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Wiener amalgams and summability of Fourier series*

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Abstract

Some recent results on a general summability method, on the so-called θ-summability is summarized. New spaces, such as Wiener amalgams, Feichtinger's algebra and modulation spaces are investigated in summability theory. Sufficient and necessary conditions are given for the norm and a.e. convergence of the θ-means.

Key Words: Wiener amalgam spaces, Feichtinger's algebra, homogeneous Banach spaces, Besov-, Sobolev-, fractional Sobolev spaces, modulation spaces, Herz spaces, Hardy-Littlewood maximal function, θ-summability of Fourier series, Lebesgue points.

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1. Introduction

In this paper we consider a general method of summation, the so called θ-summation, which is generated by a single function θ. A natural choice of θ is a function from the Wiener algebra $W(C, \ell_1)(\mathbb{R}^d)$. All concrete summability methods investigated in the literature satisfy this condition.

We shall investigate some function spaces known from other topics of analysis, for example Wiener amalgam spaces, Feichtinger's Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$, modulation and Herz spaces. Feichtinger's algebra and modulation spaces are very intensively investigated in Gabor analysis (see e.g. Feichtinger and Zimmermann [6] and Gröchenig [13]). $\mathbf{S}_0(\mathbb{R}^d)$ is the minimal (non-trivial) Banach space which is isometrically invariant under translation, modulation and Fourier transform.

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In Sections 4 and 5 we deal with norm convergence of the θ-means of multi-dimensional Fourier series and Fourier transforms. We will show that if θ is in the Wiener algebra then the θ-means $\sigma_n^\theta f$ of the Fourier series of $f \in L_2(\mathbb{T}^d)$ converge to f in L_2 norm as $n \to \infty$. Moreover, $\sigma_n^\theta f \to f$ uniformly (resp. at each point) for all $f \in C(\mathbb{T}^d)$ if and only if $\sigma_n^\theta f \to f$ in L_1 norm for all $f \in L_1(\mathbb{T}^d)$ if and only if $\hat{\theta} \in L_1(\mathbb{R}^d)$. If B is a homogeneous Banach space on $\mathbb{T}^d$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ then $\sigma_n^\theta f \to f$ in B norm for all $f \in B$. If θ is continuous and has compact support then the uniform convergence of the θ-means is equivalent to the L_1 norm convergence of the θ-means and this is equivalent to the condition $\theta \in \mathbf{S}_0(\mathbb{R}^d)$. In all cases we investigate convergence over the diagonal.

In Sections 7 and 8 the a.e. convergence of the θ-means is considered. We show that $\hat{\theta}$ is in the homogeneous Herz space $\dot{E}_q(\mathbb{R}^d)$ for some $1 < q \leqslant \infty$ if and only if the maximal operator of the θ-means of the Fourier transform of f can be estimated by the modified Hardy-Littlewood maximal function $M_p f$, where p is the dual index to q. Since M_p is of weak type (p,p) we obtain $\sigma_T^\theta f \to f$ a.e. as $T \to \infty$ for all $f \in L_r(\mathbb{R}^d)$, $p \leqslant r < \infty$. Under the condition $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$ this convergence holds also for functions from the Wiener amalgam space $W(L_p, \ell_\infty)(\mathbb{R}^d)$. The set of convergence is also characterized, the convergence holds at every p-Lebesgue point of f. The converse holds also, more exactly, $\sigma_T^\theta f(x) \to f(x)$ at each p-Lebesgue point of $f \in L_p(\mathbb{R}^d)$ (resp. of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$) if and only if $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$.

In Sections 6 and 9 we give some sufficient conditions for θ such that $\hat{\theta} \in L_1(\mathbb{R}^d)$, or $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ or $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$. More exactly, if θ is in a suitable Besov, Sobolev, fractional Sobolev, weighted Wiener amalgam or modulation space then all convergence results above hold.

Most of the proofs of the results of this survey paper can be found in Feichtinger and Weisz [5, 4]. This paper was the base of my talk given at the Fejér-Riesz Conference, June 2005, in Eger (Hungary).

2. Wiener amalgams and Feichtinger's algebra

Let us fix $d \geqslant 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d-times. We shall prove results for $\mathbb{R}^d$ or $\mathbb{T}^d$, therefore it is convenient to use sometimes the symbol $\mathbb{X}$ for either $\mathbb{R}$ or $\mathbb{T}$, where $\mathbb{T}$ is the torus. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^d u_k x_k, \qquad \|x\|_p := \Big(\sum_{k=1}^d |x_k|^p \Big)^{1/p}, \qquad |x| := \|x\|_2.$$

We briefly write L_p or $L_p(\mathbb{X}^d)$ instead of $L_p(\mathbb{X}^d, \lambda)$ space equipped with the norm (or quasi-norm) $\|f\|_p := (\int_{\mathbb{X}^d} |f|^p \, d\lambda)^{1/p}$ $(0 < p \leqslant \infty)$, where $\mathbb{X} = \mathbb{R}$ or $\mathbb{T}$ and λ is the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of the set I.

The *weak* L_p space, $L_{p,\infty}(\mathbb{X}^d)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\rho>0} \rho\lambda(|f| > \rho)^{1/p} < \infty,$$

while we set $L_{\infty,\infty}(\mathbb{X}^d) = L_\infty(\mathbb{X}^d)$. Note that $L_{p,\infty}(\mathbb{X}^d)$ is a quasi-normed space (see Bergh and Löfström [1]). It is easy to see that for each $0 < p \leqslant \infty$,

$$L_p(\mathbb{X}^d) \subset L_{p,\infty}(\mathbb{X}^d) \qquad \text{and} \qquad \|\cdot\|_{L_{p,\infty}} \leqslant \|\cdot\|_p.$$

The space of continuous functions with the supremum norm is denoted by $C(\mathbb{X}^d)$ and we will use $C_0(\mathbb{R}^d)$ for the space of continuous functions vanishing at infinity. $C_c(\mathbb{R}^d)$ denotes the space of continuous functions having compact support.

A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q^{v_s})(\mathbb{R}^d)$ $(1 \leqslant p, q \leqslant \infty)$ if

$$\|f\|_{W(L_p,\ell_q^{v_s})} := \Big(\sum_{k\in\mathbb{Z}^d} \|f(\cdot + k)\|^q_{L_p[0,1)^d} v_s(k)^q\Big)^{1/q} < \infty$$

with the obvious modification for $q = \infty$, where the weight function v_s is defined by $v_s(\omega) := (1 + |\omega|)^s$ $(\omega \in \mathbb{R}^d)$. If $s = 0$ then we write simply $W(L_p, \ell_q)(\mathbb{R}^d)$. $W(L_p, c_0)(\mathbb{R}^d)$ is defined analogously, where c_0 denotes the space of sequences of complex numbers having 0 limit, equipped with the supremum norm. If we replace the space $L_p[0, 1)^d$ by $L_{p,\infty}[0, 1)^d$ then we get the definition of $W(L_{p,\infty}, \ell_q)(\mathbb{R}^d)$. The closed subspace of $W(L_\infty, \ell_q)(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C, \ell_q)(\mathbb{R}^d)$ $(1 \leqslant q \leqslant \infty)$. The space $W(C, \ell_1)(\mathbb{R}^d)$ is called *Wiener algebra*. It is used quite often in Gabor analysis, because it provides a convenient and general class of windows (see e.g. Walnut [33] and Gröchenig [12]). As we can see later, it plays an important rule in summability theory, too.

