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A limit theorem for one-parameter alteration of two knots of B-spline curves*

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Abstract

Knot modification of B-spline curves is extensively studied in the past few years. Altering one knot value, curve points move on well-defined paths, the limit of which can be computed if the knot value tends to infinity. Symmetric alteration of two knot values can also be studied in a similar way. The extension of these limit theorems for general synchronized modification of two knots is discussed in this paper.

Key Words: B-spline curve, knots, paths

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1. Introduction

B-spline curves and surfaces are well-known geometric modeling tools in computer aided design. The definition of the k^{th} order B-spline curve is as follows (c.f.[13]):

Definition 1.1. The curve $\mathbf{s}(u)$ defined by

$$\mathbf{s}(u) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u), \quad u \in [u_{k-1}, u_{n+1}]$$

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is called B-spline curve of order $k \leqslant n$ (degree $k-1$), where the points $\mathbf{d}_l$ are called control points or de Boor-points, while $N_l^k(u)$ is the k^{th} normalized B-spline basis function, given by the following recursive functions:

$$N_j^1(u) = \begin{cases} 1 & \text{if } u \in [u_j, u_{j+1}), \\ 0 & \text{otherwise} \end{cases}$$
$$N_j^k(u) = \frac{u-u_j}{u_{j+k-1}-u_j} N_j^{k-1}(u) + \frac{u_{j+k}-u}{u_{j+k}-u_{j+1}} N_{j+1}^{k-1}(u).$$

The numbers $u_j \leqslant u_{j+1} \in \mathbb{R}$ are called knot values or simply knots, and $0/0 \doteq 0$ by definition.

In the last few years several papers dealt with knot modification of B-spline curves. From a practical point of view, optimization techniques by changing the entire knot vector have been studied in [1] and [3], while shape control algorithms for cubic B-spline curves by changing three consecutive knots have been described in [10].

Basic theoretical results of alteration of a single knot value can be found in [8] and [9], where the notion of path has been introduced for curves $\mathbf{s}(u, u_i)$ obtained by fixing the parameter value u and modifying the knot u_i. In [8] the authors proved that these paths are rational curves. In [5] these paths are extended in a way that monotonicity of knot values was not fulfilled, i.e. we let $u_i < u_{i-1}$ and $u_i > u_{i+1}$. Here we emphasize that this extension is a pure mathematical construction that is, the functions $N_l^k(u)$ obtained by this substitution are not basis functions any more. This extension, however can help us to examine the limit properties of paths. These extended paths have been studied in [5] where the following theorem has been proved:

Theorem 1.2. *Modifying the single multiplicity knot u_i of the B-spline curve $\mathbf{s}(u)$, points of the extended paths of $\mathbf{s}(u), u \in [u_{i-1}, u_{i+1})$ tend to the control points $\mathbf{d}_i$ and $\mathbf{d}_{i-k}$ as u_i tends to $-\infty$ and ∞, respectively, i.e.,*

$$\lim_{u_i \to -\infty} \mathbf{s}(u, u_i) = \mathbf{d}_i, \lim_{u_i \to \infty} \mathbf{s}(u, u_i) = \mathbf{d}_{i-k}, \forall u \in [u_{i-1}, u_{i+1}).$$

Some of the results of knot modification have been successfully extended for B-spline surfaces as well (c.f. [4], [11]).

2. Alteration of two knots

Similarly to the previous section, one can modify two (not necessarily neighboring) knots of $\mathbf{s}(u)$ as well. Let us denote the two altered knots by u_i and u_{i+z}, $((k-1) < i < i+z < (n+1))$. If their modification is independent of each other, the possible positions of each fixed point of the curve can be described as a planar region. However if the modification of the two knots is synchronized in a way that their movement depend on a single parameter, the points of the curve will move on paths. In [6] the modification of type $u_i + \lambda$ and $u_{i+z} - \lambda$ has been discussed and the following result has been proved.

