Chapter V
Dynamic Bifurcation of Nonlinear Evolution Equations

In this chapter, we mainly discuss the bifurcation of invariant sets the attractors and invariant manifolds for the finite and infinite dimensional dynamical systems. In section 5.3-5.4, we present a set of theory to deal with this problem which generalizes the Hopf bifurcation to the more general cases. We have known that the Hopf bifurcation will occur when the real parts of a pair of complex eigenvalues pass through zero. In fact, the dynamic bifurcation theory set in this chapter tells us that under some conditions the bifurcation of invariant sets likewise appears provided the real parts of some eigenvalues pass through zero. In addition, in Section 5.3 we also give a stability theorem on the bifurcation for the finite dimensional dynamical systems, which amounts to saying that there is an open and dense set $\mathcal{F}$ in the space $C_0^{3,1}(\Omega \times R, R^n)$ of the parameterized vector fields, for any vector field $v \in \mathcal{F}$ there exists a neighborhhod $O \subset C_0^{3,1}(\Omega \times R, R^n)$ of v such that all the vector fields in O have the same bifurcation structure, i.e. the same number of the bifurcation points and the same topological structure near the bifurcated invariant manifolds.

In section 5.5, we apply the theories and methods developed in section 5.3-5.4 to investigate the bifurcation of invariant manifolds for the nonlinear partial differential equations with the dissipative structure related with the mechanical and physical problems.

In the final section (section 5.6), we study the dynamical properties of $2D$ Navier-Stokes equations with the free, the Dirichlet, and the periodic boundary conditions. We find that the eigen-spaces of the Laplacian are invariant for the $2D$ Navier-Stokes equations with the Dirichlet boundary condition and the periodic boundary condition, and they are also invariant for all the considered boundary conditions in the domain having the genus zero. By virtue of this property, the problems of the global stability of the stationary solutions, the Taylor vortex type of periodic structure, and the existence of asymptotically time-periodic solution are discussed.

5.1. Examples and Introduction

5.1.1. Pendulum in a symmetric magnetic field

A famous example of the dynamic bifurcation is given by the Hopf's bifurcation. In fact, the dynamic bifurcation is very general in the nonlinear evolution equations. In order to illustrate this point, we shall in this subsection investigate the pendulum in a vertical plane with a symmetric magnetic field, which is a typical example of the dynamic attractor bifurcation.

We consider a pendulum in a verticalplane, see Fig. 5.1 below.

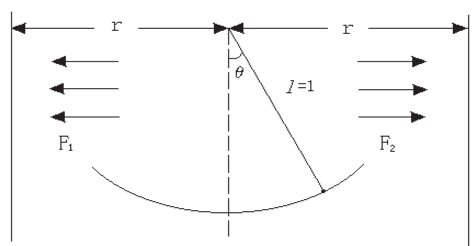


Fig. 5.1

Suppose that the length of this pendulum $l = 1$, and the one end of motion is tied a small iron ball of mass $m = 1$. The small ball moves with friction on a vertical unite circle. On the both sides of the small ball there symmetric are two magnetized plates attracting it, which have the same magnetic magnitude. The distances r from the magnetized plates to the downward vertical of the pendulum are equal, and $r \gg 1$.

From the intuition, we can see that as the magnitude λ of the magnetic fields on the both laterals is smaller to some critical value λ_0, i.e. $\lambda < \lambda_0$, under the action of friction and gravity, the pendulum will gradually stop at the position $\theta = 0$, where θ is the angle of the pendulum with the downward vertical. But, as the magnitude λ exceeds the critical value λ_0, on the both sides of the center position $\theta = 0$ two equilibrum positions $\pm\theta_\lambda \neq 0(\lambda > \lambda_0)$ will symmetrically appear, at where the small ball under the action of friction will gradually stop. And that the small ball will stop at which one of the two positions $\pm\theta_\lambda$ entirely depends on the initial state of the small ball.

In the following, we shall discuss in detail the problem. The motion equation of the pendulum in a vertical plane with symmetric magnetic field is as follows

$$\frac{d^2\theta}{dt^2} = -k\frac{d\theta}{dt} - g\sin\theta + f\cos\theta$$

with the initial condition

$$\theta(0) = \alpha_1, \quad \frac{d\theta}{dt}|_{t=0} = \alpha_2$$

where $k > 0$ is the damping coeficient, g the gravity, and f the magnetic force. By the Coulomb lows

$$\begin{aligned} f &= F_2 - F_1 = \frac{\Lambda}{(r - \sin\theta)^2} - \frac{\Lambda}{(r + \sin\theta)^2} \\ &= \frac{4r\Lambda \sin\theta}{(r^2 - \sin^2\theta)^2} \simeq \lambda \sin\theta, \qquad (by\ r \gg 1) \end{aligned}$$

where $\lambda = \frac{4\Lambda}{r^3}, \Lambda$ is propotional to the magnitude of the magnetic field. Thus, the motion equation can be approxitively expressed by

$$\frac{d^2\theta}{dt^2} = -k\frac{d\theta}{dt} - g\sin\theta + \lambda \sin\theta \cos\theta.$$

Letting $x_1 = \theta, x_2 = d\theta/dt$, then the motion equation is transformed into the following system

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -kx_2 - g\sin x_1 + \lambda \sin x_1 \cos x_1 \end{cases} \tag{5.1.1}$$

with the initial condition

$$x_1(0) = \alpha_1, x_2(0) = \alpha_2 \tag{5.1.2}$$

By the Taylor expansion

$$\lambda \sin x_1 \cos x_1 - g \sin x_1 = (\lambda - g)x_1 - \frac{1}{6}(\lambda - g)x_1^3$$
$$-\frac{1}{2}\lambda x_1^3 + o(|x_1|^3)$$

then the system (5.1.1) near $x = 0$ can be expressed as

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = (\lambda - g)x_1 - kx_2 + \frac{1}{6}(g - \lambda)x_1^3 - \frac{1}{2}\lambda x_1^3 + o(|x_1|^3) \end{cases} \tag{5.1.3}$$

It is clear that as $\lambda < g$, the eigenvalues of(5.1.3) is as follows

$$\lambda_\pm = \frac{-k \pm \sqrt{k^2 - 4(g - \lambda)}}{2}$$

whose real parts are negative. In this case the equilibrum point $x = 0$ of(5.1.1) is asymptotically stable.

As $\lambda = g$, the equation (5.1.1) near $x = 0$ is as follows

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -kx_2 - \frac{1}{2}gx_1^3 + o(|x_1|^3) \end{cases} \tag{5.1.4}$$

The eigenvalues of (5.1.4) are $\lambda_1 = 0, \lambda_2 = -k$, namely the system (5.1.4) is on the critical state. By using Theorem 5.1.2 in Subsection 5.1.4, it is easy to derive that $x = 0$ is asymptotically stable.

As $\lambda > g$, two stationary solutions $(x_1^{\pm}, x_2^{\pm}) = (\pm\theta_\lambda, 0)$ of (5.1.1) bifurcate from $x = 0$, which are as follows

$$\theta_\lambda = \cos^{-1} g/\lambda.$$

And it is easy to verify that $(\pm\theta_\lambda, 0)$ are two asymptotically stable equilibrum points.

The discussion above can be summarized as the following theorem.

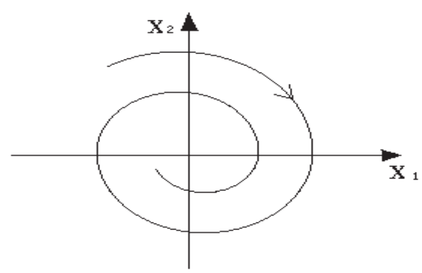


Fig. 5.2. When $\lambda \leq g, x = 0$ is an attractor

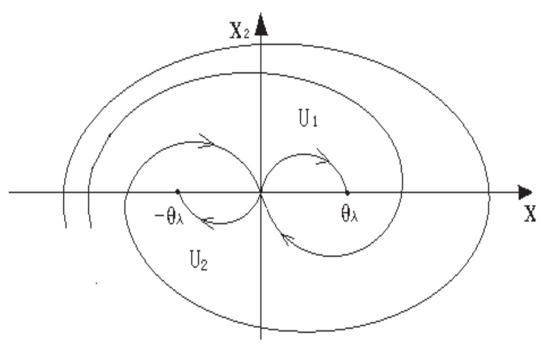


Fig 5.3. When $\lambda > g$, two attracted basins U_1 and U_2 bifurcate from the attracted basin U.

Theorem 5.1.1. There exists an open set $U \subset R^2, 0 \in U$, such that when $\lambda \leq g$, for all the initial values $(\alpha_1, \alpha_2) \in U$, the solutions of (5.1.1)(5.1.2) satisfy that

$$\lim_{t\to\infty} x(t,\alpha) = 0, \quad \alpha = (\alpha_1, \alpha_2) \in U$$

and when $\lambda > g$ with $\lambda - g$ sufficiently small, two stationary solutions $(\pm\theta_\lambda, 0)$ of (5.1.1) bifurcate from $x = 0$, and the open set U is decomposed into two open subsets $U_1, U_2 (U_1 \cap U_2 = \phi, \bar{U} = \bar{U}_1 + \bar{U}_2)$ with $(\theta_\lambda, 0) \in U_1, (-\theta_\lambda, 0) \in U_2$ and $0 \in \partial U_1 \cap \partial U_2$, such that the solutions of (5.1.1)(5.1.2) satisfy

$$\lim_{t\to\infty} (x_1(t,\alpha), x_2(t,\alpha)) = (\theta_\lambda, 0), \qquad as\ \alpha = (\alpha_1, \alpha_2) \in U_1$$

$$\lim_{t\to\infty} (x_1(t,\alpha), x_2(t,\alpha)) = (-\theta_\lambda, 0), \qquad as\ \alpha = (\alpha_1, \alpha_2) \in U_2.$$

This theorem describes the dynamic attractor bifurcation of a pendulum in a symmetric magnetic field, which can be shown by Fig 5.2-5.3.

5.1.2. Business cycles for the Kaldor's model

In this subsection, we shall use the Hopf bifurcation theorem to discuss the business cycle problems for the Kaldor's model, which is also a typical example of the dynamic attractor bifurcation. The Kaldor's model is given by

$$\begin{cases} \frac{dx}{dt} = I(x,y,\lambda) - S(x,y,\lambda) \\ \frac{dy}{dt} = I(x,y,\lambda). \end{cases} \tag{5.1.5}$$

where x is the total social income, y the total socal captal, λ the industrial technique parameter, $I(x,y,\lambda)$ the investing function and $S(x,y,\lambda)$ the savings function.

For each technique parameter λ, the social business has an equilibrum state $(x_0, y_0), x_0 > 0, y_0 > 0$, which is the function of λ. For the sack of simplicity, we assume that (x_0, y_0) is independent of λ. According to the ecnomic laws, in neighbor-hood of $(x_0, y_0) \in R^2, I$ and S satisfy that

$$\frac{\partial I}{\partial x} > 0, \quad \frac{\partial I}{\partial y} \le 0, \quad \frac{\partial S}{\partial x} > 0, \quad \frac{\partial S}{\partial y} > 0$$

By the basic conditions, in a neighborhood of $(x_0, y_0), I$ and S are taken as the following forms

$$I = a_\lambda(x - x_0) - \alpha x(y - y_0)^3$$

$$S = b_\lambda(x - x_0) + c_\lambda(y - y_0) + \beta y(x - x_0)^3 + \gamma x(y - y_0)^3.$$

where $a_\lambda, b_\lambda, c_\lambda > 0$ are the continuous function of λ, and $\alpha, \beta, \gamma > 0$ are the constants.

Thus the Kaldor's model reads as

$$\begin{cases} \frac{dx}{dt} = k_\lambda(x - x_0) - c_\lambda(y - y_0) - \delta x(y - y_0)^3 - \beta y(x - x_0)^3 \\ \frac{dy}{dt} = a_\lambda(x - x_0) - \alpha x(y - y_0)^3 \end{cases} \tag{5.1.6}$$

where $k_\lambda = a_\lambda - b_\lambda, \delta = \alpha + \gamma$. The eigenvalues of the matrix

$$A_\lambda = \begin{pmatrix} k_\lambda & -c_\lambda \\ a_\lambda & 0 \end{pmatrix}$$

are as follows

$$\lambda_\pm = \frac{k_\lambda \pm \sqrt{k_\lambda^2 - 4a_\lambda c_\lambda}}{2}$$

Near $k_\lambda = a_\lambda - b_\lambda = 0$, the eigenvalues of A_λ are complex number, and $Re\lambda_\pm = \frac{1}{2}k_\lambda$.

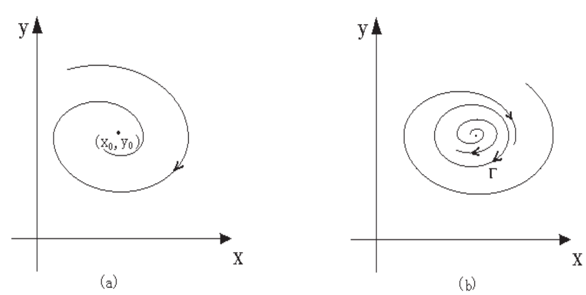
As $k_\lambda < 0$, the equilibrum point (x_0, y_0) of (5.1.6) is asymptotically stable. As $k_\lambda = 0$, the eigenvalues $\lambda_\pm = \pm i\sqrt{4a_\lambda c_\lambda}$. By Theorem 5.1.4 in Subsection 5.1.4, we can infer that (x_0, y_0) is also asymptotically stable. To see this, we notice that the divergence of the nonlinear term of (5.1.6) is negative in a neighborhood of (x_0, y_0):

$$\begin{aligned}
&\frac{\partial}{\partial x}[-\delta x(y-y_0)^3 - \beta y(x-x_0)^3] + \frac{\partial}{\partial y}[-\alpha x(y-y_0)^3] \\
&= -[3\beta y(x-x_0)^2 + 3\alpha x(y-y_0)^2] - \delta(y-y_0)^3 \\
&< 0; \quad \forall (x-x_0)^2 + (y-y_0)^2 < \epsilon \ and \ (x,y) \neq (x_0, y_0)
\end{aligned}$$

where $\epsilon > 0$ is sufficiently small.

