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Note on symmetric alteration of knots of B-spline curves

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Abstract

The shape of a B-spline curve can be influenced by the modification of knot values. Previously the effect caused by symmetric alteration of two knots have been studied on the intervals between the altered knots. Here we show how symmetric knot alteration influences the shape of the B-spline curve over the rest of the domain of definition in the case $k = 3$.

Key Words: B-spline curve, knot modification, path

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1. Introduction

The properties and capabilities of B-spline curves make them widely used in computer aided geometric design. B-spline curves are polynomial curves defined as linear combination of the control points by some polynomial functions called basis functions. These basis functions are defined in a piecewise way over a closed interval and the subdivision values of this interval are called knots. The basic definitions of the basis functions and the curve are the following.

Definition 1.1. The recursive function $N_j^k(u)$ given by the equations

$$N_j^1(u) = \begin{cases} 1 & \text{if } u \in [u_j, u_{j+1}), \\ 0 & \text{otherwise} \end{cases}$$
$$N_j^k(u) = \frac{u-u_j}{u_{j+k-1}-u_j} N_j^{k-1}(u) + \frac{u_{j+k}-u}{u_{j+k}-u_{j+1}} N_{j+1}^{k-1}(u)$$

is called normalized B-spline basis function of order k (degree $k-1$). The numbers $u_j \leqslant u_{j+1} \in \mathbf{R}$ are called knot values or simply knots, and $0/0 \doteq 0$ by definition.

Definition 1.2. The curve $\mathbf{s}(u)$ defined by

$$\mathbf{s}(u) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u), \quad u \in [u_{k-1}, u_{n+1}]$$

is called B-spline curve of order $k \leqslant n$ (degree $k-1$), where $N_l^k(u)$ is the l^{th} normalized B-spline basis function, for the evaluation of which the knots $u_0, u_1, \ldots, u_{n+k}$ are necessary. The points $\mathbf{d}_i$ are called control points or de Boor-points, while the polygon formed by these points is called control polygon.

The j^{th} arc can be written as

$$\mathbf{s}_j(u) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(u), \quad u \in [u_j, u_{j+1}).$$

The data structure of these polynomial curves include their order, control points and knot values. Obviously, any modification of these data has some effect on the shape of the curve. In case of control point repositioning the effect is widely studied (c.f. [2], [9] or [10]). The modification of any knot value influences the given curve as well. Applications of knot modifications in computer aided geometric design can be found in [1], [3], [8].

The question, how the alteration of a single knot effects the shape of the curve was studied in [4],[6],[7]. When a knot u_i is altered, points of the curve move on special curves called paths. In [6] the authors proved that these paths are rational curves. In [4] the paths have been extended allowing $u_i < u_{i-1}$ and $u_i > u_{i+1}$.

Instead of a single knot one can modify two knots at the same time. In [5] a general theorem about the extended path obtained by the symmetric alteration of two knots has been verified. (The definition of symmetric knot alteration can be found in [5].) The following statement has been shown:

Theorem 1.3 (Hoffmann-Juhász). *Symmetrically altering the knots u_i, u_{i+z}, $(z = 1, 2, \ldots, k)$, extended paths of points of the arcs $\mathbf{s}_j$, $(j = i, i+1, \ldots, i+z-1)$ converge to the midpoint of the segment bounded by the control points $\mathbf{d}_i$ and $\mathbf{d}_{i+z-k}$ when $\lambda \to -\infty$, i.e.*

$$\lim_{\lambda \to -\infty} \mathbf{s}(u, \lambda) = \frac{\mathbf{d}_i + \mathbf{d}_{i+z-k}}{2}, \quad u \in [u_i, u_{i+z}). \tag{1.1}$$

As the above theorem shows, the authors have studied the effect of symmetric alteration of two knots on the intervals between the altered knots. As the definition of the basis functions shows, the altered knots has effect on some neighbouring intervals as well.

The purpose of the present paper is to extend the above theorem, and describe the effects of the modification of two knots on the neighbouring intervals.