It is easy to see that $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and

$$W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d) \qquad (1 \leqslant p \leqslant \infty).$$

For more about amalgam spaces see e.g. Heil [14].

Translation and *modulation* of a function f are defined, respectively, by

$$T_x f(t) := f(t - x) \quad \text{and} \quad M_\omega f(t) := e^{2\pi\imath\omega\cdot t} f(t) \qquad (x, \omega \in \mathbb{R}^d).$$

Recall that the *Fourier transform* and the *short-time Fourier transform* (STFT) with respect to a window function g are defined by

$$\mathcal{F}f(x) := \hat{f}(x) := \int_{\mathbb{R}^d} f(t)e^{-2\pi\imath x\cdot t}\, dt \qquad (x \in \mathbb{R}^d, \imath = \sqrt{-1})$$

and

$$S_g f(x, \omega) := \int_{\mathbb{R}^d} f(t)\overline{g(t - x)}e^{-2\pi\imath\omega t}\, dt = \langle f, M_\omega T_x g\rangle \qquad (x, \omega \in \mathbb{R}^d),$$

respectively, whenever the integrals do exist.

Feichtinger's algebra $\mathbf{S}_0(\mathbb{R}^d)$ and the *modulation spaces* $M_1^{v_s}(\mathbb{R}^d)$ (see e.g. Feichtinger [7] and Gröchenig [13]) are intoduced by

$$M_1^{v_s}(\mathbb{R}^d) := \Big\{ f \in L^2(\mathbb{R}^d) : \|f\|_{M_1^{v_s}} := \|S_{g_0} f \cdot v_s\|_{L_1(\mathbb{R}^{2d})} < \infty \Big\} \qquad (s \geqslant 0),$$

where $g_0(x) := e^{-\pi|x|^2}$ is the Gauss function and $v_s(x, \omega) := (1 + |\omega|)^s$ $(x, \omega \in \mathbb{R}^d)$. In case $s = 0$ we write $\mathbf{S}_0(\mathbb{R}^d) := M_1(\mathbb{R}^d)$.

It is known that $\mathbf{S}_0(\mathbb{R}^d)$ is isometrically invariant under translation, modulation and Fourier transform. Actually, $\mathbf{S}_0$ is the minimal space having this property (see Feichtinger [7]). Moreover, the embeddings $\mathcal{S}(\mathbb{R}^d) \hookrightarrow M_1^{v_s}(\mathbb{R}^d)$ $(s \geqslant 0)$ and $\mathbf{S}_0(\mathbb{R}^d) \hookrightarrow W(C, \ell_1)(\mathbb{R}^d)$ are dense and continuous (see e.g. Feichtinger and Zimmermann [6] and Gröchenig [13]), where $\mathcal{S}$ denotes the Schwartz functions.

A Banach space B consisting of Lebesgue measurable functions on $\mathbb{X}^d$ is called a *homogeneous Banach space*, if

(i) for all $f \in B$ and $x \in \mathbb{X}^d$, $T_x f \in B$ and $\|T_x f\|_B = \|f\|_B$,

(ii) the function $x \mapsto T_x f$ from $\mathbb{X}^d$ to B is continuous for all $f \in B$,

(iii) the functions in B are uniformly locally integrable, i.e. for every compact set $K \subset \mathbb{X}^d$ there exists a constant C_K such that

$$\int_K |f| \, d\lambda \leqslant C_K \|f\|_B \qquad (f \in B).$$

If furthermore B is a dense subspace of $L_1(\mathbb{X}^d)$ it is called a *Segal algebra* (cf. Reiter [20]). Note that the continuous embedding into $L_1(\mathbb{X}^d)$ is a consequence of the closed graph theorem. For an introduction to homogeneous Banach spaces see Katznelson [16] or Shapiro [24]. It is easy to see that the spaces $L_p(\mathbb{X}^d)$ $(1 \leqslant p < \infty)$, $C(\mathbb{T}^d)$, $C_0(\mathbb{R}^d)$, Lorentz spaces $L_{p,q}(\mathbb{X}^d)$ $(1 < p < \infty, 1 \leqslant q < \infty)$, Hardy spaces $H_1(\mathbb{X}^d)$ (for the definitions see e.g. Weisz [37]), Wiener amalgams $W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \leqslant p, q < \infty)$, $W(L_p, c_0)(\mathbb{R}^d)$ $(1 \leqslant p < \infty)$, $W(C, \ell_q)(\mathbb{R}^d)$ $(1 \leqslant q < \infty)$ and $\mathbf{S}_0(\mathbb{R}^d)$ are homogeneous Banach spaces. Note that if B is a homogeneous Banach space on $\mathbb{R}^d$ then $B \hookrightarrow W(L_1, \ell_\infty)(\mathbb{R}^d)$ (see Katznelson [16]).

3. θ-summability of Fourier series

The θ-summation was considered in a great number of papers and books, such as Butzer and Nessel [3], Trigub and Belinsky [32], Bokor, Schipp, Szili and Vértesi [22, 2, 23, 28, 29], Natanson and Zuk [18], Weisz [35, 36, 37, 38] and Feichtinger and Weisz [5, 4]. We assume that the function θ is from the Wiener algebra $W(C, \ell_1)(\mathbb{R}^d)$. We have seen in Feichtinger and Weisz [5, 4] that this is a natural choice of θ and all summability methods considered in Butzer and Nessel [3] and Weisz [37] satisfy this condition.

Recall that for a distribution $f \in \mathcal{S}'(\mathbb{T}^d)$ the nth *Fourier coefficient* is defined by $\hat{f}(n) := f(e^{-2\pi\imath n\cdot x})$ $(n \in \mathbb{Z}^d)$. In special case, if $f \in L_1(\mathbb{T}^d)$ then

$$\hat{f}(n) = \int_{\mathbb{T}^d} f(t) e^{-2\pi\imath n\cdot t}\, dt \qquad (n \in \mathbb{Z}^d).$$

Given a function $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ the θ*-means* of a distribution f are defined by

$$\sigma_n^\theta f(x) := \sum_{j=1}^d \sum_{k_j=-\infty}^{\infty} \theta\Big(\frac{-k_1}{n_1+1},\dots,\frac{-k_d}{n_d+1}\Big)\hat{f}(k)e^{2\pi\imath k\cdot x} = \int_{\mathbb{T}^d} f(x-t) K_n^\theta(t)\, dt,$$

where $x \in \mathbb{T}^d, n \in \mathbb{N}^d$ and the θ*-kernels* K_n^θ are given by

$$K_n^\theta(t) := \sum_{j=1}^d \sum_{k_j=-\infty}^{\infty} \theta\Big(\frac{-k_1}{n_1+1},\dots,\frac{-k_d}{n_d+1}\Big)e^{2\pi\imath k\cdot t} \qquad (t \in \mathbb{T}^d).$$