Suppose that there exists a $\lambda_0 \in R$, such that

(5.1.7) $$k_\lambda = a_\lambda - b_\lambda = \begin{cases} < 0, & as\ \lambda < \lambda_0 \\ = 0, & as\ \lambda = \lambda_0 \\ > 0, & as\ \lambda > \lambda_0 \end{cases}$$



(a) as $\lambda \leq \lambda_0, (x_0, y_0)$
(b) as $\lambda > \lambda_0$, a cycleis asymptotically stable attractor Γ bifurcates from (x_0, y_0)
Fig. 5.4.

Then, under the condition (5.1.7), by the Hopf bifurcation theorem we know that as $\lambda > \lambda_0$ the equation (5.1.6) bifurcates from (x_0, y_0) a cycle attractor, see Fig. 5.4 (a) and (b) above.

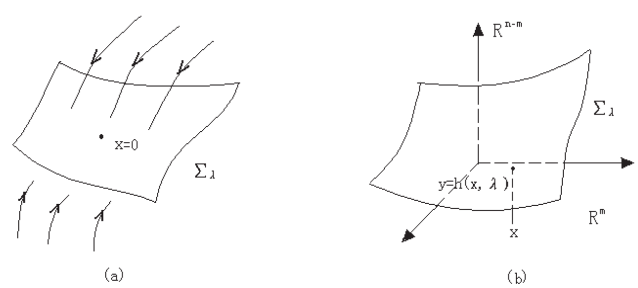
The economic explanation of this cycle attractor bifurcation of (5.1.6) is given by the following.

When the industral technique level $\lambda < \lambda_0$, the investing interest and the social income are lower, therefore the investment intention is weaker than that of the social savings, namely $a_\lambda < b_\lambda$. Thus, $k_\lambda = a_\lambda - b_\lambda < 0$. In this case, the social economy develops slowly on the stable equilibrum state.

When the technique level is promoted to exceed the critical state λ_0, i.e. $\lambda > \lambda_0$, the new industrial technology brings the higher investing interest, in this time the investing intention is stronger than that of the social savings, therefore $k_\lambda = a_\lambda - b_\lambda > 0$. In this case, the devolopment of social economy is transferred from the stable equilibrum state to the periodic fluctutation form, which is that we have seen today.

5.1.3. Basic idea of the dynamic attractor bifurcation

The above two examples show that the dynamic attractor bifurcation of a dynamical system always occurs in this case that the real parts of some eigenvalues of the dynamical system at an equilibrum point pass through from the neigative to the positive, and the other eigenvalues remain to stay in the negative. In the following, we shall illustrate the basic idea of the dynamic attractor bifurcation by considering the below equation



(a) the flows of (5.1.8)(5.1.9) in R^n are squeezed to $\sum_\lambda$
(b) $\sum_\lambda = \{(x,y) \in R^n | x \in R^m, y = h(x,\lambda)\}$
Fig. 5.5.

$$\frac{dx}{dt} = \lambda x + G_1(x,y) \tag{5.1.8}$$

$$\frac{dy}{dt} = -y + G_2(x,y). \tag{5.1.9}$$

where $x \in R^m, y \in R^{n-m}(0 < m \leq n)$, and

$$G_i(x, y) = o(|x|, |y|), i = 1, 2.$$

It is known that as $\lambda < 0$, the steady state $z = (x, y) = 0$ of (5.1.8) (5.1.9) is asymptotically stable. And when λ passes through from the negative to the positive, the flows of (5.1.8) (5.1.9) in a neighborhood of $z = 0$ in R^n are squeezed to a m-dimensional surface $\sum_\lambda(\lambda \geq 0)$, and the surface $\sum_\lambda$ is tangent to R^m at $x = 0$, which can be expressed by a function $y = h(x, \lambda), x \in R^m, y \in R^{n-m}$, see Fig 5.5. (a) and (b) above(in fact thus surface $\sum_\lambda$ exists for all $\lambda \in R$):
Obviously, the m-dimensional surface $\sum_\lambda$ is an local invariant surface, and the flows of (5.1.8)(5.1.9) in $\sum_\lambda$ are topologically equivalent to the flows of the below equation in a neighborhood of $x = 0$ in R^m,

$$\frac{dx}{dt} = \lambda x + G_1(x, h(x, \lambda)). \tag{5.1.10}$$

where $G_1(x, h(x, \lambda)) = o(|x|) \forall \lambda \geq 0$, and the function $y = h(x, \lambda)$ is continuous on λ. Suppose that as $\lambda = 0, z = 0$ is asymptotically stable for the equation (5.1.8)(5.1.9), then $x = 0$ is also asymptotically stable for the below equation

$$\frac{dx}{dt} = G_1(x, h(x, 0)) \tag{5.1.11}$$

When $\lambda > 0$ sufficiently small, the equation (5.1.10) is a small perturbation of (5.1.11), namely

$$\frac{dx}{dt} = G_1(x, h(x, 0)) + \lambda x + K(x, \lambda).$$

where $K(x, \lambda) = o(|x|)$, and

$$K(x, \lambda) = G_1(x, h(x, \lambda)) - G_1(x, h(x, 0)) \to 0 \ as \ \lambda \to 0^+.$$

Near $x = 0$, the linear term λx of the perturbation plays the key role, thereoftre the flows of (5.1.10) near $x = 0$ are "outward". But, in the place for from $x = 0$, the nonlinear term $G_1(x, h(x, 0))$ plays the key role, which implies that the flows far from $x = 0$ are "inward". Thus the outward flows and inward flows in R^m squeeze an attractor, as shown in Fig. 5.6. below.

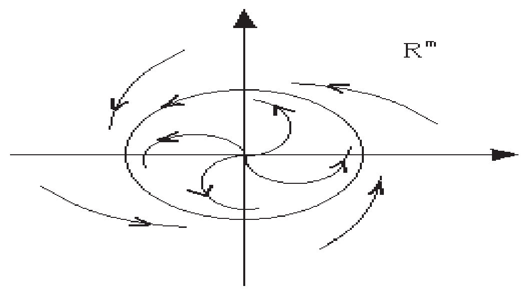


Fig. 5.6.

Because the surface $\sum_\lambda$ is attracting, thus we infer that as $\lambda > 0$ the equation (5.1.8)(5.1.9) bifurcates from $z = 0$ an attractor. In many cases, the attractor is homeomorphic to a $(m-1)-$dimensional sphere.

5.1.4. Appendix: Lyapunov stability on the critical states

From the previous discussion we can see that the determination of the asymptotical stability of steady solutions on the critical states is very important for the dynamic attractor bifurcation. Here we shall give some simple methods to treat this problem.

First, we consider the $2D$ system given by

$$\left\{ \begin{array}{l} \frac{dx_1}{dt} = G_1(x_1, x_2) \\ \frac{dx_2}{dt} = G_2(x_1, x_2) \end{array} \right. \tag{5.1.12}$$

where $G(0,0) = 0, G = \{G_1, G_2\}$. Suppose that

$$G = f + g$$

and

$$div f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 \tag{5.1.13}$$

We denote

$$V(x_1, x_2) = \int_0^{x_2} f_1(x_1, x_2) dx_2 - \int_0^{x_1} f_2(x_1, 0) dx_1$$

Then we have the following theorem.

Theorem 5.1.2. Under the condition (5.1.13), if there exists an open set $\Omega \subset R^2, 0 \in \Omega$ such that $G(x)$ has only one zero point $x = 0$ in Ω, and

i). $V(x) > 0$ for all $x \in \Omega, x \neq 0$;
ii). $f_1 g_2 - f_2 g_1 \leq 0 \quad \forall x \in \Omega$;
iii). $divg < 0, \forall x \in \Omega$ and $x \neq 0$;
then $x = 0$ is asymptotically stable for the equation (5.1.12).

Proof. From (5.1.13) it follows that

$$\frac{\partial V}{\partial x_2} = f_1; \quad \frac{\partial V}{\partial x_1} = -f_2$$

Hence, by (ii) we deduce that for the equation (5.1.12),

$$\begin{aligned} \frac{dV(x)}{dt} &= \frac{\partial V}{\partial x_1} G_1 + \frac{\partial V}{\partial x_2} G_2 \\ &= -f_2 g_1 + f_1 g_2 \\ &\leq 0, \quad \forall x \in \Omega \end{aligned}$$

which implies, by the condition i), that $V(x)$ is the Lyapnov function. Hence $x = 0$ is stable. On the other hand, by (5.1.13) and (iii) we know that (5.1.12) has no limit cycle in Ω. Thus, from the Poincare-Bendixson theorem and the fact that $x = 0$ is an isolated singular point of G, this theorem follows. The proof is complete.

The method to seek the Lyapnov function as above can be generalized to the higher dimensional systems as follows

$$\frac{dx}{dt} = G(x), x \in R^{2n}, n \geq 1 \tag{5.1.14}$$

where $x = \{x_1, \cdots, x_n, y_1, \cdots, y_n\}$, and $G(0) = 0$. Suppose that

$$G = J\nabla H + g(x)$$

where H is a Hamilton function, with $H(0) = 0$, and

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

In the same fashion as using in Theorem 5.12, we can obtain the following result.

Theorem 5.1.3. If there is an open set $U \subset R^{2n}, o \in U$, such that $G(x)$ has only one zero point $x = 0$ in U, and
i). $H(x) > 0$ for all $x \in U$ and $x \neq 0$;
ii). $\sum_{i=1}^{n} \left[\frac{\partial H}{\partial x_i} \cdot g_i + \frac{\partial H}{\partial y_i} g_{n+i} \right] < 0; \quad \forall x \in \Omega, x \neq 0$
then $H(x)$ is a Lyapunov function of (5.1.14), and $x = 0$ is asymptotically

stable.

By applying Theorem 5.1.2, we can deduce that for the equation (5.1.4) in Subsection 5.1.1, $x = 0$ is asymptotically stable. In fact, for the equation (5.1.4) we have

$$\begin{cases} G_1(x_1, x_2) = x_2 \\ G_2(x_1, x_2) = -kx_2 - \frac{1}{2}gx_1^3 + o(|x_1|^3) \end{cases}$$

Obviously, $x = 0$ is an isolated zero point of $G = \{G_1, G_2\}$. Let

$$f_1 = x_2, \quad f_2 = -\frac{1}{2}gx_1^3 + o(|x_1|^3)$$

$$g_1 = 0, g_2 = -kx_2$$

then $divf = 0$, and

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{8}gx_1^4 + o(|x_1|^4)$$

$$f_1g_2 - f_2g_1 = -kx_2^2 \leq 0$$

$$divg = -k < 0.$$

It is easy to see that the conditions i)-iii) in Theorem 5.1.2 are satisfied, hence $x = 0$ is asymptotically stable for (5.1.4).

Next, we discuss the system as follows

$$\frac{dx}{dt} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + G(x). \tag{5.1.15}$$

where $G(x) = o(|x|)$, and

$$\begin{cases} \alpha_{11} + \alpha_{22} = 0 \\ \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} > 0 \end{cases} \tag{5.1.16}$$

By (5.1.16), the eigenvalues of (5.1.15) are as follows

$$\beta_{\pm} = \pm i\sqrt{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}$$

On this critical state, the equilibrum point $x = 0$ of (5.1.15) must be one of the three cases: a center, a stable focus and an unstable focus. In the following, we give a determining theorem.

Theorem 5.1.4. Let $U \subset R^2$ be a neighborhood of $x = 0$. Under the condition (5.1.16), the following assertions hold:
i). if $divG = 0$ in U, then $x = 0$ is a center of (5.1.15);

ii). if $divG \leq 0 (\not\equiv 0)$ in U, then $x = 0$ is a stable focus;
iii). if $divG \geq 0 (\not\equiv 0)$ in U, then $x = 0$ is an unstable focus.

Proof. Because the orbits of (5.1.15) are winding in U around $x = 0$, for $x_0 \in U (x_0 \neq 0)$, take T the winding period of the point x_0 defined by (see Fig.5.7)

$$T = \min\{t > 0 | x(t, x_0) \ intersects \ with \ the \ ray \ emitted$$
$$from \ x = 0 \ passing \ through \ point \ x_0\}.$$

where $x(t, x_0)$ denotes the solution of (5.1.15) with the initial condition $x(0) = x_0$.

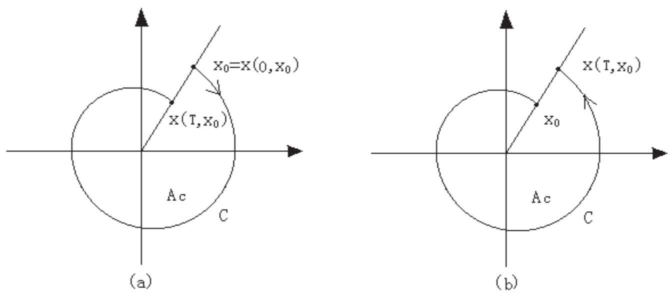


Fig. 5.7.

Let C be the closed curve enclosed by $\{x(t, x_0) | 0 \leq t \leq T\}$ and the segment connecting x_0 and $x(T, x_0)$, i.e.

$$C = \{x(t, x_0) | 0 \leq t \leq T\} \cup \{\tau x_0 + (1 - \tau) x(T, x_0) | 0 \leq \tau \leq 1\}$$

For any $t \geq 0$, we consider a homeomorphism:

$$\Phi_t : U \to \Phi_t(U)$$
$$\Phi_t(z) = x(t, z), \quad \forall z \in U$$

Let A_C be the closed region enclosed by C, and $A_t = \Phi_t(A_C)$. Obviously we have that $A_0 = \Phi_0(A_C) = A_C$, and for $t > 0$,

$$|A_t| = \begin{cases} > |A_C|, \ if \ x = 0 \ is \ an \ unstable \ focus; \\ = |A_C|, \ if \ x = 0 \ is \ a \ center \\ < |A_C|, \ if \ x = 0 \ is \ a \ stable \ focus. \end{cases}$$

On the other hand, we have

$$|A_t| = \int_{A_t} dx_1 dx_2$$

$$\frac{d}{dt}|A_t| = \int_{A_t} divG dx_1 dx_2$$

Hence we infer that

$$divG = \begin{cases} \geq 0 (\not\equiv 0) \ in \ U & \Rightarrow x = 0 \ an \ unstable \ focuse \\ = 0 \ in \ U & \Rightarrow x = 0 \ a \ center \\ \leq 0 (\not\equiv 0) \ in \ U & \Rightarrow x = 0 \ a \ stable \ focuse. \end{cases}$$

This theorem is proven.