2. Symmetric alteration of two knots: new results

In Theorem 1.3 the extended paths of points of the $\mathbf{s}_j$ arcs has been analyzed in the case $\lambda \to -\infty$. A closer look on the basis functions appearing in the above arcs shows, that the same statement holds for k=3 in the case $\lambda \to +\infty$ as well. This can be verified by the following observation: Substituting $u_i = u_i + \lambda$ and $u_{i+z} = u_{i+z} - \lambda$ into the basis functions we get that λ has the same sign in the numerator and in the denominator of the rational functions.

In Theorem 1.3 the effect of symmetric alteration of two knots is analyzed between the modified knots. However altered knots modify the curve over some neigbouring intervals as well. In the following we analyze the extended paths over these intervals.

We consider the case $k = 3$. The shape of the curve changes over four further inetrvals $([u_{i-2}, u_{i-1}), [u_{i-1}, u_i), [u_{i+z}, u_{i+z+1}), [u_{i+z+1}, u_{i+z+2}))$. On the interval $[u_{i+1}, u_{i+2})$ the only nonzero basis functions are

$$\begin{aligned} N_{i-1}^3 &= \frac{u_{i+2}-u}{u_{i+2}-u_i}\frac{u_{i+2}-u}{u_{i+2}-u_{i+1}} \\ N_i^3 &= \frac{u-u_i}{u_{i+2}-u_i}\frac{u_{i+2}-u}{u_{i+2}-u_{i+1}} + \frac{u_{i+3}-u}{u_{i+3}-u_{i+1}}\frac{u-u_{i+1}}{u_{i+2}-u_{i+1}} \\ N_{i+1}^3 &= \frac{u-u_{i+1}}{u_{i+3}-u_{i+1}}\frac{u-u_{i+1}}{u_{i+2}-u_{i+1}}. \end{aligned}$$

First we consider the case $z = 1$, i.e. we alter the u_i, u_{i+1} knots. Substituting $u_i = u_i + \lambda$ and $u_{i+1} = u_{i+1} - \lambda$ in the above functions, while the denominators of N_{i-1}^3, N_i^3 are second degree polynomials in λ the numerators are linear. In the case N_{i+1}^3 the numerator as well as the denominator is quadratic polynomial of λ, and the coefficient of λ^2 is equal to 1 both in the numerator and in the denominator. This yields

$$\lim_{\lambda\to\infty} N_{i-1}^3 = 0,\ \lim_{\lambda\to\infty} N_i^3 = 0,\ \lim_{\lambda\to\infty} N_{i+1}^3 = 1, \qquad u \in [u_{i+1}, u_{i+2}).$$

The geometric meaning of the above result is that the extended paths of arc s_{i+1} converge to the control point $\mathbf{d}_{i+1}$ when $\lambda \to \infty$. Similar calculation proofs the same statement for the case $z = 2, 3$. This yields the following lemma.

Lemma 2.1. *In the case $k = 3$ symmetrically altering the knots u_i and u_{i+z}, $(z = 1, 2, 3)$, extended paths of points of the arcs $\mathbf{s}_{i+z}, \mathbf{s}_{i-1}$ converge to the control points $\mathbf{d}_{i+z}$ and $\mathbf{d}_{i-1}$ respectively, when $\lambda \to \infty$, i.e.*

$$\lim_{\lambda\to\infty} \mathbf{s}(u, \lambda) = \mathbf{d}_{i+z}, \quad u \in [u_{i+z}, u_{i+z+1}). \tag{2.1}$$

$$\lim_{\lambda\to\infty} \mathbf{s}(u, \lambda) = \mathbf{d}_{i-1}, \quad u \in [u_{i-1}, u_i). \tag{2.2}$$