Under $\sum_{j=1}^d \sum_{k_j=-\infty}^{\infty}$ we mean the sum $\sum_{k_1=-\infty}^{\infty} \dots \sum_{k_d=-\infty}^{\infty}$. It is easy to see that

$$\begin{aligned} \sum_{j=1}^d \sum_{k_j=-\infty}^{\infty} \Big|\theta\Big(\frac{k_1}{n_1+1},\dots,\frac{k_d}{n_d+1}\Big)\Big| &\leqslant \sum_{l\in\mathbb{Z}^d} \Big(\prod_{j=1}^d (n_j+1)\Big) \sup_{x\in[0,1)^d} |\theta(x+l)| \\ &= \Big(\prod_{j=1}^d (n_j+1)\Big) \|\theta\|_{W(C,\ell_1)} < \infty, \end{aligned}$$

and hence $K_n^\theta \in L_1(\mathbb{T}^d)$. We will always suppose that $\theta(0) = 1$.

Now we present some well known one-dimensional summability methods as special cases of the θ-summation. For more examples see Feichtinger and Weisz [5, 4].

Example 3.1 (Fejér summation). Let

$$\theta(x) := \begin{cases} 1-|x| & \text{if } 0 \leqslant |x| \leqslant 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

$$\sigma_n^\theta f(x) := \sum_{k=-n}^{n} (1 - \frac{|k|}{n+1})\hat{f}(k)e^{2\pi\imath k\cdot x}.$$

Example 3.2 (Riesz summation). Let

$$\theta(x) := \begin{cases} (1-|x|^\gamma)^\alpha & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

for some $0 \leqslant \alpha, \gamma < \infty$. The Riesz operators are given by

$$\sigma_n^\theta f(x) := \sum_{k=-n}^{n} \Big(1 - \Big|\frac{k}{n+1}\Big|^\gamma\Big)^\alpha \hat{f}(k)e^{2\pi\imath k\cdot x}.$$

Example 3.3 (Weierstrass summation). Let

$$\theta(x) = e^{-|x|^\gamma} \qquad (0 < \gamma < \infty),$$

$$\sigma_n^\theta f(x) := \sum_{k=-\infty}^{\infty} e^{-(\frac{|k|}{n+1})^\gamma} \hat{f}(k) e^{2\pi\imath k\cdot x}.$$

The most known form of the Weierstrass means are

$$W_r^\theta f(x) := \sum_{k=-\infty}^{\infty} r^{|k|^\gamma} \hat{f}(k) e^{2\pi\imath k\cdot x} \qquad (0 < r < 1).$$

Example 3.4 (Generalized Picar and Bessel summations). Let

$$\theta(x) = \frac{1}{(1+|x|^\gamma)^\alpha}$$

for some $0 < \alpha, \gamma < \infty$ such that $\alpha\gamma > 1$. The θ-means are given by

$$\sigma_n^\theta f(x) := \sum_{k=-\infty}^{\infty} \frac{1}{\left(1 + (\frac{|k|}{n+1})^\gamma\right)^\alpha} \hat{f}(k) e^{2\pi\imath k\cdot x}.$$

Example 3.5 (de La Vallée-Poussin summation). Let

$$\theta(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant 1/2, \\ -2x+2 & \text{if } 1/2 < x \leqslant 1, \\ 0 & \text{if } x > 1. \end{cases}$$

Example 3.6. Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m$ and $\beta_0, \ldots, \beta_m$ $(m \in \mathbb{N})$ be real numbers, $\beta_0 = 1$, $\beta_m = 0$. Suppose that $\theta(\alpha_j) = \beta_j$ $(j = 0, 1, \ldots, m)$, $\theta(x) = 0$ for $x \geqslant \alpha_m$, θ is a polynomial on the interval $[\alpha_{j-1}, \alpha_j]$ $(j = 1, \ldots, m)$.

4. Norm Convergence of the θ-means of Fourier series

In this section we collect some results about the norm convergence of $\sigma_{\mathbf{n}}^\theta f$ as $n \to \infty$. The proofs of the theorems can be found in Feichtinger and Weisz [5]. Note that $\mathbf{x}$ denotes the vector $(x, \ldots, x) \in \mathbb{R}^d$ $(x \in \mathbb{R})$.

Theorem 4.1. *If $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\theta(0) = 1$ then for all $f \in L_2(\mathbb{T}^d)$*

$$\lim_{n\to\infty} \sigma_{\mathbf{n}}^\theta f = f \qquad \text{in } L_2(\mathbb{T}^d) \text{ norm.}$$

If the Fourier transform of θ is integrable then the θ-means can be written as a singular integral of f and the Fourier transform of θ in the following way.

Theorem 4.2. *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\hat\theta \in L_1(\mathbb{R}^d)$ then*

$$\sigma_{\mathbf{n}}^{\theta} f(x) = (n+1)^d \int_{\mathbb{R}^d} f(x-t)\, \hat\theta\big((n+1)t\big)\, dt$$

for all $x \in \mathbb{T}^d$, $n \in \mathbb{N}$ and $f \in L_1(\mathbb{T}^d)$.

For the uniform and L_1 norm convergence of $\sigma_{\mathbf{n}}^{\theta} f \to f$ a sufficient and necessary condition can be given.

Theorem 4.3. *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\theta(0)=1$ then the following conditions are equivalent:*

(i) $\hat\theta \in L_1(\mathbb{R}^d)$,

(ii) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iii) $\sigma_{\mathbf{n}}^{\theta} f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iv) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *in* $L_1(\mathbb{T}^d)$ *norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$.

One part of the preceding result is generalized for homogeneous Banach spaces.

Theorem 4.4. *Assume that B is a homogeneous Banach space on $\mathbb{T}^d$. If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0)=1$ and $\hat\theta \in L_1(\mathbb{R}^d)$ then for all $f \in B$*

$$\lim_{n\to\infty} \sigma_{\mathbf{n}}^{\theta} f = f \qquad \text{in } B \text{ norm.}$$

Since $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ implies $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\hat\theta \in \mathbf{S}_0(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$, the next corollary follows from Theorems 4.3 and 4.4.

Corollary 4.5. *If $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ and $\theta(0)=1$ then*

(i) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(ii) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *in* $L_1(\mathbb{T}^d)$ *norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$,

(iii) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *in* B *norm for all* $f \in B$ *as* $n \to \infty$ *if* B *is a homogeneous Banach space.*

If θ has compact support then $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ is equivalent to the conditions $\theta, \hat\theta \in L_1(\mathbb{R}^d)$ (see Feichtinger and Zimmermann [6]). This implies

Corollary 4.6. *If $\theta \in C(\mathbb{R}^d)$ has compact support and $\theta(0)=1$ then the following conditions are equivalent:*

(i) $\theta \in \mathbf{S}_0(\mathbb{R}^d)$,

(ii) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iii) $\sigma_{\mathbf{n}}^{\theta} f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iv) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *in* $L_1(\mathbb{T}^d)$ *norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$.