5.2. Some Related Concepts and Lemmas

5.2.1. Invariant sets and attractors

Let H, H_1 be the Hilbert spaces, and $H_1 \hookrightarrow H$ be a compact and dense embedding. Hereafter, we always concern the dynamical properties of the below abstract nonlinear evolution equations.

(5.2.1)
$$\begin{cases} \frac{du}{dt} = L_\lambda u + G(u, \lambda) \\ u(0) = \phi \end{cases}$$

where $\lambda \in R$ is a parameter, and

$$\begin{cases} L_\lambda : H_1 \to H \ is \ a \ linear \ completely \ continuous \ field \\ G(\cdot, \lambda) : H_1 \to H \ is \ a \ continuous \ operator \ with \\ G(x, \lambda) = o(\|x\|_{H_1}), \quad \forall \lambda \in R \end{cases}$$

When $H_1 = H = R^n (n \geq 1)$, the equation (5.2.1) is the usual system of ordinary differential equations.

Definition 5.2.1. A set $\sum \subset H_1$ is called an invariant set of (5.2.1), if $u(t, \sum) = \sum, \forall t \geq 0$, here $u(t, \phi)$ is the solution of (5.2.1).

We know that the system (5.2.1) generates an operator semigroup $S(t), t \geq 0$, which enjoys the properties (Cf.[Te]):

$$\begin{cases} S(t) : H_1 \to H_1 \ a \ continuous \ mapping, \forall t \geq 0 \\ S(0) = I \ (identity \ in H_1) \\ S(t+s) = S(t) \cdot S(s), \quad \forall s, t \geq 0 \end{cases}$$

The solution of (5.2.1) can be expressed as

$$u(t, \phi) = S(t)\phi, \quad t \geq 0$$

Thus, an invariant set $\sum$ of (5.2.1) can be equivalently stated as

$$S(t)\Sigma = \Sigma, \forall t \geq 0$$

Definition 5.2.2. A set $\sum \subset H_1$ is termed an attractor of(5.2.1), if $\sum$ is a compact invariant set of (5.2.1), and there exists a neighborhood $U \subset H_1$ of $\sum$, such that for any $\phi \in U$ we have

$$dist(u(t,\phi),\Sigma) \to 0 \ in \ H - norm, as \ t \to \infty. \quad (5.2.2)$$

The largest open set U satisfying (5.2.2) is called the attracted basin of $\sum$.

If $\sum \subset H_1$ is an invariant set (or an attractor) of (5.2.1) which is homeomorphic to a m-dimensional sphere S^m, then we say that $\sum$ is a S^m-invariant set (or a S^m-attractor).

For $\phi \in H_1$(or for $\sum \subset H_1$), we define the ω-limit of ϕ (or $\sum$) by

$$\omega(\phi) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)\phi}$$

or

$$\omega(\Sigma) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)\Sigma}$$

where the closures are taken in H_1. Likewise, when it exists, the α-limit set of $\phi \in H_1$ and $\sum \subset H_1$ are defined by

$$\alpha(\phi) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(-t)\phi}$$

and

$$\alpha(\Sigma) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(-t)\Sigma}$$

The following lemmas can be found in [Te].

Lemma 5.2.3. Suppose that for some subset $\sum \subset H_1, \sum \neq \phi$, and for some $t_0 > 0$, the set $\bigcup_{t\geq t_0} S(t)\sum$ is relatively compact in H. Then $\omega(\sum)$ is nonempty, compact, and invariant. Similarly, if the sets $S(-t)\sum, t \geq 0$ are nonempty and for some $t_0 > 0, U_{t\geq t_0} S(-t)\sum$ is relatively compact, then $\alpha(\sum)$ is nonempty, compact, and invariant.

The following lemma shows how to obtain the existence of an attractor. To this end, we introduce a related concept of absorbing set.

Definition 5.2.4. Let $\sum \subset H_1$ be a subset and U an open set containing $\sum$. We say that $\sum$ is absorbing in U if the orbit of any bounded set of U enter into $\sum$ after a certain time (which may depend on the set):

$$\begin{cases} \forall B_0 \subset U, B_0 \textit{ bounded} \\ \exists t_0(B_0) \textit{ such that } S(t)B_0 \subset \sum, \forall t \geq t_0(B_0) \end{cases}$$

Lemma 5.2.5. Suppose that the operators $S(t)$ are uniformly compact for t larg, i.e. for any bounded set B there exists t_0 such that $U_{t\geq t_0} S(t)B$ is relatively compact in H_1. We also assume that there exists an open set U and a bounded set B of U such that B is absorbing in U. Then the ω-limit set of $B, \sum = \omega(B)$ is an attractor which attracts the bounded sets of U, and it is the maximum attractor in U. Furthermore, if U is connected, then $\sum$ is connected too.

5.2.2. Center manifolds

Let us consider the system of ordinary differential equations as follows

$$\begin{cases} \frac{dx}{dt} = Ax + G_1(x, y, \lambda) \\ \frac{dy}{dt} = By + G_2(x, y, \lambda) \end{cases} \tag{5.2.3}$$

where $x \in R^m, y \in R^{n-m} (0 < m \leq n), A$ and B are respectively the $m \times m$ and $(m-m) \times (n-m)$ matrices, $G_i(x, y, \lambda)(i = 1, 2)$ are continuous on λ, and $C^r (r \geq 1)$ on $(x, y) \in R^m \times R^{n-m}$, moreover

$$G_i(x, y, \lambda) = o(\|x\|, \|y\|), \quad \forall \lambda \in R, (i = 1, 2) \tag{5.2.4}$$

The following are the well known center manifold theorems, which can be be found in [CH].

Theorem 5.2.6. Suppose that all the eigenvalues of A have non-negative real parts, and all the eigenvalues of B have negative (or positive) real parts. Then, for the system (5.2.3) with the condition (5.2.4), there exists a C^r function

$$h(\cdot, \lambda) : \Omega \to R^{n-m}; \quad \Omega \subset R^m \textit{ a neighborhood of } x = 0$$

such that $h(x, \lambda)$ is continuous on λ, and
i). $h(0, \lambda) = 0, h'_x(0, \lambda) = 0$;
ii). the set

$$M_\lambda = \{(x, y) | \quad x \in \Omega \subset R^m, y = h(x, \lambda)\}$$

called the local center manifold, is a local invariant manifold of (5.2.3);

iii). if M_λ is positive invariant (or negative invariant), namely $z(t,\phi) \in M_\lambda(z(-t,\phi) \in M_\lambda), \forall t \geq 0$ provided $\phi \in M_\lambda$, then M_λ is an attracting set of (5.2.3)(or a repelling set) , i.e. there is a neighborhood $U \subset R^n$ of M_λ, as $\phi \in U$, we have

$$\lim_{t\to\infty} dist(z(t,\phi), M_\lambda) = 0$$

or

$$\lim_{t\to\infty} dist(z(-t,\phi), M_\lambda) = 0$$

where $z(t,\phi) = \{x(t,\phi), y(t,\phi)\}$ is the solution of (5.2.3) with the initial condition $z(0,\phi) = \phi$.

The center manifold theorem amounts to saying that there is a m-dimensional surface $M_\lambda \subset R^n$ tangent to the eigenspace R^m of A at $x = 0$, which is invariant and attracting under the orbits of (5.2.3), see Fig. 5.5 (a) and (b).

Although, as we know, the local center manifold may not be unique, we have the following result, see [CH].

Theorem 5.2.7. There is a neighborhood $U \subset R^n$ of zero such that every invariant set (5.2.3) in U belong to the intersection of all local center manifolds in U.

This theorem ensure us to apply Theorem 5.2.6 to the bifurcation of invariant sets without the trouble of non-uniqueness

5.2.3. Global stable manifolds

We continue to consider the system (5.2.3) with the condition (5.2.4). In the following, we give the global stable manifold theorem (see [HPS]) which plays an important role on the investigation of the structure of attractor in bifurcation.

Definition 5.2.8. Let M, N be two differentiable manifolds. A C^1 mapping $h : M \to N$ is an immersion if for every point $x \in M$, the rank of the tangent mapping

$$Dh(x) : T_xM \to T_{h(x)}N$$

equals to the dimension of M

$$rankDh(x) = dimM, \quad \forall x \in M.$$

The image $h(M) \subset N$ is called an immersion submanifold.

Theorem 5.2.9. Let all the eigenvalues of A have positive real parts, and all the eigenvalues of B have neigative real parts. Then, there uniquely exist the injective immersions

$$\begin{cases} h_1 : R^m \to R^n \\ h_2 : R^{n-m} \to R^n \end{cases}$$

which satisfy:
i). $h_1(0) = 0, h_2(0) = 0$;
ii). $h_1(R^m)$ and $h_2(R^m)$ are respectively the unstable and stable manifolds of (5.2.3) at the singular point $(x, y) = 0$:

$$W^u = h_1(R^m) = \{x \in R^n | \lim_{t\to\infty} s(-t)x = 0\}$$

$$W^s = h_2(R^{n-m}) = \{x \in R^n | \lim_{t\to\infty} s(t)x = 0\}$$

where $S(t)$ is the semigroup generated by (5.2.3);
iii). W^u and W^s are respectively tangent to the eigenspace of A and B at $z = (x, y) = 0$:

$$\begin{cases} T_{z=0}W^u = R^m \\ T_{z=0}W^s = R^{n-m} \end{cases}$$

5.3. Bifurcation of Attractors and Invariant Manifolds of the Finite Dimensional Vector Fields

In this section, we shall discuss the dynamic bifurcation of attractors and s^m-invariant manifold for the following finite dimensional systems

$$\frac{dx}{dt} = A_\lambda x + G(x, \lambda), \quad \lambda \in R, x \in R^n \quad (n \geq 2) \tag{5.3.1}$$

where $G : R^n \times R \to R^n$ is $c^r (r \geq 1)$ on the argument $x \in R^n$ and continuous on $\lambda \in R$ with

$$G(x, \lambda) = o(|x|), \quad \forall \lambda \in R \tag{5.3.2}$$

and

$$A_\lambda = \begin{pmatrix} a_{11}(\lambda) & \cdots & a_{1n}(\lambda) \\ \vdots & & \vdots \\ a_{n1}(\lambda) & \cdots & a_{nn}(\lambda) \end{pmatrix} \tag{5.3.3}$$

is a $n \times n$ matrix, a_{ij} are the continuous function of λ. Let all the eigenvalues (counting the multiplicities) of (5.3.3) are given by

$$\beta_1(\lambda), \cdots, \beta_n(\lambda) \tag{5.3.4}$$

We know that the eigenvalues $\beta_i(\lambda)(1 \leq i \leq n)$ are continuous on λ.

5.3.1. Equilibrium attractor bifurcation

The example of pendulum in a symmetric magnetic field in Subsection 5.1.1 well describes the phenomena of equilibrum attractor bifurcation. Now, we investigate more generally the equilibrum attractor bifurcation for the systems (5.3.1).

Suppose that the eigenvalues (5.3.4) satisfy

$$\begin{cases} \beta_1(\lambda) = \begin{cases} < 0, & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0 \end{cases} \\ Re\beta_i(\lambda_0) < 0, \forall 2 \leq i \leq n \end{cases} \tag{5.3.5}$$

The following is the equilibrum attractor bifurcation theorem.

Theorem 5.3.1. Under the condition (5.3.5), we also assume that $G(x, \lambda)$ is analytic at $x = 0$, and $x = 0$ is asymptotically stable for the system (5.3.1) with $\lambda = \lambda_0$. Then there exists an open set $U \subset R^n$ with $0 \in U$, such that as $\lambda > \lambda_0$ the system (5.3.1) bifurcates from $x = 0$ exactly two equilibrum points $x_1, x_2 \in U$, and the open set U is decomposed into two open sets U_1 and U_2,

$$\bar{U} = \bar{U}_1 + \bar{U}_2, \quad U_1 \cap U_2 = \phi, \ and \ 0 \in \bar{U}_1 \cap \bar{U}_2$$

with $x_i \in U_i (i = 1, 2)$, such that

$$\lim_{t \to \infty} x(t, \phi) = x_i, \ as \ \phi \in U_i (i = 1, 2).$$

where $x(t, \phi)$ is the solution of (5.3.1) with $x(0, \phi) = \phi$.

Proof. Under an appropriate coordinate transformation, the system (5.3.1) can be rewritten as the following form

$$\begin{cases} \frac{dz}{dt} = \beta_1(\lambda) z + g_1(z, y, \lambda) \\ \frac{dy}{dt} = B_\lambda y + g_2(z, y, \lambda) \end{cases} \tag{5.3.6}$$

where $z \in R^1, y \in R^{n-1}$, and by (5.3.2)

$$g_i(z, y, \lambda) = o(|z|, |y|), i = 1, 2, \forall \lambda \in R \tag{5.3.7}$$

and B_λ a $(n-1)\times(n-1)$ matrix whose eigenvalues are $\beta_j(\lambda)(2\le j\le n)$. By the center manifold theorem (Theorem 5.2.6) there is an one dimensional manifold

$$\sum_\lambda = \{(z, h(z,\lambda))| \quad z\in R^1, \quad h(z,\lambda)\in R^{m-1}, \lambda\in R\}$$

where

$$h(\cdot,\lambda): I \to R^{n-1} \ is\ artitrary\ order\ differentiable,$$

$$(5.3.8) \qquad h(0,\lambda)=0, \quad \frac{dh(0,\lambda)}{dz}=0, \quad I=(-a,a)\ for\ some\ a>0$$

and $\sum_\lambda$ is local invariant for the system (5.3.1).