Theorem 2.1. *Symmetrically altering the knots* u_i *and* u_{i+z} $(z \in \{1, 2, \ldots, k\}$, *where* k *is the order of the original B-spline curve), extended paths of points of the arcs* $\mathbf{s}_j$, $(j = i, i+1, \ldots, i+z-1)$ *converge to the midpoint of the segment bounded by the control points* $\mathbf{d}_i$ *and* $\mathbf{d}_{i+z-k}$ *when* $\lambda \to -\infty$, *i.e.*

$$\lim_{\lambda \to -\infty} \mathbf{s}(u, \lambda) = \frac{\mathbf{d}_i + \mathbf{d}_{i+z-k}}{2}, \quad u \in [u_i, u_{i+z}). \tag{2.1}$$

In this paper we extend this result for a more general movement of knots. Let the modification of the two knots be described by the following way:

$$\begin{aligned} u_i &= u_i + t\lambda \\ u_j &= u_j - (1-t)\lambda, \end{aligned}$$

where $t \in [0, 1]$ is fixed and $\lambda \in \mathbb{R}$ is a running parameter. If one intend to preserve the monotonicity of the knot values, only $\lambda \in [-c, c]$, $c = \min\{u_i - u_{i-1}, u_{i+1} - u_i, u_j - u_{j-1}, u_{j+1} - u_j\}$ is allowed, but in case of extended paths the parameter can be any real number.

3. The limit theorem

For the synchronized motion described in the previous section the following statement holds.

Theorem 3.1. *Modifying the knots*

$$u_i = u_i + t\lambda, \quad u_{i+z} = u_{i+z} - (1-t)\lambda, \quad (z = 1, 2, \ldots, k) \tag{3.1}$$

the points of the extended paths of $\mathbf{s}(u)$, $u \in [u_i, u_{i+z})$ *tend to a point of the line segment* $\mathbf{d}_i\mathbf{d}_{i+z-k}$ *the barycentric coordinates of which are* t *and* $(1-t)$, *i.e.*

$$\lim_{\lambda \to -\infty} \mathbf{s}(u, t, \lambda) = t\mathbf{d}_i + (1-t)\mathbf{d}_{i+z-k}, \quad u \in [u_i, u_{i+z}), t \in [0, 1].$$

Proof. At first we prove that if $u \in [u_i, u_{i+z})$, then for $z = 1, 2, ..., k-1$

$$\begin{aligned} \lim_{\lambda \to -\infty} N_{i+z-k}^k(u, t, \lambda) &= (1-t) \\ \lim_{\lambda \to -\infty} N_i^k(u, t, \lambda) &= t \\ \lim_{\lambda \to -\infty} N_j^k(u, t, \lambda) &= 0, \quad (j \neq i, i+z-k), \end{aligned}$$

and for $z = k$

$$\begin{aligned} \lim_{\lambda \to -\infty} N_i^k(u, t, \lambda) &= 1 \\ \lim_{\lambda \to -\infty} N_j^k(u, t, \lambda) &= 0, \quad (j \neq i). \end{aligned}$$

We prove the statement by induction on k. For the sake of simplicity the variables of the basis functions are omitted.

i) $k = 3$

On the interval $[u_i, u_{i+1})$ the basis function is of the following form

$$N_i^3 = \frac{u - u_i}{u_{i+2} - u_i} \frac{u - u_i}{u_{i+1} - u_i}.$$

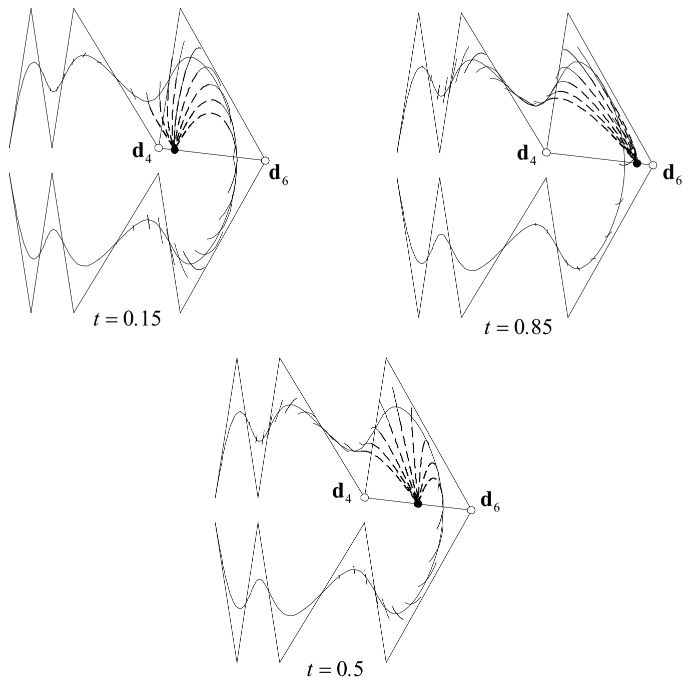

Figure 1: A cubic ($k = 4$) B-spline curve and its paths for various values of t, ($i = 6, z = 2$).