5. Norm convergence of the θ-means of Fourier transforms

All the results above can be shown for non-periodic functions $f \in L_p(\mathbb{R}^d)$. Suppose first that $f \in L_p(\mathbb{R}^d)$ for some $1 \leqslant p \leqslant 2$. The Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(u) e^{2\pi \imath x \cdot u} \, du \qquad (x \in \mathbb{R}^d)$$

holds if $\hat{f} \in L_1(\mathbb{R}^d)$.

In the investigation of Fourier transforms we can take a larger space than $W(C, \ell_1)(\mathbb{R}^d)$, we will assume that $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. The *$\theta$-means* of $f \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant 2)$ are defined by

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \theta\Big(\frac{-t_1}{T_1}, \dots, \frac{-t_d}{T_d}\Big) \hat{f}(t) e^{2\pi \imath x \cdot t} \, dt = \int_{\mathbb{R}^d} f(x-t) K_T^\theta(t) \, dt$$

where $x \in \mathbb{R}^d, T \in \mathbb{R}_+^d$ and

$$K_T^\theta(x) = \int_{\mathbb{R}^d} \theta\Big(\frac{-t_1}{T_1}, \dots, \frac{-t_d}{T_d}\Big) e^{2\pi \imath x \cdot t} \, dt = \Big(\prod_{j=1}^d T_j\Big) \hat{\theta}(T_1 x_1, \dots, T_d x_d),$$

$(x \in \mathbb{R}^d)$. Thus the θ-means can rewritten as

$$\sigma_T^\theta f(x) = \Big(\prod_{j=1}^d T_j\Big) \int_{\mathbb{R}^d} f(x-t) \hat{\theta}(T_1 t_1, \dots, T_d t_d) \, dt \tag{5.1}$$

which is the analogue to Theorem 4.2. Note that $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ implies $\theta \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty)$. Now we formulate Theorem 4.1 for Fourier transforms.

Theorem 5.1. *If $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ and $\theta(0) = 1$ then for all $f \in L_2(\mathbb{R}^d)$*

$$\lim_{T \to \infty} \sigma_{\mathbf{T}}^\theta f = f \qquad \text{in } L_2(\mathbb{R}^d) \text{ norm.}$$

Since σ_T^θ is defined only for $f \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant 2)$, instead of Theorem 4.3 we have

Theorem 5.2. *If $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ and $\theta(0) = 1$ then the following conditions are equivalent:*

(i) $\hat{\theta} \in L_1(\mathbb{R}^d)$,

(ii) $\sigma_{\mathbf{T}}^\theta f \to f$ *in* $L_1(\mathbb{R}^d)$ *norm for all* $f \in L_1(\mathbb{R}^d)$ *as* $T \to \infty$.

If θ has compact support then $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ is also an equivalent condition.

If $\hat{\theta} \in L_1(\mathbb{R}^d)$, the definition of the θ-*means* extends to $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$ by

$$\sigma_T^\theta f := f * K_T^\theta \qquad (T \in \mathbb{R}_+^d),$$

where $*$ denotes the convolution. Note that $\theta \in L_1(\mathbb{R}^d)$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ imply $\theta \in C_0(\mathbb{R}^d)$.

The analogue of Theorem 4.4 follows in the same way:

Theorem 5.3. *Assume that B is a homogeneous Banach space on $\mathbb{R}^d$. If $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ (e.g. $\theta \in \mathbf{S}_0(\mathbb{R}^d)$) then for all $f \in B$*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f = f \qquad \text{in } B \text{ norm.}$$

Since the space $C_u(\mathbb{R}^d)$ of uniformly continuous bounded functions endowed with the supremum norm is also a homogeneous Banach space, we have

Corollary 5.4. *If f is a uniformly continuous and bounded function, $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f = f \qquad \text{uniformly.}$$

6. Sufficient conditions

In this section we give some sufficient conditions for a function θ, which ensures that $\hat{\theta} \in L_1(\mathbb{R}^d)$, resp. $\theta \in \mathbf{S}_0(\mathbb{R}^d)$. As mentioned before $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ implies also that $\hat{\theta} \in L_1(\mathbb{R}^d)$. Recall that $\mathbf{S}_0(\mathbb{R}^d)$ contains all Schwartz functions. If $\theta \in L_1(\mathbb{R}^d)$ and $\hat{\theta}$ has compact support or if $\theta \in L_1(\mathbb{R}^d)$ has compact support and $\hat{\theta} \in L_1(\mathbb{R}^d)$ then $\theta \in \mathbf{S}_0(\mathbb{R}^d)$.

Sufficient conditions can be given with the help of Sobolev, fractional Sobolev and Besov spaces, too. For a detailed description of these spaces see Triebel [31], Runst and Sickel [21], Stein [26] and Grafakos [11].

A function $\theta \in L_p(\mathbb{R}^d)$ is in the *Sobolev space* $W_p^k(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty, k \in \mathbb{N})$ if $D^\alpha \theta \in L_p(\mathbb{R}^d)$ for all $|\alpha| \leqslant k$ and

$$\|\theta\|_{W_p^k} := \sum_{|\alpha|\leqslant k} \|D^\alpha \theta\|_p < \infty,$$

where D denotes the distributional derivative.

This definition is extended to every real s in the following way. The *fractional Sobolev space* $\mathcal{L}_p^s(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty, s \in \mathbb{R})$ consists of all tempered distribution θ for which

$$\|\theta\|_{\mathcal{L}_p^s} := \|\mathcal{F}^{-1}((1 + |\cdot|^2)^{s/2}\hat{\theta})\|_p < \infty.$$

It is known that $\mathcal{L}_p^s(\mathbb{R}^d) = W_p^k(\mathbb{R}^d)$ if $s = k \in \mathbb{N}$ and $1 < p < \infty$ with equivalent norms.