It is not difficult to see that the topological structure of the orbits of (5.3.6) in $\sum_\lambda$ is equivalent to that of the below equation in $I\subset R$ for all $\lambda\in R$

$$(5.3.9) \qquad \frac{dz}{dt} = \beta_1(\lambda)z + f(z,\lambda), \quad z\in I=(-a,a).$$

where $f(z,\lambda)=g_1(z,h(z,\lambda),\lambda)$. By (5.3.7) and (5.3.8), we have

$$(5.3.10) \qquad \frac{\partial f(0,\lambda)}{\partial z} = 0, \quad \forall\lambda\in R$$

By the assumptions, $x=0$ is asymptotically stable for (5.3.6) at $\lambda=\lambda_0$, therefore $z=0$ is also asymptotically stable for the following equation (note $\beta_1(\lambda_0)=0$),

$$(5.3.11) \qquad \frac{dz}{dt} = f(z,\lambda_0), \quad z\in(-a,a), \quad a>0$$

We need to show that there is a $K\ge 2$, such that

$$(5.3.12) \qquad f(z,\lambda_0) = \alpha z^k + o(|z|^k), \quad \alpha\ne 0$$

By the hypothese, $G(x,\lambda_0)$ is analytic at $x=0$, and so is the vector field $\{g_1(z,y,\lambda_0), g_2(z,y,\lambda_0)\}$, for the sake of simplicity, we drop the sign λ_0. Because $x=0$ is asymptotically stable for (5.3.6) at $\lambda=\lambda_0$, in a neighborhood of $z=0$, we have

$$(5.3.13) \qquad \frac{dy}{dz} = \frac{By+g_2(z,y)}{g_1(z,y)} \ne 0 \quad for\ (z,y)=(z,h(z))$$

If (5.3.12) is not valid, then

$$\frac{d^k g_1(z,h(z))}{dz^k}|_{z=0} = 0, \quad \forall 1\le k<\infty$$

We denote $N(z)$ the class of functions that

$$\frac{\partial^k N(0)}{\partial z^k} = 0, \quad \forall 1 \le k < \infty.$$

From (5.3.13) it follows that

$$Bh(z) + g_2(z, h(z)) = 0 \ mod(N(z))$$

Because $g_2(z, y)$ is analytic, from (5.3.13) we can infer that

$$h(z) = h_1(z) + N(z)$$

where $h_1(z)$ is an analytic function which satisfies

$$h_1(z) + B^{-1} g_2(z, h_1(z)) = 0 \tag{5.3.14}$$

((5.3.14) can be ensured by the implicity theorem). On the other hand we have

$$\begin{aligned} g_1(z, h(z)) &= g_1(z, h_1(z) + N(z)) \\ &= g_1(z, h_1(z)), \ mod(N(z)) \\ &= 0, \ mod(N(z)) \end{aligned}$$

Since $g_1(z, y)$ and $y = h_1(z)$ are analytic, the function $g_1(z, h_1(z))$ is also analytic. Hence we have

$$g_1(z, h_1(z)) = 0 \tag{5.3.15}$$

The analytic function satisfying (5.3.14) and (5.3.15) does not exist. Otherwise $(z(t), y(t)) = (z_0, h_1(z_0))$ satisfies

$$\begin{cases} \frac{dz}{dt} = g_1(z, y) \\ \frac{dy}{dt} = By + g_2(z, y) \\ z(0) = z_0, \quad y(0) = h_1(z_0) \end{cases}$$

$\forall z_0 \in (-a, a)$. It is a contradiction with that $(z, y) = (0, 0)$ is asymptotically stable for (5.3.6). Thus we obtain the equality (5.3.12).

By the asymptotic stability of $z = 0$ for (5.3.11), from (5.3.12) it follows that

$$\begin{cases} k = \text{ an add number} \\ \alpha < 0 \end{cases}$$

which implies that the equation (5.3.9) bifurcates from $z = 0$ exactly two asymptotically stable equilibrum point $z_1, z_2 \in I = (-a, a)$ for $\lambda - \lambda_0 > 0$ sufficiently small, and $z = 0$ is unstable. Hence, by Theorem 5.2.7, the two points $x_i = \{z_i, h(z_i, \lambda)\}(i = 1, 2)$ are asymptotically stable singular points of (5.3.6) because $\sum_\lambda$ are positive invariant for all $\lambda - \lambda_0$ sufficiently small, therefore are attracting.

It is easy to see, by the stable manifold theorem that as $\lambda < \lambda_0, x = 0$ has a $(n-1)$-dimensional stable manifold M_s^{n-1} and an one-dimensional unstable manifold M_u^1. Obviously, M_s^{n-1} divides the open set U into two parts U_1 and U_2 with $x_i \in U_i$, and x_i are attracting in $U_i (i = 1, 2)$, for instance see Fig. 5.3. The proof is complete.

If the vector field $G(x, \lambda)$ is not analytic, in general Theorem 5.3.1 is not valid. But, we can still obtain some criterion for the equilibrum attractor bifurcation of (5.3.1).

First of all, we recall some simple properties of matrix. Let A be a n-order matrix, A^* its conjugate matrix. We know that A^* and A have the same eigenvalues. Let $\beta_j (1 \leq j \leq n)$ be all the eigenvalues of A(counting multiplicities). We say the vectors $\xi_j \in R^n (1 \leq j \leq n)$ are eigenvectors of A, if they satisfy

$$\begin{aligned} &(A - \beta_j I)^{k_j} \xi_j = 0, \quad \text{as } \beta_j \text{ are real numbers} \\ &(A - \beta_j I)^{k_j} (\xi_j + i\xi_{j+1}) = 0 \\ &(A - \beta_{j+1} I)^{k_j} (\xi_j - i\xi_{j+1}) = 0 \quad \text{as } \beta_{j+1} = \bar{\beta}_j \text{ are complex numbers} \end{aligned}$$

for some $1 \leq k_j \leq m_j, m_j$ the multiplicity of β_j.

Let $\xi_j = \{\xi_{j_1}, \cdots, \xi_{j_n}\}$ and $\xi_k^* = \{\xi_{k_1}^*, \cdots, \xi_{k_n}^*\}$ be respectively the eigenvectors of A and A^*, and

$$P = \begin{pmatrix} \xi_{11} & \xi_{21} & \cdots & \xi_{n1} \\ \xi_{12} & \xi_{22} & \cdots & \xi_{n2} \\ \vdots & \vdots & & \vdots \\ \xi_{1n} & \xi_{2n} & \cdots & \xi_{nn} \end{pmatrix}, \quad P^* = \begin{pmatrix} \xi_{11}^* & \xi_{12}^* & \cdots & \xi_{1n}^* \\ \xi_{21}^* & \xi_{22}^* & \cdots & \xi_{2n}^* \\ \vdots & & & \vdots \\ \xi_{n1}^* & \xi_{n2}^* & \cdots & \xi_{nn}^* \end{pmatrix}$$

Then we have the properties as follows

$$\begin{cases} P^* = P^{-1} \\ P^* A P = J \end{cases}$$

where J is the Jordan form of A, and ξ_j^*, ξ_j are corresponding to the same eigenvalue β_j. If

$$\beta_{j+1} = \beta_{j+2} = \cdots = \beta_{j+m_j} = \beta$$

then the eigenvectors ξ_{j+l} and $\xi_{j+l}^* (1 \leq l \leq m_j)$ satisfy

$$(A - \beta I)\xi_{j+l} = \xi_{j+l+1}, \quad \text{as } l < m_j, \text{ and } (A - \beta I)\xi_{j+m_j} = 0$$

$$(A^* - \beta I)\xi_{j+l}^* = \xi_{j+l-1}^*, \quad \text{as } l > 1, \text{ and } (A^* - \beta I)\xi_{j+1}^* = 0$$

The properties are in fact the anather expression of the Jordan theorem and the Fredholm's alternative theorem.

Now we return to continue our discusion on the bifurcation of (5.3.1). Let ξ_j and $\xi_j^*(1 \le j \le n)$ be respectively the eigenvectors of A_λ and A_λ^* at $\lambda = \lambda_0$. We take the coordinate transformation

$$\begin{cases} x = P\widetilde{x}, \quad \widetilde{x} = \{z_1, \cdots, z_n\} \\ x = z_1\xi_1 + \cdots + z_n\xi_n \end{cases}$$

Under the transformation above, the equation (5.3.1) at $\lambda = \lambda_0$ can be written as to read

$$\begin{cases} \frac{dz}{dt} = \widetilde{G}_1(z, y) \\ \frac{dy}{dt} = Ly + \widetilde{G}_2(z, y) \end{cases} \tag{5.3.16}$$

where $\{z, y\} = \widetilde{x}, z = z_1, y = \{z_2, \cdots, z_n\}, L$ is the $(n-1)$-order submatrix of the Jordon form of A_{λ_0} corresponding to the eigenvalues $\beta_j(\lambda_0)(2 \le j \le n)$, and

$$\widetilde{G}_1(z, y) =< G(x(z, y)), \xi_1^* >$$

$$\widetilde{G}_2(z, y) =< G(x(z, y)), \xi^* >, \quad \xi^* = (\xi_2^*, \cdots, \xi_n^*)$$

Suppose that the functions $\widetilde{G}_1$ and $\widetilde{G}_2$ have the expansions

$$\begin{cases} \widetilde{G}_1(z, y) = \rho_1 z^{k_1} + o(|z|^{k_1}) + yg_1(z, y), & \rho_1 \ne 0 \\ \widetilde{G}_2(z, y) = \rho_2 z^{k_2} + o(|z|^{k_2}) + yg_2(z, y), & \rho_2 \in R^{n-1} \end{cases} \tag{5.3.17}$$

where $1 < k_1 < \infty, 1 < k_2 \le \infty$(as $k_2 = \infty$, it means that $\widetilde{G}_2 = yg_2(z, y) mod(N(z))$), and $g_i(0, 0) = 0 (i = 1, 2)$. Then we have

Theorem 5.3.2. Under the conditions (5.3.5) and (5.3.17), if $k_1 \le k_2$ in (5.3.17),and $x = 0$ is asymptotically stable for (5.3.1) at $\lambda = \lambda_0$, then the conclusion of Theorem 5.3.1 holds true.

Proof. We only need to show that the function $f(z) = \widetilde{G}_1(z, h(z))$ has the expansion (5.3.12). By (5.3.16), in a neighborhood of $z = 0$ we have

$$\begin{aligned} \frac{dy}{dz} &= \frac{Ly + \widetilde{G}_2(z, y)}{\widetilde{G}_1(z, y)} \\ &= \frac{Ly + \rho_2 z^{k_2} + yg_2(z, y) + o(|z|^{k_2})}{\rho_1 z^{k_1} + yg_1(z, y) + o(|z|^{k_1})} \end{aligned}$$

where $y = h(z)$. By the fact that

$$\frac{dy}{dz}|_{z=0} = \frac{dh(0)}{dz} = 0$$

as $k_1 \leq k_2 (\rho_1 \neq 0)$, we can infer that

$$h(z) \sim z^{2k_1 - 1} \; as \; z \to 0$$

By $k_1 > 1$, it implies that

$$f(z) = \widetilde{G}_1(z, h(z)) = \rho_1 z^{k_1} + o(|z|^{k_1}), \quad \rho_1 \neq 0$$

Thus the expansion (5.3.12) is derived. The proof is complete.

5.3.2. Bifurcation of the higher dimensional attractors

Naturally, the bifurcation theorem of equilibrum attractors(Theorem 5.3.1) suggests us to investigate further the dynamic bifurcation of the higher dimensional attractors. To this end, we here give the difinition of bifurcation of the invariant sets.

Definition 5.3.3. We say that the system (5.3.1) bifurcates from $(x, \lambda) = (0, \lambda_0)$ an invariant set $\sum$, if there exists a sequence $\{\sum_{\lambda_n}\}$ of invariant sets of (5.3.1), $o \notin \sum_{\lambda_n}$ with $\sum_{\lambda_n}$ homeomorphic to $\sum$, such that

$$\begin{cases} \lim_{n\to\infty} \lambda_n = \lambda_0 \\ \lim_{n\to\infty} d(\sum_{\lambda_n}, 0) = \lim_{n\to\infty} \max_{x\in\Sigma_{\lambda_n}} |x| = 0 \end{cases}$$

Suppose that the eigenvalues (5.3.4) satisfy

$$(5.3.18) \quad \begin{cases} Re\beta_i(\lambda) = \begin{cases} < 0, & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0. \end{cases} & (1 \leq i \leq m) \\ Re\beta_j(\lambda_0) < 0, \quad \forall m+1 \leq j \leq n \quad (\; if \; n > m) \end{cases}$$

where $1 \leq m \leq n$.

The following is the main theorem in this section.

Theorem 5.3.4. Under the condition (5.3.18), if $x = 0$ is asymptotically stable for (5.3.1) at $\lambda = \lambda_0$, then the following assertions hold true for $\lambda - \lambda_0 > 0$ sufficiently small.

i). The system (5.3.1) bifurcates from $x = 0$ an attractor $\sum_\lambda$ with $m - 1 \leq dim \sum_\lambda \leq m$, which is connected as $m > 1$.

ii). $\sum_\lambda$ is the limit of a family of manifolds $M_\tau (0 \leq \tau < \infty)$:

$$\sum_\lambda = \bigcap_{\tau \geq 0} M_\tau, \quad M_{\tau_2} \subset M_{\tau_1}, \quad \forall \tau_2 \geq \tau_1$$

where M_τ possesses the hometopy type of m-annulus for all $\tau \geq 0$.

iii). If $\sum_\lambda$ is a finit simplicial complex, then $\sum_\lambda$ is a deformation retract of a manifold having the homotopy type of m-annulus. Therefore, $\sum_\lambda$ have the homotopy type of S^{m-1}, (if $m=1, \sum_\lambda$ is homotopicaly equivalent to two distinct points).

iv). If the equilibrum points of (5.3.1) in $\sum_\lambda$ are finite, then the following index formula is valid

$$\sum_{x_i\in\sum_\lambda} ind[-(A_\lambda+G),x_i]=\begin{cases} 2, & m=\ odd \\ 0, & m=\ even\end{cases}$$

In order to prove this theorem, we need the following technical lemmas, the first one of which is also useful in the orbit analysis of vector fields for the other purposes.