Substituting equations (3.1) into this function the numerator as well as the denominator will be quadratic in λ. The main coefficient of the numerator is t^2 independently of z. For $z = 1$ the main coefficient in the denominator can be calculated by applying

$$\begin{aligned} u_{i+2} - u_i &= u_{i+2} - (u_i + t\lambda) = u_{i+2} - u_i - t\lambda \\ u_{i+1} - u_i &= u_{i+1} - (1-t)\lambda - (u_i + t\lambda) = u_{i+1} - u_i - \lambda \end{aligned}$$

which turn to be t, while for $z = 2$ applying

$$u_{i+2} - u_i = u_{i+2} - (1-t)\lambda - (u_i + t\lambda) = u_{i+2} - u_i - \lambda$$

$$u_{i+1} - u_i \quad = \quad u_{i+1} - (u_i + t\lambda) = u_{i+1} - u_i - t\lambda$$

the main coefficient is t again. If $z = 3$, then due to $u_i = u_i + t\lambda$ the main coefficient is t^2. Thus we obtain, that

$$\lim_{\lambda \to -\infty} N_i^3 = \begin{cases} t, \text{ if } z = 1, 2 \\ 1, \text{ if } z = 3 \end{cases} \qquad u \in [u_i, u_{i+1}) \, .$$

On the interval $[u_i, u_{i+1})$ the other two basis functions are of the form

$$\begin{aligned} N_{i-2}^3 &= \frac{u_{i+1} - u}{u_{i+1} - u_{i-1}} \frac{u_{i+1} - u}{u_{i+1} - u_i} \\ N_{i-1}^3 &= \frac{u - u_{i-1}}{u_{i+1} - u_{i-1}} \frac{u_{i+1} - u}{u_{i+1} - u_i} + \frac{u_{i+2} - u}{u_{i+2} - u_i} \frac{u - u_i}{u_{i+1} - u_i}. \end{aligned}$$

Similar calculation leads to the main coefficients and to the limits of these two functions, which are $(1-t)$ and 0 for $z = 1$, 0 and $(1-t)$ for $z = 2$ while 0 in both cases for $z = 3$. For the rest of the indices $(j \neq i-2, i-1, i)$ $N_j^3 \equiv 0$ always holds. Thus the proof is ready for $k = 3$ on the interval $[u_i, u_{i+1})$. For the intervals $[u_{i+1}, u_{i+2})$ and $[u_{i+2}, u_{i+3})$ the statement can be proved in an analogous way.

ii) Suppose that for $\forall u \in [u_i, u_{i+z})$

$$\lim_{\lambda \to -\infty} N_i^{k-1} = \begin{cases} t, \text{ if } z = 1, ..., k-2 \\ 1, \text{ if } z = k-1 \end{cases} \tag{3.2a}$$

$$\lim_{\lambda \to -\infty} N_{i+z-k+1}^{k-1} = \begin{cases} (1-t), \text{ if } z = 1, ..., k-2 \\ 1, \text{ if } z = k-1 \end{cases} \tag{3.2b}$$

$$\lim_{\lambda \to -\infty} N_j^{k-1} = 0, (j \neq i, i+z-k+1) \, , \text{ if } z = 1, ..., k-1. \tag{3.2c}$$

holds. At first we prove that the assumptions (3.2a)-(3.2c) yield

$$\lim_{\lambda \to -\infty} N_i^k = \begin{cases} t, \text{ if } z = 1, ..., k-1 \\ 1, \text{ if } z = k \end{cases} \qquad u \in [u_i, u_{i+z}) \, . \tag{3.3}$$