In order to define the Besov spaces take a non-negative Schwartz function $\psi \in \mathcal{S}(\mathbb{R})$ with support $[1/2, 2]$ which satisfies $\sum_{k=-\infty}^{\infty} \psi(2^{-k}s) = 1$ for all $s \in \mathbb{R} \setminus \{0\}$. For $x \in \mathbb{R}^d$ let

$$\phi_k(x) := \psi(2^{-k}|x|) \quad \text{for} \quad k \geqslant 1 \quad \text{and} \quad \phi_0(x) = 1 - \sum_{k=1}^{\infty} \phi_k(x).$$

The *Besov space* $B_{p,r}^s(\mathbb{R}^d)$ $(0 < p, r \leqslant \infty, s \in \mathbb{R})$ is the space of all tempered distributions f for which

$$\|f\|_{B_{p,r}^s} := \Big(\sum_{k=0}^{\infty} 2^{ksr} \|(\mathcal{F}^{-1}\phi_k) * f\|_p^r \Big)^{1/r} < \infty.$$

The Sobolev, fractional Sobolev and Besov spaces are all quasi Banach spaces and if $1 \leqslant p, r \leqslant \infty$ then they are Banach spaces. All these spaces contain the Schwartz functions. The following facts are known: in case $1 \leqslant p, r \leqslant \infty$ one has

$$W_p^m(\mathbb{R}^d), B_{p,r}^s(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \qquad \text{if} \quad s > 0, m \in \mathbb{N},$$

$$W_p^{m+1}(\mathbb{R}^d) \hookrightarrow B_{p,r}^s(\mathbb{R}^d) \hookrightarrow W_p^m(\mathbb{R}^d) \qquad \text{if} \quad m < s < m+1,$$

$$B_{p,r}^s(\mathbb{R}^d) \hookrightarrow B_{p,r+\epsilon}^s(\mathbb{R}^d), B_{p,\infty}^{s+\epsilon}(\mathbb{R}^d) \hookrightarrow B_{p,r}^s(\mathbb{R}^d) \qquad \text{if} \quad \epsilon > 0,$$

$$B_{p_1,1}^{d/p_1}(\mathbb{R}^d) \hookrightarrow B_{p_2,1}^{d/p_2}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \qquad \text{if} \quad 1 \leqslant p_1 \leqslant p_2 < \infty.$$

Theorem 6.1.

(i) If $1 \leqslant p \leqslant 2$ and $\theta \in B_{p,1}^{d/p}(\mathbb{R}^d)$ then $\hat{\theta} \in L_1(\mathbb{R}^d)$ and

$$\|\hat{\theta}\|_1 \leqslant C \|\theta\|_{B_{p,1}^{d/p}}.$$

(ii) If $s > d$ then $\mathcal{L}_1^s(\mathbb{R}^d) \hookrightarrow \mathbf{S}_0(\mathbb{R}^d)$.

(iii) If d' denotes the smallest even integer which is larger than d and $s > d'$ then

$$B_{1,\infty}^s(\mathbb{R}^d) \hookrightarrow W_1^{d'}(\mathbb{R}^d) \hookrightarrow \mathbf{S}_0(\mathbb{R}^d).$$

The embedding $W_1^2(\mathbb{R}) \hookrightarrow \mathbf{S}_0(\mathbb{R})$ follows from (iii). With the help of the usual derivative we give another useful sufficient condition for a function to be in $\mathbf{S}_0(\mathbb{R}^d)$.

A function θ is in $V_1^k(\mathbb{R})$ $(k \geqslant 2, k \in \mathbb{N})$, if there are numbers $-\infty = a_0 < a_1 < \ldots < a_n < a_{n+1} = \infty$ such that $n = n(\theta)$ is depending on θ and

$$\theta \in C^{k-2}(\mathbb{R}), \quad \theta \in C^k(a_i, a_{i+1}), \quad \theta^{(j)} \in L_1(\mathbb{R})$$

for all $i = 0, \ldots, n$ and $j = 0, \ldots, k$. Here C^k denotes the set of k-times continuously differentiable functions. The norm of this space is introduced by

$$\|\theta\|_{V_1^k} := \sum_{j=0}^{k} \|\theta^{(j)}\|_1 + \sum_{i=1}^{n} |\theta^{(k-1)}(a_i + 0) - \theta^{(k-1)}(a_i - 0)|$$

where $\theta^{(k-1)}(a_i \pm 0)$ denote the right and left limits of $\theta^{(k-1)}$. These limits do exist and are finite because $\theta^{(k)} \in C(a_i, a_{i+1}) \cap L_1(\mathbb{R})$ implies

$$\theta^{(k-1)}(x) = \theta^{(k-1)}(a) + \int_a^x \theta^{(k)}(t)\, dt$$

for some $a \in (a_i, a_{i+1})$. Since $\theta^{(k-1)} \in L_1(\mathbb{R})$ we establish that $\lim_{-\infty} \theta^{(k-1)} = \lim_{\infty} \theta^{(k-1)} = 0$. Similarly, $\theta^{(j)} \in C_0(\mathbb{R})$ for $j = 0, \ldots, k-2$.

Of course, $W_1^2(\mathbb{R})$ and $V_1^2(\mathbb{R})$ are not identical. For $\theta \in V_1^2(\mathbb{R})$ we have $\theta' = D\theta$, however, $\theta'' = D^2\theta$ only if $\lim_{a_i+0} \theta' = \lim_{a_i-0} \theta'$ $(i = 1, \ldots, n)$.

We generalize the previous definition for the d-dimensional case as follows. For $d > 1$ and $k \geqslant 2$ let $\theta \in V_1^k(\mathbb{R}^d)$ if θ is even in each variable and

$$\theta \in C^{k-2}(\mathbb{R}^d), \quad \theta \in C^k([0,\infty)^d \setminus \{(0,\ldots,0)\}), \quad \partial_1^{i_1} \cdots \partial_d^{i_d} \theta(t) \in L_1([0,\infty)^l)$$

for each $i_j = 0, \ldots, k$ $(j = 1, \ldots, d)$ and fixed $0 < t_{m_1}, \ldots, t_{m_{d-l}} < \infty$ $(1 \leqslant m_1 < m_2 < \cdots < m_{d-l} \leqslant d)$ and $1 \leqslant l \leqslant d$.

Theorem 6.2. *If* $\theta \in V_1^2(\mathbb{R}^d)$ *then* $\theta \in \mathbf{S}_0(\mathbb{R}^d)$.

The next Corollary follows from the definition of $\mathbf{S}_0(\mathbb{R}^d)$.

Corollary 6.3. *If each* $\theta_j \in V_1^2(\mathbb{R})$ $(j = 1, \ldots, d)$ *then* $\theta := \prod_{j=1}^d \theta_j \in \mathbf{S}_0(\mathbb{R}^d)$.

7. A.e. convergence of the θ-means of Fourier transforms

For the a.e. convergence we will investigate first Fourier transforms rather than Fourier series, because the theorems for Fourier transforms are more complicated. The proofs of the results can be found in Feichtinger and Weisz [4].

$L_p^{loc}(\mathbb{X}^d)$ $(1 \leqslant p \leqslant \infty)$ denotes the space of measurable functions f for which $|f|^p$ is locally integrable, resp. f is locally bounded if $p = \infty$. For $1 \leqslant p \leqslant \infty$ and $f \in L_p^{loc}(\mathbb{X}^d)$ let us define a generalization of the *Hardy-Littlewood maximal function* by

$$M_p f(x) := \sup_{x \in I} \Big(\frac{1}{|I|} \int_I |f|^p \, d\lambda \Big)^{1/p} \qquad (x \in \mathbb{X}^d)$$

with the usual modification for $p = \infty$, where the supremum is taken over all cubes with sides parallel to the axes. If $p = 1$, this is the usual Hardy-Littlewood maximal function. The following inequalities follow easily from the case $p = 1$, which can be found in Stein [27] or Weisz [37]:

$$\|M_p f\|_{L_{p,\infty}} \leqslant C_p \|f\|_p \qquad (f \in L_p(\mathbb{X}^d)) \tag{7.1}$$

and

$$\|M_p f\|_r \leqslant C_r \|f\|_r \qquad (f \in L_r(\mathbb{X}^d), p < r \leqslant \infty). \tag{7.2}$$

The first inequality holds also if $p = \infty$.