Let $F(\cdot,\lambda)\in C^r(\Omega,R^n)(r\geq 1)$ be a family of vector fields, $\Omega\subset R^n$ an open set, and $F(x,\lambda)$ continuously depends on $\lambda\in R$.

We say that $\Gamma\subset R^n$ is an orbit curve of a vector field F if Γ is an union of the orbits $\gamma_i(i=1,2,\cdots)$ of F, i.e. $\Gamma=\bigcup_i\bar{\gamma}_i$, and they are connected in order each other (see Fig. 5.8 below) in this way that the end point of γ_i is the starting point of γ_{i+1}. Each of γ_i or consists of all singular points of F, which is called the singular orbit, or an orbit connected by the singular points of F, which is called the non-singular orbit.

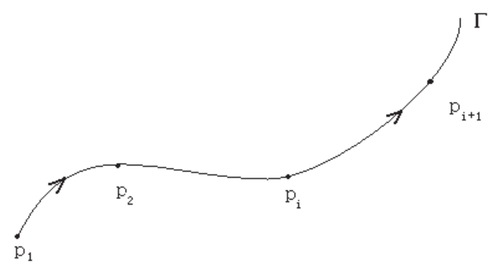


Fig. 5.8. an orbit curve Γ

According to the definition, in an orbit curve Γ, if γ_i and γ_{i+1} are the non-singular orbits, then the limit sets of γ_i and γ_{i+1} satisfy that

$$\omega(\gamma_i)=\alpha(\gamma_{i+1}).$$

The starting point p_1 of Γ need not a singular point.

Lemma 5.3.5. Let Γ_λ be an orbit curve of $F(x,\lambda)$ with the starting point p_λ. If $p_\lambda\to p_0(\lambda\to 0)$, then Γ_λ converge to an orbit curve Γ_0 of $F(x,0)$ with the starting point p_0.

Proof. Because $F(x,\lambda)$ is continuous on λ, it is easy to see that if the singular points z_λ of $F(x,\lambda)$ converge to z_0 as $\lambda \to 0$, then z_0 must be a singular point of $F(x,0)$. Hence it sufficies to prove this lemma only for the case that $\{\Gamma_\lambda\}$ is a sequence of the non-singular orbits with bounded length (if Γ_λ are the orbits possessing infinite length, then we can divid Γ_λ into countable segments with finite length $l_i^\lambda (i = 1, 2, \cdots)$, and then proceed for each l_i^λ in the same fashion).

Let $\Gamma_\lambda = \bar{\gamma}_\lambda$ be a closure of a non-singular orbit γ_λ of $F(x,\lambda)$, with the starting point p_λ and the end point q_λ, where $\{q_\lambda\} \subset R^n$ is bounded. By the definition, q_λ is a singular point of $F(x,\lambda)$. Without loss of generality, we assume that the starting points p_λ of Γ_λ are the singular points, then Γ_λ is a complete orbit $x_\lambda(t)$ of $F(x,\lambda)$ satisfiying

$$\begin{cases} \frac{dx_\lambda}{dt} = F(x_\lambda, \lambda) \\ x_\lambda(t) \to p_\lambda, t \to -\infty \\ x_\lambda(t) \to q_\lambda, t \to +\infty \end{cases} \tag{5.3.19}$$

If we define

$$x_\lambda(-\infty) = p_\lambda, \quad x_\lambda(\infty) = q_\lambda$$

the the closed segment Γ_λ is parameterized by

$$\Gamma_\lambda = \{x_\lambda(t) | t \in [-\infty, \infty]\}.$$

In order to avoid the infinity interval, let

$$t = tg\tau$$

Then in new variable τ, (5.3.19) becomes

$$\frac{dy_\lambda(\tau)}{d\tau} = F(y_\lambda, \lambda)\frac{dtg\tau}{d\tau}, \quad \tau \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$y_\lambda(\tau) = p_\lambda, \quad \tau \to -\frac{\pi}{2}$$

$$y_\lambda(\tau) = q_\lambda, \quad \tau \to \frac{\pi}{2}$$

$$y_\lambda(\tau) = x_\lambda(tg\tau) \tag{5.3.20}$$

Obviously, the solution of (5.3.20) satisfies

$$y_\lambda(\tau) = p_\lambda + \int_{-\frac{\pi}{2}}^{\tau} F(y_\lambda, \lambda)\frac{dtg\tau}{d\tau} d\tau \tag{5.3.21}$$

By (5.3.21) we have

$$\begin{aligned}
|y_\lambda(\tau_1) - y_\lambda(\tau_2)| &= |\int_{\tau_1}^{\tau_2} F(y_\lambda, \lambda)\frac{dtg\tau}{d\tau}d\tau| \\
&= (t_i = tg\tau_i) \\
&\le \int_{tg^{-1}t_1}^{tg^{-1}t_2} |F(x_\lambda, \lambda)|dt \\
&\le C|tg^{-1}t_2 - tg^{-1}t_1| \\
&\le C|\tau_2 - \tau_1|
\end{aligned}$$

Namely $y_\lambda : [-\frac{\pi}{2}, \frac{\pi}{2}] \to R^n$ are equicontinuous. By the Arzela-Ascoli theorem, $\{x_\lambda(tg)\}$ has a convergent subsequence for $\lambda \to 0$, still denote it by $\{x_\lambda(tg)\}$:

$$x_\lambda(tg\tau) \to x_0(tg\tau) \quad inC[-\frac{\pi}{2}, \frac{\pi}{2}] \; as \; \lambda \to 0$$

$$x_0(tg) \in C[-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$x_0(tg(-\frac{\pi}{2})) = p_0$$

$$x_0(tg(\frac{\pi}{2})) = q_0$$

It is then easy to see that $x_0(t)$ can be consided as a continuous curve in R^n connecting p_0 and q_0, and defined on $(-\infty, \infty)$ such that

$$\begin{cases} x_0(t) \to p_0, \; as \; t \to -\infty \\ x_0(t) \to q_0, \; as \; t \to \infty \end{cases}$$

In fact, $x_0(t)$ is an union of complete orbits of $F(x, 0)$. To see this we infer from (5.3.19) that

$$x_\lambda(t) = x_\lambda(t_1) + \int_{t_1}^{t} F(x_\lambda, \lambda)dt$$

Passing to the limit $\lambda \to 0$, we obtain

$$x_0(t) = x_0(t_1) + \int_{t_1}^{t} F(x_0, 0)dt$$

Hence

$$\frac{dx_0}{dt} = F(x_0, 0), \quad \forall t \in (-\infty, \infty).$$

Consequently, $\Gamma_0 = \{x_0(t)|t \in [-\infty, \infty]\}$ is an orbit curve of $F(x, 0)$, which is the limit of $\{\Gamma_\lambda\}$ for $\lambda \to 0$. The proof is complete.

Lemma 5.3.6. Suppose that $x = 0$ is an asymptotically stable singular point of $F(x,0)$, then there is an open set $\Omega \subset R^n, 0 \in \Omega$, such that the ω-limit sets of Ω satisfy

$$\left\{ \begin{array}{l} \lim_{\lambda \to 0} d(\omega_\lambda(\Omega), 0) = 0 \\ d(\omega_\lambda(\Omega), 0) = \sup_{x \in \omega_\lambda(\Omega)} |x| \\ \omega_\lambda(\Omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S_\lambda(t)\Omega}. \end{array} \right. \tag{5.3.22}$$

where $S_\lambda(t)$ is the operator semigroup generated by $F(x,\lambda)$.

Proof. Because $S_\lambda(t) : R^n \to R^n$ is a homeomorphism for all $t \in R, S_\lambda(t)\Omega$ is an open set and $\partial[S_\lambda(t)\Omega] = [S_\lambda(t)\partial\Omega]$. Hence

$$\partial[\bigcup_{t \geq \tau} S_\lambda(t)\Omega] \subset \overline{\bigcup_{t \geq \tau} S_\lambda(t)\partial\Omega}$$

Therefore, to prove (5.3.22) it is sufficient to verify that

$$\lim_{\lambda \to 0} d(\omega_\lambda(\partial\Omega), 0) = 0 \tag{5.3.23}$$

Assuming that the equality is false, we shall deduce a contradiction. We divide this proof into two cases.

Case a). The distances between $\omega_\lambda(\partial\Omega)$ and $x = 0$ satisfy

$$\lim_{\lambda \to 0} dist(\omega_\lambda(\partial\Omega), 0) = \lim_{\lambda \to 0} \inf_{x \in \omega_\lambda(\partial\Omega)} |x| = 0 \tag{5.2.24}$$

By (5.2.24), there is a sequence of $p_\lambda \in \omega_\lambda(\partial\Omega)$ such that $p_\lambda \to 0$(as $\lambda \to 0$). Because (5.3.23) is not true, and $\omega_\lambda(\partial\Omega)$ is a connected and invariant set of $F(x,\lambda)$(by Lemma 5.2.3), there is a number $\delta > 0$ and an orbit curve $\Gamma_\lambda \subset \omega_\lambda(\partial\Omega)$ of $F(x,\lambda)$ with the starting point p_λ, such that

$$\left\{ \begin{array}{l} \Gamma_\lambda \bigcap \partial B_\delta(0) \neq \phi, \quad \forall \lambda \neq 0, \\ B_\delta(0) = \{x \in R^n | |x| < \delta\} \end{array} \right. \tag{5.3.25}$$

On the other hand, by Lemma 5.3.5, we have

$$\Gamma_\lambda \to \Gamma_0, \ as \ \lambda \to 0$$

where Γ_0 is an orbit curve of $F(x,0)$ with the starting point $x = 0$, and from (5.3.25) it follows that

$$\Gamma_0 \bigcap \partial B_\delta(0) \neq \phi$$

Thus, we read a contradiction with the assumption that $x = 0$ is an asymptotically stable singular point of $F(x,0)$.

Case b) There is a number $\delta > 0$, such that

$$dist(\omega_\lambda(\partial\Omega), 0) \geq \delta > 0, \quad \forall\lambda \neq 0 \tag{5.3.26}$$

If $\omega_\lambda(\partial\Omega)\bigcap\Omega \neq \phi$, we take a point $p_\lambda \in \omega(\partial\Omega)\bigcap\Omega$ and an orbit curve $\Gamma_\lambda \subset \omega_\lambda(\partial\Omega)$ with the starting point p_λ. By (5.3.25) we have

$$dist(\Gamma_\lambda, 0) \geq \delta > 0 \tag{5.3.27}$$

Because $\{p_\lambda\}$ is bounded, let $p_\lambda \to p_0 \in \bar{\Omega}(\lambda \to 0)$, then the orbit curves Γ_λ, by Lemma 5.3.5, converge to Γ_0, an orbit curve of $F(x, 0)$, with the starting point $p_0 \in \bar{\Omega}$. From (5.3.27) it follows

$$dist(\Gamma_0, 0) \geq \delta > 0$$

which is a contradiction with that the solution $u(t, p_0)$ of $F(x, 0)$ satisfies that $\lim_{t\to\infty} u(t, p_0) = 0, \forall p_0 \in \bar{\Omega}$ (we take $\bar{\Omega} \subset R^n$ in the attracted basin of $x = 0$).

If $\omega_\lambda(\partial\Omega)\bigcap\Omega = \phi$, then we take

$$\Gamma_\lambda = \{x_\lambda(t)| \quad \frac{dx_\lambda}{dt} = F(x_\lambda, \quad \lambda), x_\lambda(0) = \psi \in \partial\Omega, \quad 0 \leq t < \infty\}$$

Hence, the ω-limit sets

$$\omega(\Gamma_\lambda)\bigcap\Omega = \psi \tag{5.3.28}$$

On the other hand, $\Gamma_\lambda \to \Gamma_0(\lambda \to 0)$, an orbit curve of $F(x, 0)$ with the end point of Γ_0 is $x = 0$, a contradiction with (5.3.28). Thus, this lemma is proven.

Now, we return to prove Theorem 5.3.4.

The Proof of Theorem 5.3.4. Under a proper coordinate transformation, the system (5.3.1) can be rewritten as

$$\begin{cases} \frac{dx}{dt} = B_\lambda x + g_1(x, y, \lambda) \\ \frac{dy}{dt} = C_\lambda x + g_2(x, y, \lambda) \end{cases} \tag{5.3.29}$$

where $x \in R^m, y \in R^{n-m}, B_\lambda$ is the $m \times m$ matrix with the eigenvalues $\beta_1(\lambda), \cdots, \beta_m(\lambda), C_\lambda$ is the $(n-m) \times (n-m)$ matrix with the eigenvalues $\beta_{m+1}(\lambda), \cdots, \beta_n(\lambda)$, and

$$g_i(x, y, \lambda) = o(|x|, |y|), \quad \forall\lambda \in R, (i = 1, 2) \tag{5.3.30}$$

For the sake of simplicity, we assume that $\lambda_0 = 0$, i.e.

$$\beta_i(\lambda) = \begin{cases} < 0, & \lambda < 0 \\ = 0, & \lambda = 0 \\ > 0, & \lambda > 0 \end{cases} \quad (1 \leq i \leq m)$$

$$\beta_j(0) < 0, \quad \forall m+1 \leq j \leq m.$$

Let $h(x, \lambda)$ be the function defined as in the center manifold theorem (Theorem 5.2.6), and $M_\lambda = \{(x, y)|y = h(x, \lambda), x \in \Omega \subset R^m\}$ be a center manifold of (5.3.29). It is known that the topological structure of the orbits of (5.3.29) in M_λ is equivalent to that of the below system in $\Omega \subset R^m$

$$\frac{dx}{dt} = B_\lambda x + f(x, \lambda), \quad x \in \Omega \subset R^m \tag{5.3.31}$$

where

$$f(x, \lambda) = g_1(x, h(x, \lambda), \lambda)$$

While, by (5.3.30) and the property i) in Theorem 5.2.5, we have

$$f(x, \lambda) = o(|x|), \quad \forall \lambda \in R.$$

By the assumption, $z = 0 (z = (x, y))$ is asymptotically stable for (5.3.29) at $\lambda = 0$, therefore, $x = 0$ is also asymptotically stable for the following system

$$\frac{dx}{dt} = B_0 x + f(x, 0), \quad x \in \Omega \subset R^m$$

Let $F(x, \lambda) = B_\lambda x + f(x, \lambda)$. Then $F(x, \lambda)$ is continuous on λ, and $x = 0$ is an asymptotically stable singular point of $F(x, 0)$. By Lemma 5.3.6, there exist an open set $\Omega_0 \subset R^m, 0 \in \Omega_0 \subset \Omega$, and constants $r, \lambda_1 > 0$, such that $\mathcal{B}_r = \{x \in R^m||x| < r\}$ is an obsorbing set of $F(x, \lambda)$ in Ω_0 for all $|\lambda| < \lambda_1$. Therefore, from Lemma 5.2.5 it follows that

$$\mathcal{A}_\lambda = \omega_\lambda(\bar{\mathcal{B}}_r) \subset \mathcal{B}_r, \quad \forall |\lambda| < \lambda_1 \tag{5.3.32}$$

is an attractor of (5.3.31) in $\Omega_0 \subset R^m$.