The symmetric modification of knots u_i and u_{i+z}, $(z = 1, 2, 3)$ effects the shape of the arcs s_{i+z+1}, s_{i-2} as well. On the interval $[u_{i+2}, u_{i+3})$ the only nonzero basis functions are

$$\begin{aligned} N_i^3 &= \frac{u_{i+3} - u}{u_{i+3} - u_{i+1}} \frac{u_{i+3} - u}{u_{i+3} - u_{i+2}} \\ N_{i+1}^3 &= \frac{u - u_{i+1}}{u_{i+3} - u_{i+1}} \frac{u_{i+3} - u}{u_{i+3} - u_{i+2}} + \frac{u_{i+4} - u}{u_{i+4} - u_{i+2}} \frac{u - u_{i+2}}{u_{i+3} - u_{i+2}} \\ N_{i+2}^3 &= \frac{u - u_{i+2}}{u_{i+4} - u_{i+2}} \frac{u - u_{i+2}}{u_{i+3} - u_{i+2}}. \end{aligned}$$

First we consider the case $z = 1$, i.e. we substitute $u_i = u_i + \lambda$ and $u_{i+1} = u_{i+1} - \lambda$ in the above functions. A short calculation shows that the alteration of the mentioned knots does not effect the basis function N_{i+2}^3. In the case N_i^3 while λ does not appear in the numerator, the denominator is a linear polynomial of λ. N_{i+1}^3 is the sum of two rational functions. While both the numerator and the denominator are linear polynomials of λ in one of the terms, the other one is not effected by the knot alteration.

Consequently, for $u \in [u_{i+2}, u_{i+3})$ the following equalities hold:

$$\begin{aligned} \lim_{\lambda\to\infty} N_i^3 &= 0 \\ \lim_{\lambda\to\infty} N_{i+1}^3 &= 1 + \frac{u_{i+4} - u}{u_{i+4} - u_{i+2}} \frac{u - u_{i+2}}{u_{i+3} - u_{i+2}} \\ \lim_{\lambda\to\infty} N_{i+2}^3 &= N_{i+2}^3. \end{aligned}$$

Similar calculation proofs the same statement for the case $z = 2, 3$. This yields to the following lemma.

Lemma 2.2. *In the case $k = 3$ symmetrically altering the knots u_i and u_{i+z}, $(z = 1, 2, 3)$, extended paths of points of the arcs $\mathbf{s}_{i+z+1}, \mathbf{s}_{i-2}$ converge to*

$$\begin{aligned} \lim_{\lambda\to\infty} \mathbf{s}(u, \lambda) &= \mathbf{d}_{i+z}(1 + \frac{u_{i+z+3} - u}{u_{i+z+3} - u_{i+z+1}} \frac{u - u_{i+z+1}}{u_{i+z+2} - u_{i+z+1}}) + \mathbf{d}_{i+z+1} N_{i+z+1} \\ &= \mathbf{d}_{i+z} + (\mathbf{d}_{i+z}(u_{i+z+3} - u) + \mathbf{d}_{i+z+1}(u - u_{i+z+1}))\cdot \\ &\quad \cdot \frac{u - u_{i+z+1}}{(u_{i+z+3} - u_{i+z+1})(u_{i+z+2} - u_{i+z+1})}, \quad u \in [u_{i+z+1}, u_{i+z+2}). \end{aligned}$$

$$\begin{aligned} \lim_{\lambda\to\infty} \mathbf{s}(u, \lambda) &= \mathbf{d}_{i-4} N_{i-4} + \mathbf{d}_{i-3}(\frac{u - u_{i-3}}{u_{i-1} - u_{i-3}} \frac{u_{i-1} - u}{u_{i-1} - u_{i-2}} + 1) \\ &= \mathbf{d}_{i-3} + (\mathbf{d}_{i-3}(u - u_{i-3}) + d_{i-4}(u_{i-1} - u))\cdot \\ &\quad \cdot \frac{u_{i-1} - u}{(u_{i-1} - u_{i-3})(u_{i-1} - u_{i-2})}, \quad u \in [u_{i-2}, u_{i-1}). \end{aligned}$$

The alteration of knots u_i, u_{i+z} does not change the shape of any further arcs. Therefore, the effect of symmetric alteration of two knots in the case $= 3$ is fully explored on the domain of definition of the B-spline curve.

Obviously the above two lemmas are valid for the case $\lambda \to -\infty$. Here we note, that the limits of the basis functions have been considered by substituting $u_i + \lambda, u_{i+z} - \lambda$ to the computed functions. Since these functions are defined recursively, the limit can be interpreted in a different way as well.

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