The space $\dot{E}_q(\mathbb{R}^d)$ contains all functions $f \in L_q^{loc}(\mathbb{R}^d)$ for which

$$\|f\|_{\dot{E}_q} := \sum_{k=-\infty}^{\infty} 2^{kd(1-1/q)} \|f \mathbf{1}_{P_k}\|_q < \infty,$$

where $P_k := \{2^{k-1} \leqslant |x| < 2^k\}$, $(k \in \mathbb{Z})$. These spaces are special cases of the Herz spaces [15] (see also Garcia-Cuerva and Herrero [9]). The non-homogeneous version of the space $\dot{E}_q(\mathbb{R}^d)$ was used by Feichtinger [8] to prove some Tauberian theorems. It is easy to see that

$$L_1(\mathbb{R}^d) = \dot{E}_1(\mathbb{R}^d) \hookleftarrow \dot{E}_q(\mathbb{R}^d) \hookleftarrow \dot{E}_{q'}(\mathbb{R}^d) \hookleftarrow \dot{E}_\infty(\mathbb{R}^d), \qquad 1 < q < q' < \infty.$$

To prove pointwise convergence of the θ-means we will investigate the *maximal operator*

$$\sigma_\square^\theta f := \sup_{T>0} |\sigma_{\mathbf{T}}^\theta f|.$$

If $\hat{\theta} \in L_1(\mathbb{R}^d)$ then (5.1) implies

$$\|\sigma_\square^\theta f\|_\infty \leqslant \|\hat{\theta}\|_1 \|f\|_\infty \qquad (f \in L_\infty(\mathbb{R}^d)).$$

In the one-dimensional case Torchinsky [30] proved that if there exists an even function η such that η is non-increasing on $\mathbb{R}_+$, $|\hat{\theta}| \leqslant \eta$, $\eta \in L_1$ then $\sigma_\square^\theta$ is of weak type $(1,1)$ and a.e. convergence holds. Under similar conditions we will generalize this result for the multi-dimensional setting. First we introduce an equivalent condition.

Theorem 7.1. *For $\theta \in L_1(\mathbb{R}^d)$ let $\eta(x) := \sup_{\|t\|_r \geqslant \|x\|_r} |\hat{\theta}(t)|$ for some $1 \leqslant r \leqslant \infty$. Then $\hat{\theta} \in \dot{E}_\infty(\mathbb{R}^d)$ if and only if $\eta \in L_1(\mathbb{R}^d)$ and*

$$C^{-1}\|\eta\|_1 \leqslant \|\hat{\theta}\|_{\dot{E}_\infty} \leqslant C\|\eta\|_1.$$

Theorem 7.2. *Let $\theta \in L_1(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$ and $1/p + 1/q = 1$. If $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$ then*

$$\|\sigma_\square^\theta f\|_{L_{p,\infty}} \leqslant C_p \|\hat{\theta}\|_{\dot{E}_q} \|f\|_p$$

for all $f \in L_p(\mathbb{R}^d)$. Moreover, for every $p < r \leqslant \infty$,

$$\|\sigma_\square^\theta f\|_r \leqslant C_r \|\hat{\theta}\|_{\dot{E}_q} \|f\|_r \qquad (f \in L_r(\mathbb{R}^d)).$$

The proof of this theorem follows from the pointwise inequality

$$\sigma_\square^\theta f(x) \leqslant C \|\hat{\theta}\|_{\dot{E}_q} M_p f(x) \tag{7.3}$$

and from (7.1) and (7.2). Inequality (7.3) is proved in Feichtinger and Weisz [4].

Theorem 7.2 and the usual density argument due to Marcinkiewicz and Zygmund [17] imply

Corollary 7.3. *If* $\theta \in L_1(\mathbb{R}^d)$, $\theta(0)=1$, $1 \leqslant p \leqslant \infty$, $1/p+1/q=1$ *and* $\hat\theta \in \dot E_q(\mathbb{R}^d)$ *then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^{\theta} f = f \qquad a.e.$$

if $f \in L_r(\mathbb{R}^d)$ *for* $p \leqslant r < \infty$ *or* $f \in C_0(\mathbb{R}^d)$.

Note that $\dot E_q(\mathbb{R}^d) \supset \dot E_{q'}(\mathbb{R}^d)$ whenever $q < q'$. If $\hat\theta$ is in a smaller space (say in $\dot E_\infty(\mathbb{R}^d)$) then we get convergence for a wider class of functions (namely for $f \in L_r(\mathbb{R}^d)$, $1 \leqslant r \leqslant \infty$).

In order to generalize the last theorem and corollary for the larger space $W(L_1,\ell_\infty)(\mathbb{R}^d)$, we have to define the *local Hardy-Littlewood maximal function* by

$$m_p f(x) := \sup_{0<r\leqslant 1} \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f|^p \, d\lambda \Big)^{1/p} \qquad (x \in \mathbb{R}^d),$$

where $f \in L_p^{loc}(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$ and $B(x,r)$ denotes the ball with center x and radius r. It is easy to see that inequalities (7.1) and (7.2) imply

$$\|m_p f\|_{W(L_{p,\infty},\ell_s)} \leqslant C_p \|f\|_{W(L_p,\ell_s)} \qquad (f \in W(L_p,\ell_s)(\mathbb{R}^d)) \tag{7.4}$$

and

$$\|m_p f\|_{W(L_r,\ell_s)} \leqslant C_r \|f\|_{W(L_r,\ell_s)} \qquad (f \in W(L_r,\ell_s)(\mathbb{R}^d)) \tag{7.5}$$

for all $p < r \leqslant \infty$ and $1 \leqslant s \leqslant \infty$. Recall that

$$\|f\|_{W(L_{p,\infty},\ell_\infty)} = \sup_{k\in\mathbb{Z}^d} \sup_{\rho>0} \rho\, \lambda(|f|>\rho, [k,k+1))^{1/p}.$$

Theorem 7.4. *Let* $\theta \in L_1(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$ *and* $1/p+1/q=1$. *If* $\hat\theta \in \dot E_q(\mathbb{R}^d)$ *then*

$$\|\sigma_{\square}^{\theta} f\|_{W(L_{p,\infty},\ell_\infty)} \leqslant C_p \|\hat\theta\|_{\dot E_q} \|f\|_{W(L_p,\ell_\infty)}$$

for all $f \in W(L_p,\ell_\infty)(\mathbb{R}^d)$. *Moreover, for every* $p < r \leqslant \infty$,

$$\|\sigma_{\square}^{\theta} f\|_{W(L_r,\ell_\infty)} \leqslant C_r \|\hat\theta\|_{\dot E_q} \|f\|_{W(L_r,\ell_\infty)} \qquad (f \in W(L_r,\ell_\infty)(\mathbb{R}^d)).$$

It is easy to see that

$$M_p f \leqslant C m_p f + C_p \|f\|_{W(L_p,\ell_\infty)} \qquad (1 \leqslant p \leqslant \infty).$$

The proof of Theorem 7.4 follows from (7.3)–(7.5).