In addition, by the global stable manifold theorem (Theorem 5.2.9) we have the global unstable manifold M^u_λ at $x = 0$ of (5.3.31) in R^m for all $0 < \lambda < \lambda_1$, and

$$M^u_\lambda = I(R^m, \lambda) \subset R^m$$

where

$$I(\cdot, \lambda) : R^m \to R^m \quad (I(0, \lambda) = 0)$$

is an injective immersion. From (5.3.32) we can see that

$$M^u_\lambda \subset \mathcal{A}_\lambda \subset \mathcal{B}_r, \quad \forall 0 < \lambda < \lambda_1$$

which implies that M^u_λ is homeomorphic to a m-dimensional open disk(notice that the closure of M^u_λ may not be homeomorphic to a m-dimensional closed disk, because $M^u_\lambda \subset R^m$ is an immersion. Hence we can claim that

$$\begin{cases} \sum_\lambda = \mathcal{A}_\lambda / M^u_\lambda \ \textit{is an attractor of}\ (5.3.29)\ \textit{for}\ 0 < \lambda < \lambda_1, \\ m - 1 \leq \textit{dim} \sum_\lambda \leq m;\ \textit{and} \\ \lim_{\lambda \to 0^+} d(\sum_\lambda, 0) = 0, \quad (\textit{by}\ (5.3.22)) \end{cases}$$

Thus, we obtain the conclusion i).

In the following, we prove the conclusion ii). We denote

$$D_\tau = \overline{\bigcup_{t\geq\tau} S_\lambda(t)\bar{\mathcal{B}}_r} \quad (\tau \geq 0)$$

It is clear that D_τ is shrunk for $\tau \geq 0$, or

(5.3.33) $$D_{\tau_1} \subset D_{\tau_2} \ as \ \tau_1 > \tau_2 \geq 0$$

By the semigroup properties of $S_\lambda(t)$, it is easy to see that

(5.3.34) $$\begin{cases} D_\tau = S_\lambda(\tau)D_0 & \forall \tau \geq 0 \\ D_0 = \overline{\bigcup_{t\geq 0} S_\lambda(t)\bar{\mathcal{B}}_r} \end{cases}$$

Obviously, D_τ is homeomorphic to $D_0, \forall \tau \geq 0$, and $\bar{\mathcal{B}}_r \subset D_0$. We divide the proof into the following several steps.

Claim 1. D_0 is a closure of an open set.

This is easily seen by that

$$D_0 = \overline{\bigcup_{t\geq 0} S_\lambda(t)\bar{\mathcal{B}}_r} = \overline{\bigcup_{t\geq 0} S_\lambda(t)\mathcal{B}_r}$$

and $\bigcup_{t\geq 0} S_\lambda(t)\mathcal{B}_r$ is an open set because $S_\lambda(t)\mathcal{B}_r$ are open sets for all $t \geq 0$.

Claim 2. D_0 is a m-dimensional manifold with boundary.

Suppose that the claim is false, then there exists a point $x \in \partial D_0$ such that, for any neighborhood O of x in $R^m, D_0 \bigcap O$ is not homeomorphic to $R^m_+ = \{(x_1, \cdots, x_m) \in R^m | x_m > 0\}$, or

(3.3.35) $$\begin{cases} D_0 \bigcap \bar{O} = \bar{O}_1 + \bar{O}_2, & O_1 \ and \ O_2 \ are \ open \ sets, and \\ O_1 \bigcap O_2 = \phi, & x \in \bar{O}_1 \bigcap \bar{O}_2 \end{cases}$$

which can be shown as in Fig. 5.9 below

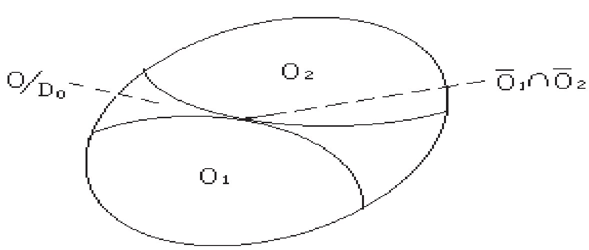


Fig. 5.9.

From (5.3.35) we can see that

$$(5.3.36) \qquad \begin{cases} O/D_0 \neq \phi \textit{ is an open set}, \bar{O}_1 \bigcap \bar{O}_2 \neq \phi, \textit{ and} \\ \bar{O}_1 \bigcap \bar{O}_2 \subset \partial(O/D_0) = \partial D_0 \bigcap O. \end{cases}$$

Because $S_\lambda(\tau) : R^m \to R^m$ is a homeomorphism $\forall \tau \in R$, the case that $S_\lambda(\tau)\bar{O}_1 \subset O_1$ and $S_\lambda(\tau)\bar{O}_2 \subset O_2$ for some $\tau \in R$ does not occur, see Fig 5.10(a). Hence, from (5.3.33) and (5.3.34) we can deduce that there is at least a point $x_0 \in \bar{O}_1 \bigcap \bar{O}_2$ such that

$$(5.3.37) \qquad S_\lambda(\tau)x_0 \in \bar{O}_1 \bigcap \bar{O}_2, \quad \forall \tau \geq 0$$

Otherwise, from (5.3.36) we can see that there are $\tau_0 > 0$ and points $y \in O_1$(or $y \in O_2$), such that

$$S_\lambda(\tau_0)y \in O/D_0$$

see Fig. 5.10(b), which is a contradiction with (5.3.33).

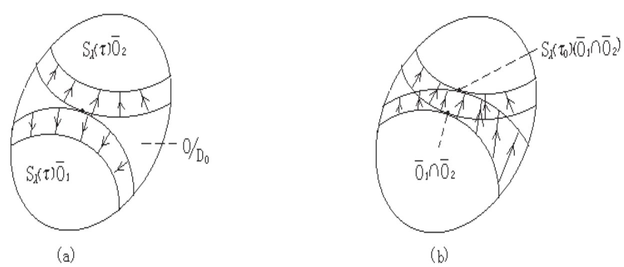


Fig 5.10.

Because $\bar{O}_1 \bigcap \bar{O}_2 \bar{\in} \mathcal{B}_r$, from (5.3.37) we can get that

$$d(S_\lambda(\tau)(\bar{O}_1 \bigcap \bar{O}_2), 0) \geq r > 0, \quad \forall \lambda, \tau \geq 0$$

Thus, we read a contradiction with that

$$\lim_{\tau \to \infty \lambda \to 0} d(S_\lambda(\tau)D_0, 0) = 0 \quad (\textit{by } (5.3.22))$$

Claim 3. D_0 is homotopically equivalent to a disk.

We know that $\bar{\mathcal{B}}_r$ is a m-disk, and $\bar{\mathcal{B}}_r \subset D_0$. We shall show that D_0 and $\bar{\mathcal{B}}_r$ have the same homotopy type. By (5.3.32), there is a $T > 0$ such that

$$D_T = S_\lambda(T)D_0 \subset \bar{\mathcal{B}}_r$$

For the two mappings

$$h : D_0 \to \bar{\mathcal{B}}_r \; defined \; as \; h(x) = S_\lambda(T)x$$

$$i : \bar{\mathcal{B}}_r \to D_0 \; an \; inclusion \; mapping$$

we define the homotopies

$$H : D_0 \times [0, T] \to D_0$$

$$H \circ i : \bar{\mathcal{B}}_r \times [0, T] \to \bar{\mathcal{B}}_r$$

where the homotopy H is defined by

$$H(x, t) = S_\lambda(t)x, \quad \forall x \in D_0, t \in [0, T].$$

It is easy to see that the two homotopies H and $H \circ i$ give that

$$id \simeq i \circ h : D_0 \to D_0$$

$$id \simeq h \circ i : \bar{\mathcal{B}}_r \to \bar{\mathcal{B}}_r$$

Hence, D_0 and $\bar{\mathcal{B}}_r$ are homotopy equivalence.

Claim 4. $\sum_\lambda$ is the limit of a family of manifolds having the homotopy type of a m-annulus.

By (5.3.33), we have

$$\left\{ \begin{array}{l} \lim_{\tau \to \infty} D_\tau = \mathcal{A}_\lambda \\ M_\lambda^u \subset \mathcal{A}_\lambda \subset D_\tau, \forall \tau \geq 0 \end{array} \right. \tag{5.3.38}$$

Because $M_\lambda^u = I(R^m, \lambda)$ is bounded, $I(\cdot, \lambda) : R^m \to R^m$ is an injective immersion, for any $\mathcal{B}_\rho = \{x \in R^m | |x| < \rho\} (0 < \rho)$,

$$B_\rho = I(\mathcal{B}_\rho, \lambda), \quad 0 < \rho < \infty$$

is an embedding open m-disk, and

$$\left\{ \begin{array}{l} B_{\rho_1} \subset B_{\rho_2} \; as \; \rho_2 < \rho_1 \\ d(\partial B_\rho, \partial M_\lambda^u) = \max_{x \in \partial B_\rho} dist(x, \partial M_\lambda^u) \to 0, (\rho \to \infty). \end{array} \right. \tag{5.3.39}$$

From (5.3.8) and (5.3.39) we can obtain

$$\left\{ \begin{array}{l} \sum_\lambda \subset D_{\tau_2} \backslash B_{\rho_2} \subset D_{\tau_1} \backslash B_{\rho_1}, \quad \forall \tau_2 \geq \tau_1, \rho_2 \geq \rho_1 \\ D_\tau \backslash B_\rho \to \sum_\lambda = \mathcal{A}_\lambda \backslash M_\lambda^u, \quad \tau \to \infty, \rho \to \infty \end{array} \right. \tag{5.3.40}$$

Obviously, $D_\tau \backslash B_\rho$ is homotopically equivalent to a m-annulus for all $\tau \geq 0, \rho > 0$. The conclusion ii) is verified.

We are now in a position to prove the conclusion iii). We need to show that $\sum_\lambda$ is a deformation retract of $D_0 \backslash B_1, (B_1 = I(\mathcal{B}_1, \lambda))$.

It is easy to see that any embedding submanifold $M \subset D_0 \backslash B_1$, which is homeomorphic to $D_0 \backslash B_1$, is a deformation retract of $D_0 \backslash B_1$ provided $\sum_\lambda \subset M$.

Let M be a smooth manifold with boundary. For each point $x \in \partial M$ we denote

$$Z(x, \lambda) = \text{ the point } Z \in M, \text{ which lies on the inward}$$

$$\text{normal line starting from } x, \text{ and the arc length}$$

$$\text{from } z \text{ to } x \text{ is } \lambda (\lambda \geq 0).$$

Obviously, $z(x, 0) = x$.

By (5.3.40), we can take a sequence of smooth submanifolds $\{M_n\}$ of $D_0 \backslash B_1$, such that

$$\left\{ \begin{array}{l} \sum_\lambda \subset M_{n+1} \subset M_n \subset D_0 \backslash B_1, \forall n \geq 1 \\ M_n \text{ homeomorphic to } D_0 \backslash B_1, \text{ and} \\ \lim_{n \to \infty} M_n = \sum_\lambda \end{array} \right. \tag{5.3.41}$$

Moreover, the sequence $\{M_n\}$ possesses the properties that, for any point $x \in \partial M_n$, there exists a number $\lambda_n(x) \geq 0$ such that for all $x, y \in \partial M_n, x \neq y$, the line segment

$$l_x = \{z(x, \lambda) | 0 \leq \lambda \leq \lambda_n(x)\}$$

does not intersect with the line segment l_y, moreover, the points in the line segment $l_x (x \in \partial M_n)$ satisfy

$$z(x, \lambda) \bar{\in} M_{n+1}, \quad \forall 0 \leq \lambda < \lambda_n(x), \text{ as } \lambda_n(x) > 0; \text{ and}$$

$$z(x, \lambda_n(x)) \in \partial M_{n+1} \qquad (if\ \lambda_n(x) = 0, \text{ then } x \in \partial M_n \bigcap \partial M_{n+1})$$

The properties can be ensured by the procedure that letting the smooth manifold $M_1 \subset D_0 \backslash B_1$ shrinks along its inward normal direction, then we intercept the manifold M_2, and again, from M_2 we get M_3 in the same fashion, and so on.

Thus, $\forall x \in \partial M_1$, we can define a curve

$$L_x = \bigcup_{n=1}^{\infty} l_{x_n}, \quad x_1 = x, \quad x_{n+1} = z(x_n, \lambda_n(x_n)).$$

Namely L_x is the union of the line segments l_{x_n}, in where the end point x_{n+1} of l_{x_n} is the starting point of $l_{x_{n+1}}$. Because $\sum_\lambda$ is a finite simplicial complex, the length of L_x is finite for all $x \in \partial M_1$, otherwise the number of simpleces in $\sum_\lambda$ can not be finite. It is not difficult to see that

$$L_x \bigcap L_y = \phi, \forall x \neq y, x, y \in \partial M_1$$

$$L_x \bigcap \sum_\lambda = \phi, \forall x \in \partial M_1$$

and by (5.3.41), the end point q_x of $L_x = \bigcup_{n=1}^{\infty} l_{x_n}$ satisfies

$$\lim_{n\to\infty} y_n = q_x \in \Sigma_\lambda, (y_n \in l_{x_n}).$$

The property is guaranted by the finite length of L_x. On the other hand, we can see that

$$M_1 = \sum_\lambda \bigcup (\bigcup_{x\in\partial M_1} L_x)$$

Then, we define the mapping $H : M_1 \times [0, 1] \to M_1$ by

$$H(y,t) = \begin{cases} y, & y \in \sum_\lambda, \quad \forall 0 \leq t \leq 1 \\ p(y,t), & y \in L_x \end{cases}$$

where $p(y, t)$ is the point $p \in L_x$ that the arc length along L_x from y to p is $tr(y)$, where $r(y)$ is the length of L_x from y to q_x the end point. It is clear that H is continuous, and

$H(\cdot, 0) = id : M_1 \to M_1$
$H(\cdot, 1) : M_1 \to \Sigma_\lambda,\ and$
$H \circ i = id : \Sigma_\lambda \to \Sigma_\lambda,$

where $i : \sum_\lambda \to M_1$ is an inclusion mapping. Hence, $\sum_\lambda$ is a deformation retract of M_1. The conclusion iii) is proven.