By definition

$$N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1}^{k-1}(u) \, .$$

Due to (3.2a) the limit of the first term is t if $z \leq k-2$. If $z = 1$ then $N_{i+1}^{k-1}(u) \equiv 0$, thus the limit of the second term equals 0, while for $z = 2, ..., k-2$ the limit also equals 0 due to (3.2c). For $z = k-1$ the limit of the fraction in the first term equals t since

$$\begin{aligned} u - u_i &= u - u_i - t\lambda \\ u_{i+k-1} - u_i &= u_{i+k-1} - \lambda + t\lambda - u_i - t\lambda = u_{i+k-1} - u_i - \lambda. \end{aligned}$$

But (3.2a) yields $\lim_{\lambda\to-\infty} N_i^{k-1} = 1$, thus the limit of the first term is t again. Taking into account equation (3.2c) the limit of the second term is 0.

Finally, for $z = k$ the proof is analogous to that of Theorem 1.2, thus we proved (3.3).

Now applying (3.2a)-(3.2c) we verify

$$\lim_{\lambda\to-\infty} N_{i+z-k}^k = \begin{cases} (1-t), \text{ if } z = 1, ..., k-1 \\ 1, \text{ if } z = k \end{cases} \qquad u \in [u_i, u_{i+z}). \tag{3.4}$$

By definition

$$N_{i+z-k}^k(u) = \frac{u - u_{i+z-k}}{u_{i+z-1} - u_{i+z-k}} N_{i+z-k}^{k-1}(u) + \frac{u_{i+z} - u}{u_{i+z} - u_{i+z-k+1}} N_{i+z-k+1}^{k-1}(u).$$

Due to (3.2c) the limit of the first term equals 0 for $z \le k-1$. The limit of the fraction in the second term is 1 for $z = 1, ..., k-2$, while (3.2b) yields $\lim_{\lambda\to-\infty} N_{i+z-k+1}^{k-1}(u) = (1-t)$. For $z = k-1$ applying

$$\begin{aligned} u_{i+k-1} - u &= u_{i+z} - (1-t)\lambda - u \\ u_{i+k-1} - u_i &= u_{i+k-1} - \lambda + t\lambda - u_i - t\lambda = u_{i+k-1} - u_i - \lambda \end{aligned}$$

the limit of the fraction in the second term equals $(1-t)$, while due to (3.2b) $\lim_{\lambda\to-\infty} N_{i+z-k+1}^{k-1}(u) = 1$. Thus the limit of the second term is always equal to $(1-t)$. The case $z = k$ is analogous to the proof of Theorem 1.2 again, thus (3.4) is verified.

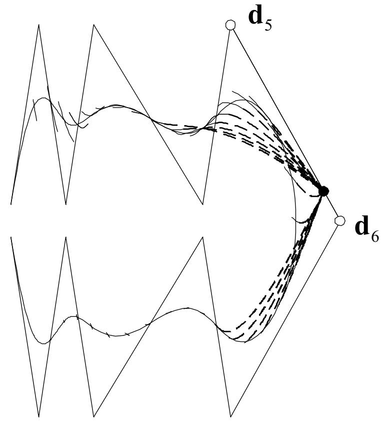

Figure 2: A quadric ($k = 5$) B-spline curve and its paths for $t = 0.85$, $(i = 6, z = 4)$.

Finally, we prove that assuming (3.2a)-(3.2c)

$$\lim_{\lambda\to-\infty} N_j^k = 0, (j \ne i, i+z-k), \forall z,\ u \in [u_i, u_{i+z}) \tag{3.5}$$

holds. By definition

$$N_j^k(u) = \frac{u - u_j}{u_{j+k-1} - u_j} N_j^{k-1}(u) + \frac{u_{j+k} - u}{u_{j+k} - u_{j+1}} N_{j+1}^{k-1}(u).$$

The limit of the first term equals 0 (if $j = i + z - k + 1$ then $j + k - 1 = i + z$, thus the limit of the fraction is 0, while for the other cases the limit of the basis function in the first term is 0 due to (3.2c)). The limit of the second term equals 0 as well, (for $j + 1 = i$ the limit of the fraction equals 0, while for the rest of the cases (3.2c) yields $\lim_{\lambda \to -\infty} N_{j+1}^{k-1}(u) = 0$). Thus (3.5) has also been verified and this completes the proof. □

Figure 1 demonstrates the result for cubic curves, while Figure 2 shows an example for a higher order curve.

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