Corollary 7.5. *If* $\theta \in L_1(\mathbb{R}^d)$, $\theta(0)=1$, $1 \leqslant p < \infty$, $1/p+1/q=1$ *and* $\hat\theta \in \dot E_q(\mathbb{R}^d)$ *then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^{\theta} f = f \quad a.e.$$

if $f \in W(L_p,c_0)(\mathbb{R}^d)$.

Note that $W(L_p, c_0)(\mathbb{R}^d)$ contains all $W(L_r, c_0)(\mathbb{R}^d)$ spaces for $p \leqslant r \leqslant \infty$.

We can characterize the set of convergence in the following way. Lebesgue differentiation theorem says that

$$\lim_{h\to 0} \frac{1}{|B(0,h)|} \int_{B(0,h)} f(x+u)\, du = f(x)$$

for a.e. $x \in \mathbb{X}^d$, where $f \in L_1^{loc}(\mathbb{X}^d)$, $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = \mathbb{R}$. A point $x \in \mathbb{X}^d$ is called a *p-Lebesgue point* (or a Lebesgue point of order p) of $f \in L_p^{loc}(\mathbb{X}^d)$ if

$$\lim_{h\to 0} \Big(\frac{1}{|B(0,h)|} \int_{B(0,h)} |f(x+u) - f(x)|^p \, du \Big)^{1/p} = 0 \qquad (1 \leqslant p < \infty)$$

resp.

$$\lim_{h\to 0} \sup_{u\in B(0,h)} |f(x+u) - f(x)| = 0 \qquad (p = \infty).$$

Usually the 1-Lebesgue points, called simply Lebesgue points are considered (cf. Stein and Weiss [25] or Butzer and Nessel [3]). One can show that almost every point $x \in \mathbb{X}^d$ is a p-Lebesgue point of $f \in L_p^{loc}(\mathbb{X}^d)$ if $1 \leqslant p < \infty$, which means that almost every point $x \in \mathbb{R}^d$ is a p-Lebesgue point of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$. $x \in \mathbb{X}^d$ is an ∞-Lebesgue point of $f \in L_\infty^{loc}(\mathbb{X}^d)$ if and only if f is continuous at x. Moreover, all r-Lebesgue points are p-Lebesgue points, whenever $p < r$.

Stein and Weiss [25, p. 13] (see also Butzer and Nessel [3, pp. 132-134]) proved that if $\eta(x) := \sup_{|t|\geqslant|x|} |\hat{\theta}(t)|$ and $\eta \in L_1(\mathbb{R}^d)$ then one has convergence at each Lebesgue point of $f \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty)$. Using the $\dot{E}_q$ spaces we generalize this result.

Theorem 7.6. *Let $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$, $1 \leqslant p \leqslant \infty$ and $1/p + 1/q = 1$. If $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$ then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^{\theta} f(x) = f(x)$$

for all p-Lebesgue points of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$.

Note that $W(L_1, \ell_\infty)(\mathbb{R}^d)$ contains all $L_p(\mathbb{R}^d)$ spaces and amalgam spaces $W(L_p, \ell_q)(\mathbb{R}^d)$ for the full range $1 \leqslant p, q \leqslant \infty$.

If f is continuous at a point x then x is a p-Lebesgue point of f for every $1 \leqslant p \leqslant \infty$.

Corollary 7.7. *Let $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$, $1 \leqslant p \leqslant \infty$ and $1/p + 1/q = 1$. If $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$ and $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ is continuous at a point x then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^{\theta} f(x) = f(x).$$

Recall that $\dot{E}_1(\mathbb{R}^d) = L_1(\mathbb{R}^d)$ and $W(L_\infty, \ell_\infty)(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. If f is uniformly continuous then we have uniform convergence (see Corollary 5.4).

Let us consider converse-type problems. The partial converse of Theorem 7.2 is given in the next result.

Theorem 7.8. *Let $\theta \in L_1(\mathbb{R}^d)$, $\hat{\theta} \in L_1(\mathbb{R}^d)$, $1 \leqslant p < \infty$ and $1/p + 1/q = 1$. If*

$$\sigma_{\square}^{\theta} f(x) \leqslant C M_p f(x) \tag{7.6}$$

for all $f \in L_p(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ then $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$.

The converse of Theorem 7.6 reads as follows.

Theorem 7.9. *Suppose that $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$, $\hat{\theta} \in L_1(\mathbb{R}^d)$, $1 \leqslant p < \infty$ and $1/p + 1/q = 1$. If*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^{\theta} f(x) = f(x) \tag{7.7}$$

for all p-Lebesgue points of $f \in L_p(\mathbb{R}^d)$ then $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$.

Corollary 7.10. *Suppose that $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$, $\hat{\theta} \in L_1(\mathbb{R}^d)$, $1 \leqslant p < \infty$ and $1/p + 1/q = 1$. Then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^{\theta} f(x) = f(x)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{R}^d)$ (resp. of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$) if and only if $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$.

If we take the supremum in the maximal θ-operator over a cone, say over $\{T \in \mathbb{R}_+^d : 2^{-\tau} \leqslant T_i/T_j \leqslant 2^{\tau}; i, j = 1, \ldots, d\}$ for some fixed $\tau \geqslant 0$:

$$\sigma_c^{\theta} f := \sup_{\substack{2^{-\tau} \leqslant T_i/T_j \leqslant 2^{\tau} \\ i,j=1,\ldots,d}} |\sigma_T^{\theta} f|,$$

then all the results above can be shown for σ_c^{θ}. In this case, under the conditions above we obtain the convergence $\sigma_T^{\theta} f \to f$ a.e. as $T \to \infty$ and $2^{-\tau} \leqslant T_i/T_j \leqslant 2^{\tau}$ $(i, j = 1, \ldots, d)$. This convergence has been investigated in a great number of papers (e.g. in Marcinkiewicz and Zygmund [17], Zygmund [39], Weisz [34, 36, 37]). For more details see Feichtinger and Weisz [4]. The unrestricted convergence of $\sigma_T^{\theta} f$, i.e. as $T_j \to \infty$ for each $j = 1, \ldots, d$, is also investigated in that paper.