Finally, we show the conclusion iv). By the topological degree theory, we know that the Brouwer degree of the vector fields in (5.3.1) satisfies

$$deg(-(A_\lambda + G), \Omega_0, 0) = 1, \quad \forall -\lambda_1 < \lambda < \lambda_1 \tag{5.3.42}$$

where $\lambda_1 > 0$ and $\Omega_0 \subset R^m$ are defined as in (5.3.32). Because $\sum_\lambda$ is the maximum attractor of (5.3.1) in $\Omega_0 \backslash \{0\}, (0 < \lambda < \lambda_1)$, all the non-zero singular points of $A_\lambda + G$ in Ω_0 are in $\sum_\lambda$, and if which are finite, then we have

$$deg(-(A_\lambda + G), \Omega_0, 0) = ind(-(A_\lambda + G), 0)+$$

$$+ \sum_{x_i \in \sum_\lambda} ind(-(A_\lambda + G), x_i) \tag{5.3.43}$$

On the other hand, by (5.3.18) we have

$$ind(-(A_\lambda + G), 0) = \begin{cases} 1, & m = \ even \\ -1, & m = \ odd \end{cases} \qquad (5.3.44)$$

Consequently, from (5.3.42)-(5.3.44) we infer the conclusion iv).

The proof of Theorem 5.3.4 is complete.

Remark 5.3.7. From the conclusion iv) of Theorem 5.3.4, we see that if the attractor $\sum_\lambda$ has no singular points of (5.3.1), then the dimension m must be an even number.

Remark 5.3.8. If the following conclusion holds true

$$(A) \qquad \begin{cases} If\ D \subset R^m\ is\ a\ contractible\ m-dimensional\ compact \\ manifold\ with\ boundary,\ then\ D\ is\ a\ m-disk \end{cases}$$

then the conclusions ii) and iii) of Theorem 5.3.4 are rewritten as

ii)′ $\sum_\lambda = \bigcap_{\tau \geq 0} M_\tau, M_{\tau_2} \subset M_{\tau_1} \forall \tau_2 \geq \tau_1, M_\tau$ are the m-annuluses.

iii)′. If $\sum_\lambda$ is a finite simplicial complex, then $\sum_\lambda$ is a deformation retract of a m-annulus.

When $1 \leq m \leq 3$, the conclusion (A) obviously holds true. The conclusion (A) is related to the Poincare Conjecture (which has been proven except the dimension three), which amounts to saying that if M^m is a closed simply connected manifold with the integral homology of the m-sphere S^m, then M^m is homeomorphic to S^m.

In fact, for a m-dimensional compact manifold with boundary $D \subset R^m (\partial D$ is a $(m-1)$-dimensional closed compact manifold), we have the short exact homological sequence

$$0 \to H_{m-k-1}(D, Z) \to H_k(\partial D, Z) \to H_k(D, Z) \to 0 \qquad (5.3.45)$$

$\forall 0 \leq k \leq m$. When D is contractible, from (5.3.45) we get

$$H_k(\partial D, Z) = \begin{cases} Z, & k = m-1, 0 \\ 0, & k \neq m-1, 0, \end{cases}$$

If we can prove that ∂D is simply connected, then the conclusion (A) is a special case of the Poincare Conjecture.

5.3.3. S^1-invariant sets

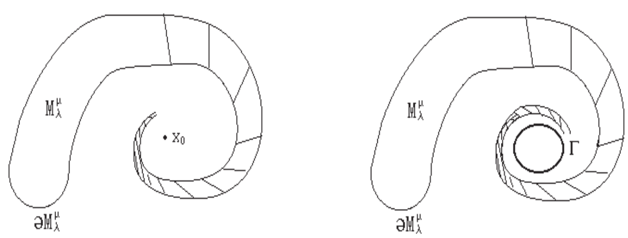
The topolgical structure of the attractors $\sum_\lambda$ in Theorem 5.3.4 may be very complex. But it is interesting to investigate the problem that under what conditions the attractor $\sum_\lambda$ is a S^k-invariant set, or contains a S^k-invariant set ($k \geq 1$). Here, we consider the simplest case of the dimension $m = 2$ in Theorem 5.3.4, and an immediate result can be obtained.

Theorem 5.3.9 Under the conditions of Theorem 5.3.4, if $m = 2$, then we have the following conclusions.

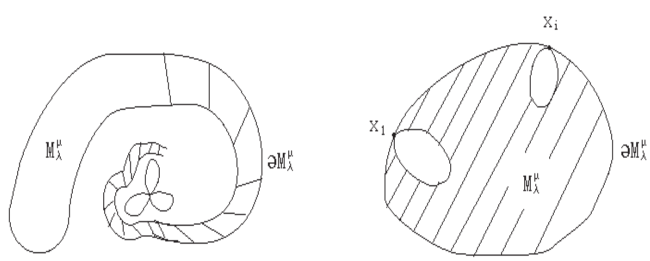
i). $\sum_\lambda$ contains at least a S^1-invariant set.

ii). If $\sum_\lambda$ has no-singular points of (5.3.1), then $\sum_\lambda$ is either an annulus or a periodic orbit, therefore, $\sum_\lambda$ contains at least one periodic orbit.

Proof. We know that the boundaries of M^u_λ and $R^m \backslash \mathcal{A}_\lambda$ are in $\sum_\lambda$, namely $\partial M^u_\lambda \bigcup \partial(R^m \backslash \mathcal{A}_\lambda) \subset \sum_\lambda$, and ∂M^u_λ and $\partial(R^m \backslash \mathcal{A}_\lambda)$ are invariant set. By the Poincare-Bendixson theorem, if $\sum_\lambda$ is not a finite simplicial complex, then there must exists some limit set of the orbits in ∂M^u_λ(or in $\partial(R^m \backslash \mathcal{A}_\lambda)$) which is one of a focus, a cycle and a k-petal, for instance see Fig.5.11 (a)-(c).



(a) x_0 is a focus
(b) Γ is a cycle



(c) Γ is a k-petal
(d) x_1 are self-intersections, which are singular points

Fig. 5.11.

Obviously, if the limit sets of orbits in ∂M^u_λ(or in $\partial(R^m \backslash \mathcal{A}_\lambda)$) contain a cycle or a k-petal Γ, then $\Gamma \subset \sum_\lambda$ contains a S^1-invariant set, and if all the limit sets are focuses, then ∂M_λ^u is homeomorphic to a cycle S^1. When $\sum_\lambda$ is a finite simplicial complex, then ∂M_λ^u is a cycle S^1 with the self-intersection (see Fig 5.11(d)). It is easy to see that, in this case, $\sum_\lambda$ must contain a S^1-invariant set, for instance, in Fig 5.11(d) the union of the arcs $x_i\widehat{\ x_{i+1}}$ in ∂M_λ^u is a S^1-invariant set. Thus, the conclusion i) is proven.

It is not difficult to see that when $\sum_\lambda$ is not a finite simplicial complex, or $\sum_\lambda$ is a finite simplicial complex but one of ∂M_λ^u and $\partial(R^m \backslash \mathcal{A}_\lambda)$ is a cycle S^1 with a non-empty self-intersection, then $\sum_\lambda$ contains the equilibrum points of (5.3.1). Hence, if $\sum_\lambda$ has no singular points, then $\sum_\lambda$ must be either a 2-annulus or a cycle S^1, and the conclusion ii) follows. The proof is complete.

In above, we always discuss the bifurcation of attractors, which occurs in the case that the real parts of some eigenvalues pass through zero from the neigative to the positive and the others stay on the negative. Naturally we wonder whether a vector field $F(x, \lambda)$ will bifurcate from a singular point x_0 an invariant set so long as the real parts of some eigenvalues of $DF(x_0, \lambda)$ pass through zero. When the number m of the real parts of eigenvalues passing through zero (counting multiplicity) is odd, we know the problem is positive because there is bifurcation of singular points from x_0, by the bifurcation theorem (Theorem 4.1.1). But, when m=even, we know little except the Hopf bifurcation. In the following, we give a generalized version of the Hopf bifurcation, which is positive to the above problem for $m = 2$.

Suppose the eigenvalues (5.3.4) satisfy that

$$\begin{cases} Re\beta_i(\lambda) = \begin{cases} < 0(\ or\ > 0), & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0\ (or\ < 0), & \lambda > \lambda_0 \end{cases} \quad (i = 1, 2) \\ Re\beta_j(\lambda_0) \neq 0, \quad \forall 3 \leq j \leq n \end{cases} \tag{5.3.46}$$

We then have the following theorem

theorem 5.3.10. Under the condition (5.3.46), if the system (5.3.1) has no bifurcation of the equilibrum points from $(x, \lambda) = (0, \lambda_0)$, then (5.3.1) must bifurcate from $(0, \lambda_0)$ a periodic orbit.

Proof. Without loss of generality, we assume that $\lambda_0 = 0$. Under the condition (5.3.46), the system (5.3.1), in a proper coordinate system, can be rewritten as the form (5.3.29) with $m = 2$, and by the center manifold theorem (Theorem 5.2.6), the bifurcation of (5.3.1) is completely determined by the following two dimensional system

$$\frac{dx}{dt} = B_\lambda x + g(x, \lambda), \tag{5.3.47}$$

where $x \in R^2$, and

$$g(x, \lambda) = g_1(x, h(x, \lambda), \lambda) = o(|x|), \quad \forall \lambda \in R.$$

while

$$\textit{the eigenvalues of } B_\lambda = \{\beta_1(\lambda), \beta_2(\lambda)\}.$$

where $h : \Omega \to R^{n-2}(\Omega \subset R^2)$ is defined as in Theorem 5.2.6. By the condition (5.3.46), we know that

$$ind(B_\lambda + g(\cdot, \lambda), 0) = 1, \quad \textit{as } \lambda \neq 0$$

By the assumption, the system (5.3.1) has no bifurcation of the singular points, therefore we obtain that $x = 0$ is an isolated singular point of the system (5.3.47) with $\lambda = 0$, and

$$ind(B_0 + g(\cdot, 0), 0) = 1 \tag{5.3.48}$$

On the other hand, by the Poincare formula for a two-dimensional vector field (see [CH]), we have

$$ind(B_0 + g(\cdot, 0), 0) = 1 + \frac{1}{2}(e - h)$$

where e=number of elliptic regions, and h =number of hyperbolic regions. And the elliptic, hyperbolic and parabolic regions E, H and P in a neighborhood $U \subset R^2$ of $x = 0$ are defined as (see Fig 5.12)

$$\begin{aligned}
&E = \{x \in U | S(t)x \in U, \forall t \in R, \textit{ and the limit set } \omega(x) = 0, \alpha(x) = 0\} \\
&H = \{x \in U | \omega(x) \textit{ and } \alpha(x) \bar{\in} U\} \\
&P = \{x \in U | \omega(x) = 0, \alpha(x) \bar{\in} U; \textit{ or } \alpha(x) = 0, \omega(x) \bar{\in} U; \\
&\ \textit{or } S(t)x \in U \quad \forall t \in R, \textit{ and } \omega(x), \alpha(x) \neq 0\};
\end{aligned}$$

where $S(t)$ is the operator semi-groups generated by $B_0 + g(x, 0)$.

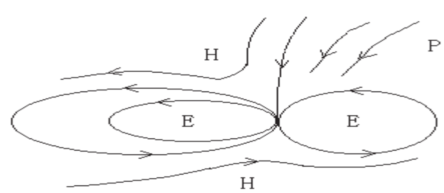


Fig. 5.12.

By (5.3.48), for the vector field $B_0 + g(\cdot, 0)$, we have $e = h$. Because the α and ω-limit sets of every orbit in the elliptic regions are $x = 0$, it is not difficult to see that $e = 0$, otherwise, in the some fashion as used in Theorem 5.3.4, we can derive the bifurcation of singular points of (5.3.1). Hence, about the singular point $x = 0$ of $B_0 + g(\cdot, 0)$ there are only the parabolic regions. A singular point with the parabolic regions must be one of the following three types

a) $x = 0$ is a stable focus or node;

b) $x = 0$ is an unstable focus or node;

c) $x = 0$ is a stable but not asymptotically stable singular point.

In the two dimensional system, the case c) implies that there exists a sequence of periodic orbits $\{\Gamma_n\} \subset R^2$, such that $\Gamma_n \to \{0\}$ as $n \to \infty$, thus the sequence $\{\Gamma_n\}$ is the bifurcation of the poriodic orbits of (5.3.47). Obviously, the case b) for the vector field $B_0 + g(\cdot, 0)$ is the case a) for $-(B_0 + g(\cdot, 0))$. For the case a) and b), we can derive this theorem in the same manmer as used in Theorem 5.3.4 and Theorem 5.3.9. The proof is complete.

Remark 5.3.11. In the condition (5.3.46), if the eigenvalues $\beta_1(\lambda) = a(\lambda) + ib(\lambda), \beta_2(\lambda) = a(\lambda) - ib(\lambda)$, and $b(\lambda_0) \neq 0$, then Theorem 5.3.10 is the Hopf bifurcation theorem. In fact, in this case, the matrix (5.3.3) is non-degenerate:

$$deg A|_{\lambda=\lambda_0} \neq 0$$

Hence, by the inverse function theorem, the system (5.3.1) has no bifurcation of the equilibrum points.

