi/3}^c F(\theta) d\theta \le \frac{2}{\sqrt{3}} \int_{\pi/3}^c F(\theta)\sin\theta\, d\theta &= \frac{2}{\sqrt{3}}\left(\frac{1}{2} - \int_0^{\pi/3} F(\theta)\sin\theta\, d\theta\right)\\ &= \frac{2}{\sqrt{3}}\int_0^{\pi/3}(1-F(\theta))\sin\theta\, d\theta \le \int_0^{\pi/3}(1-F(\theta))d\theta.\end{aligned}$$

This shows that

$$\int_0^c F(\theta) d\theta \le \frac{\pi}{3},$$

and it is easily checked that equality occurs exactly if $F(\theta) = 1$ a.e. on $[0,\pi/3]$. This inequality, combined with the corresponding inequality that is obtained by applying essentially the same argument to the interval $[c,\pi]$ yields the desired conclusion.

Proof of the theorem. It is obvious that $\alpha(K) \ge 1$. Moreover, if $\alpha(K) = 1$ then it follows from Lemma 1 that $L(K^+(\theta)) = L(K^-(\theta))$ (for all θ). It is known (see [3, Theorem 4.5.9]) that this can only happen if K is centrally symmetric. But since K is of constant width it must be a circular disc.

For the proof of the right-hand inequality of (1) we let $h(\theta)$ denote the support function of K in the direction $u(\theta)$ and assume, as we may, that the width of K is 1. If $h(\theta)$ is twice continuously differentiable then the radius of curvature, say $r(\theta)$, is given by $r(\theta) = h(\theta) + h''(\theta)$. Furthermore, since the width $h(\theta) + h(\theta+\pi)$ of K equals 1 we obtain the well-known fact that $r(\theta) + r(\theta+\pi) = 1$, which, in turn, implies that

$$r(\theta) \le 1. \tag{2}$$

We also note that integration by parts, together with the above representation of $r(\theta)$ in terms of the support function, yields

$$\int_0^\pi r(\theta)\sin\theta\, d\theta = h(0) + h(\pi). \tag{3}$$

Next we prove that for every θ

$$A(K^+(\theta)) - A(K^-(\theta)) \le \frac{\pi}{6} \tag{4}$$

To show this, it is convenient to assume that K is positioned so that $\theta = 0$ and that $[0,1]$ is a diameter of K. In addition, we may assume that h is twice continuously differentiable. This is justified by a theorem of Schneider [5] which shows that any convex body of constant width can be approximated (in the Hausdorff metric) by convex bodies having the same constant width and whose support functions possess the desired regularity property. Then, $h(0) = 1$, $h(\pi) = 0$ and in view of (2) and (3) we can apply Lemma 2 with $F(\theta) = r(\theta)$ to infer that

$$L(K^+(0)) = \int_0^\pi r(\theta)d\theta \le \frac{2\pi}{3}.$$

Combining this with Lemma 1 and the fact that $L(K^+(0)) + L(K^-(0)) = \pi$ we deduce the desired conclusion

$$A(K^+(0)) - A(K^-(0)) = \frac{1}{2}\big(L(K^+(0)) - L(K^-(0))\big) \le L(K^+(0)) - \frac{\pi}{2} \le \frac{\pi}{6}.$$

Let now A_o denote the area of a Reuleaux triangle of width 1, i.e., $A_o = (\pi - \sqrt{3})/2$, and assume that K is not a Reuleaux triangle. Then $A(K) > A_o$ and it follows that for every θ,

$$A(K^+(\theta)) + A(K^-(\theta)) > A_o.$$

Hence, (4) implies that for every θ

$$\frac{\big(A(K^+(\theta))/A(K^-(\theta))\big) - 1}{\big(A(K^+(\theta))/A(K^-(\theta))\big) + 1} = \frac{A(K^+(\theta)) - A(K^-(\theta))}{A(K^+(\theta)) + A(K^-(\theta))} < \frac{\pi}{6A_o},$$

or, equivalently, that

$$\frac{A(K^+(\theta))}{A(K^-(\theta))} < \frac{\pi + 6A_o}{6A_o - \pi} = \alpha_o.$$

The theorem follows now by taking the maximum over all θ.

We finally remark that a more careful analysis, that takes into account the statement in Lemma 2 regarding the occurrence of equality, reveals that in (4) equality holds exactly if K is a Reuleaux triangle. Thus if we define $\max\{A(K^+(u)) - A(K^-(u)) : u \in C\}$ as another measure of asymmetry we obtain a result analogous to that of the above theorem.

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