8. A.e. convergence of the θ-means of Fourier series

All the results of Section 7 holds also for Fourier series. In this case we define the *maximal operator* of the θ-means by

$$\sigma_{\square}^{\theta} f := \sup_{n \in \mathbb{N}} |\sigma_{\mathbf{n}}^{\theta} f|.$$

Similarly to Theorem 7.4 we have

Theorem 8.1. *Let $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$ and $1/p+1/q=1$. If $\hat\theta \in \dot E_q(\mathbb{R}^d)$ then*

$$\|\sigma_\square^\theta f\|_{L_{p,\infty}} \leqslant C_p \|\hat\theta\|_{\dot E_q} \|f\|_p$$

for all $f \in L_p(\mathbb{T}^d)$. Moreover, for every $p < r \leqslant \infty$,

$$\|\sigma_\square^\theta f\|_r \leqslant C_r \|\hat\theta\|_{\dot E_q} \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

The analogue of Theorems 7.6, 7.9 and Corollary 7.10 reads as follows.

Theorem 8.2. *Let $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0)=1$, $\hat\theta \in L_1(\mathbb{R}^d)$, $1 \leqslant p < \infty$ and $1/p+1/q=1$. Then*

$$\lim_{n\to\infty} \sigma_{\mathbf{n}}^\theta f(x) = f(x)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{T}^d)$ if and only if $\hat\theta \in \dot E_q(\mathbb{R}^d)$.

Corollary 8.3. *Let $\theta \in L_1(\mathbb{R}^d)$, $\theta(0)=1$, $1 \leqslant p \leqslant \infty$ and $1/p+1/q=1$. If $\hat\theta \in \dot E_q(\mathbb{R}^d)$ and $f \in L_p(\mathbb{T}^d)$ is continuous at a point $x \in \mathbb{T}^d$ then*

$$\lim_{n\to\infty} \sigma_{\mathbf{n}}^\theta f(x) = f(x).$$

9. Besov, modulation and Sobolev spaces

The next theorem was proved in Herz [15], Peetre [19] and Girardi and Weis [10].

Theorem 9.1. *If $1 \leqslant p \leqslant 2$, $1/p+1/q=1$ and $\theta \in B_{p,1}^{d/p}(\mathbb{R}^d)$ then $\hat\theta \in \dot E_q(\mathbb{R}^d)$ and*

$$\|\hat\theta\|_{\dot E_q} \leqslant C_p \|\theta\|_{B_{p,1}^{d/p}}.$$

Theorem 9.1 implies the following result.

Corollary 9.2. *If $\theta \in L_1(\mathbb{R}^d) \cap B_{p,1}^{d/p}(\mathbb{R}^d)$ for some $1 \leqslant p \leqslant 2$ and $\theta(0)=1$ then Theorems 7.2, 7.4, 7.6 and Corollaries 7.3, 7.5 and 7.7 hold.*

For the connection between the $\dot E_q(\mathbb{R}^d)$ and amalgam spaces we have proved the following result.

Theorem 9.3. *If $f \in W(L_q, \ell_1^{v_{d/p}})(\mathbb{R}^d)$ for some $1 \leqslant q \leqslant \infty$, $1/p+1/q=1$ then $f \in \dot E_q(\mathbb{R}^d)$ and*

$$\|f\|_{\dot E_q} \leqslant C_q \|f\|_{W(L_q,\ell_1^{v_{d/p}})}.$$

Corollary 9.4. *Let $\theta \in L_1(\mathbb{R}^d)$, $\theta(0)=1$, $1 \leqslant p \leqslant \infty$ and $1/p+1/q=1$. If $\hat\theta \in W(L_q, \ell_1^{v_{d/p}})(\mathbb{R}^d)$ then Theorems 7.2, 7.4, 7.6 and Corollaries 7.3, 7.5 and 7.7 hold.*

In particular, if $q = \infty$ then we get the condition $\hat{\theta} \in W(C, \ell_1^{v_d})(\mathbb{R}^d)$. Note that if $\theta \in L_1(\mathbb{R}^d)$ and $\hat{\theta}$ has compact support then Corollary 9.4 holds. Actually $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ in this case (see e.g. Feichtinger and Zimmerman [6]).

If θ is in a suitable modulation space then the θ-means converge a.e. to f. Indeed, using Theorem 9.3 we can show that $\hat{\theta} \in W(L_\infty, \ell_1^{v_d})(\mathbb{R}^d) \subset \dot{E}_\infty(\mathbb{R}^d)$ if $\theta \in M_1^{v_d}(\mathbb{R}^d)$ and

$$C^{-1}\|\hat{\theta}\|_{\dot{E}_\infty} \leqslant \|\hat{\theta}\|_{W(L_\infty, \ell_1^{v_d})} \leqslant C\|\theta\|_{M_1^{v_d}}.$$

Theorem 9.5. *If* $\theta \in M_1^{v_d}(\mathbb{R}^d)$ *and* $\theta(0) = 1$ *then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. *Moreover,*

$$\sup_{\rho>0} \rho\, \lambda(\sigma_\square^\theta > \rho) \leqslant C\|\theta\|_{M_1^{v_d}}\|f\|_1 \qquad (f \in L_1(\mathbb{R}^d)),$$

$$\|\sigma_\square^\theta f\|_{W(L_{1,\infty}, \ell_\infty)} \leqslant C\|\theta\|_{M_1^{v_d}}\|f\|_{W(L_1,\ell_\infty)} \qquad (f \in W(L_1, \ell_\infty)(\mathbb{R}^d))$$

and, for every $1 < p \leqslant \infty$,

$$\|\sigma_\square^\theta f\|_p \leqslant C_p\|\theta\|_{M_1^{v_d}}\|f\|_p \qquad (f \in L_p(\mathbb{R}^d)),$$

$$\|\sigma_\square^\theta f\|_{W(L_p,\ell_\infty)} \leqslant C_p\|\theta\|_{M_1^{v_d}}\|f\|_{W(L_p,\ell_\infty)} \qquad (f \in W(L_p, \ell_\infty)(\mathbb{R}^d)).$$

Similarly to Theorem 6.2 we give sufficient conditions for θ to be in the modulation space $\theta \in M_1^{v_d}(\mathbb{R}^d)$.

Theorem 9.6. *If* $\theta \in V_1^k(\mathbb{R}^d)$ $(k \geqslant 2)$ *then* $\theta \in M_1^{v_s}(\mathbb{R}^d)$ *for all* $0 \leqslant s < k-1$.

Corollary 9.7. *If each* $\theta_j \in V_1^k(\mathbb{R})$ $(k \geqslant 2, j = 1, \ldots, d)$ *then* $\theta := \prod_{j=1}^d \theta_j \in M_1^{v_s}(\mathbb{R}^d)$ *for all* $0 \leqslant s < k-1$.

$V_1^2(\mathbb{R})$ is not contained in $M_1^{v_1}(\mathbb{R})$, however, the same results hold as in Theorem 9.5.

Corollary 9.8. *If* $\theta \in V_1^2(\mathbb{R})$ *then* $\hat{\theta} \in \dot{E}_\infty(\mathbb{R})$ *and Theorem 9.5 hold.*

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