5.3.4. Remarks on bifurcation of invariant manifolds

Now, let us concern the following problems, which are interesting and relatively difficult.

1). Do the attractors $\sum_\lambda$ in Theorem 5.3.4 contain at least a S^{m-1}-invariant manifold?

2). What conditions do the vector fields of (5.3.1) satisfy so that the invariant set $\sum_\lambda$ is a S^{m-1}-attractor?

3). If $\sum_\lambda$ has no singular points, then does $\sum_\lambda$ contain an invariant manifold?

4). Under the below condition

$$\begin{cases} Re\beta_i(\lambda) = \begin{cases} < 0 (> 0), & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0 (< 0), & \lambda > \lambda_0 \end{cases} (1 \leq i \leq m) \\ Re\beta_j(\lambda_0) \neq 0, \forall m+1 \leq j \leq n \end{cases} \tag{5.3.49}$$

does the system (5.3.1) bifurcate from $(x, \lambda) = (0, \lambda_0)$ an invariant manifold? If does not, then what additional conditions do we need to imposed in order for (5.3.1) to do so?

With the problems in mind, here we present some open quasitions and suggestions as follows.

Problem 5.3.12. Under the conditions of Theorem 5.3.4, if the vector fields $G(x, \lambda 0$ in (5.3.1) are analytic, then the bifurcated invariant set $\sum_\lambda$ is a S^{m-1}-attractor.

Conjecture 5.3.13. Under the conditions of Theorem 5.3.4, if the bifurcated attractor $\sum_\lambda$ of (5.3.1) has no singular points, then $\sum_\lambda$ must contain at least an invariant manifold.

This conjecture is based on the observation that the boundary ∂M_λ^u of the unstable manifold $M_\lambda^u \subset R^m$ of (5.3.1) at $x = 0$ is a $(m-1)$-sphere S^{m-1} with the self-intersection, which is invariant in $\sum_\lambda$. If ∂M_u^λ has no singular points, then the self-intersection in ∂M_u^λ, which is invariant too, either is empty, thus $\partial M_u^\lambda = S^{m-1}$, or contains some invariant manifolds with dimensions $k(1 \leq k \leq m-2)$.

We referred the fourth problem to the follow conjecture.

Conjecture 5.3.14. Under the condition (5.3.49), the system (5.3.1) bifurcates from $(x, \lambda) = (0, \lambda_0)$ at least a k-dimensional invariant manifold $\sum(0 \leq k \leq m-1)$, and as $k = 0, \sum$ consists of the singular points of (5.3.1).

When $m = 2$, this conjecture holds true, which is Theorem 5.3.10. When m=odd, the conjecture is trivial because there always exists the bifurcation of singular points.

In order to describe the problem 4), we introduce a definition on the stable, unstable and neutral sets, which are the analogues of the stable, unstable and center manifolds. We denote the vector fields of (5.3.1) by

$$F(x, \lambda) = A_\lambda x + G(x, \lambda), \quad x \in R^n, \lambda \in R.$$

Definition 5.3.15. Let $S_\lambda(t) : R^n \to R^n$ be the operator semigroups generated by $F(x, \lambda), U \subset R^n$ be a neighborhood of $x = 0$. We respectively call the sets

$$\begin{aligned}
\Omega_\lambda^s(U) &= \{x \in U | S_\lambda(t)x \in U \ \forall t \geq 0, \ and \ \omega(x) = \{0\}\} \\
\Omega_\lambda^u(U) &= \{x \in U | S_\lambda(-t)x \in U \ \forall t \geq 0, \ and \ \alpha(x) = \{0\}\} \\
\Omega_\lambda^n(U) &= \{x \in U | S_\lambda(t)x \in U \ \forall t \in R, \ and \ \alpha(x), \omega(x) \neq \{0\}\},
\end{aligned}$$

the stable, unstable and neutral sets of $F(x, \lambda)$ in U.

Obviously, we have

$$dim\Omega^s_\lambda + dim\Omega^u_\lambda + dim\Omega^n_\lambda \geq n.$$

And if the sets $\Omega^s_\lambda, \Omega^u_\lambda$ and Ω^n_λ are manifolds with dimension ≥ 1, then $x = 0$ is an interior point of them, and as $x = 0$ is an isolated singular point of $F(x, \lambda)$, we have

$$\begin{aligned} ind(F^s(\cdot, \lambda), 0) &= deg(F^s(\cdot, \lambda), \Omega^s_\lambda, 0) = (-1)^k \\ ind(F^u(\cdot, \lambda), 0) &= deg(F^u(\cdot, \lambda), \Omega^u_\lambda, 0) = 1 \\ ind(F^n(\cdot, \lambda), 0) &= deg(F^n(\cdot, \lambda), \Omega^n_\lambda, 0) = 1 \end{aligned}$$

where $k = dim\Omega^s_\lambda$, and F^s, F^u, F^n are respectively the restrictions of F on $\Omega^s_\lambda, \Omega^u_\lambda$ and Ω^n_λ. Furthermore, we have

Theorem 5.3.16. Let $x = 0$ be an isolated singular point of $F(x, \lambda)$ at $\lambda = \lambda_0$, and $U \subset R^n$ be a neighborhood of $x = 0$. If all the stable, unstable and neutral sets in U are manifolds, then

$$dim\Omega^s_{\lambda_0} + dim\Omega^u_{\lambda_0} + dim\Omega^n_{\lambda_0} = n$$

$$ind(F(\cdot, \lambda), 0) = (-1)^k, \quad k = dim\Omega^s_{\lambda_0}.$$

We now return to analyse the problem 4). Conjecture 5.3.14 is based on the following facts. By the center manifold theorem, the bifurcation of invariant manifolds of $F(x, \lambda)$ can be reduced to that of the vector fields in R^m defined as in (5.3.31)

$$F_1(x, \lambda) = B_\lambda x + f(x, \lambda), \quad x \in R^m.$$

By the condition (5.3.49), the neutral set of $F_1(x, \lambda_0)$ equals to the neutral set of $F(x, \lambda_0)$. If the neutral set of $F_1(x, \lambda_0)$ has the dimension smaller than m:

$$dim\Omega^n_{\lambda_0} < m$$

then the stable and unstable sets of $F_1(x, \lambda_0)$ satisfy

$$dim\Omega^s_{\lambda_0} + dim\Omega^u_{\lambda_0} \geq 1 \tag{5.3.50}$$

By using lemma 5.3.5, in a similar manner as used in Theorem 5.3.4 and Theorem 5.3.10, from (5.3.50) we can derive that $F_1(x, \lambda)$ have the bifurcation of invariant set at $(0, \lambda_0)$. Thus, we summarize to a theorem as follows

Theorem 5.3.17. Under the condition (5.3.49), if the dimension of the neutral set of $F(x, \lambda)$ at $\lambda = \lambda_0$ is smaller than m, then the system (5.3.1) bifurcates from $(0, \lambda_0)$ an invariant set.

Thus, Conjecture 5.3.14 derectly depends on Conjecture 5.3.13 and the conjecture that if the neutral set of $F(x,\lambda_0)$ has $\dim\Omega^n_{\lambda_0} = m(m\text{=even})$, then there is a sequence of invariant manifolds $\{\Gamma_n | n = 1, 2, \cdots\} \subset \Omega^n_{\lambda_0}$ of $F(x,\lambda_0)$ such that $\Gamma_n \to \{0\}$ as $n \to \infty$.

5.3.5. Stability on bifurcation

Let $\Omega \subset R^n$ be an open set, $O \in \Omega$, and $\lambda \in R$ a parameter. We denote the space of parameterized vector fields by

$$C^{k,1}_0(\Omega \times R, R^n) = \{F : \Omega \times R \to R^n | F(0,\lambda) = 0, \forall\lambda \in R\}$$

with the norm

$$\|F\|_{C^{k,1}} = \sup_{\lambda\in R}\sup_{\Omega}[\sum_{p=0}^{k} |D^p_x F| + |D_\lambda F| + |D^2_{\lambda x} F|].$$

Obviously, if $F \in C^{k,1}_0(\Omega \times R, R^n)(k \geq 1)$, then F is k-th differentiable on $x \in \Omega$ and differentiable on $\lambda \in R$. Moreover, F can be expressed as

$$F(\cdot,\lambda) = A_\lambda + G(\cdot,\lambda)$$

where $G(\cdot,\lambda)$ and A_λ satisfy (5.3.2)-(5.3.4), and the simple real and simple complex eigenvalues of A_λ are differentiable on λ(see [Ka]), which can be expanded by

$$\beta_i(\lambda) = \alpha^i_0 + \alpha^i_1\lambda + o(|\lambda|), \quad (1 \leq i \leq n).$$

Definition 5.3.18. Let $F_1, F_2 \in C^{k,1}_0(\Omega \times R, R^n), \lambda_i \in R$ be a bifurcation point of invariant set Γ^i_ρ of $F_i(x,\rho)(i = 1, 2)$. We say that the both bifurcation points λ_1 and λ_2 have the same structure if F_1 and F_2 are locally topologically equivalent at $\Gamma^1_{\rho_1}$ and $\Gamma^2_{\rho_2}(\rho_1 - \lambda_1 = \rho_2 - \lambda_2)$, i.e. there are neighborhoods $U_i \subset R^n$ of $\Gamma^i_{\rho_i}(i = 1, 2)$ and a homeomorphism $\phi : U_1 \to U_2$ such that ϕ takes the orbits of F_1 in U_1 to orbits of F_2 in U_2, preserving orientation.

Definition 5.3.19. Let $F \in C^{k,1}_0(\Omega \times R, R^n), \lambda_0$ be a bifurcation point of $F(x,\lambda)$. We say that the bifurcation point λ_0 is stable in $C^{k,1}_0(\Omega \times R, R^n)$, if $\forall\epsilon > 0$ sufficiently small, there is a neighborhood $O \subset C^{k,1}_0(\Omega \times R, R^n)$ of F such that $\forall F_1 \in O, F_1$ has a bifurcation point λ_1 with $|\lambda_1 - \lambda_0| < \epsilon$, where λ_1 and λ_2 have the same structure. If all the bifurcation points of F are stable in $C^{k,1}_0(\Omega \times R, R^n)$, then we say that F is stable on bifurcation in $C^{k,1}_0(\Omega \times R, R^n)$, or F is a vector field with stable bifurcation.

Definition 5.3.20. Let $F \in C_0^{k,1}(\Omega\times R, R^n)(k \geq 1), \lambda_0 \in R$ be a parameter of F.
i). λ_0 is called an eigen-parameter with multiplicity m of F, if $DF(0, \lambda_0)$ has the eigenvalues with zero real parts, the sum of whos multiplicities is m.
ii). λ_0 is simple if either the multiplicity $m = 1$, or $m = 2$ and the two eigenvalues passing through zero at λ_0 are the conjugate imarginary number $\beta_1(\lambda_0) = \overline{\beta_2(\lambda_0)} = i\beta(\beta \neq 0)$.
iii). If the simple eigenvalues passing through zero in (5.3.49) satisfy

$$\frac{\partial Re\beta_i(\lambda_0)}{\partial\lambda} \neq 0, \quad 1 \leq i \leq m \quad (m = 1,\ or\ = 2)$$

then λ_0 is called to be regular.

The following is a basic theorem on stable bifurcation points

Theorem 5.3.21. Let $F \in C_0^{3,1}(\Omega \times R, R^n)$, and λ_0 be a simple eigen-parameter of $F(x, \lambda)$. There exists a number $b(\lambda_0)$, called the bifurcation number of F at λ_0, which continuously depends on the module $\|F\|_{C^{3,1}}$, such that the following assertions holds true.
i). λ_0 is a stable bifurcation point of F if and only if λ_0 is regular and the bifurcation number $b(\lambda_0) \neq 0$.
ii). If λ_0 has multiplicity $m = 1$, and $b(\lambda_0) \neq 0$, then $F(x, \lambda)$ bifurcates an unique branch of singular points $x^*(\lambda)$, which are hyperbolic for $\lambda \neq \lambda_0$(i.e. the eigenvalues of $DF(x^*, \lambda)$ have the non-zero real parts), and have k -dimensional stable manifold for $\lambda < \lambda_0$(or $\lambda_0 < \lambda$), $(k + 1)$-dimensional stable manifold for $\lambda_0 < \lambda$(or $\lambda < \lambda_0$), depending the manner passing through zero of eigenvalues in (5.3.49), where k=number of the eigenvalues of $Re\beta_j(\lambda_0) < 0$.
iii). If λ_0 has multiplicity $m = 2, F(x, \lambda)$ bifurcates an unique branch of periodic orbits Γ_λ for $\lambda < \lambda_0$
(or $\lambda_0 < \lambda$) as $b(\lambda_0) > 0$, and for $\lambda_0 < \lambda$(or $\lambda < \lambda_0$) as $b(\lambda_0) < 0$, which are hyperbolic for $\lambda \neq \lambda_0$.

Proof. First, we consider the case that λ_0 has multiplicity $m = 1$. Thus, in a neighborhood of $(x, \lambda) = (0, \lambda_0), F(z, \lambda)$ can be expressed, in a proper coordinate system, as

$$F(z, \lambda) = \begin{pmatrix} \beta_1(\lambda) & 0 \\ 0 & B_\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} G_1(x, y, \lambda) \\ G_2(x, y, \lambda) \end{pmatrix}, \tag{5.3.51}$$

where $z = (x, y), x \in R^1 \bigcap \Omega, y \in R^{n-1} \bigcap \Omega, G_1 : \Omega \times R \to R, G_2 : \Omega \times R \to R^{n-1}$ are C^3 functions on $x \in \Omega$, and

$$G_i(x, y, \lambda) = o(|x|, |y|), \quad (i = 1, 2) \tag{5.3.52}$$

while, B_λ is the $(n-1)$ order matrix having the eigenvalues $\{\beta_j(\lambda)\}|2 \leq j \leq n\}$ defined as in (5.3.49).

By the center manifold theorem, there is a function

$$y = h(x,\lambda), h(0,\lambda) = 0, h'_x(0,\lambda) = 0 \tag{5.3.53}$$

which is C^3 on $x \in R^1 \bigcap \Omega$, and continuous on λ, such that the bifurcation of (5.3.51) at λ_0 is equivalent to that of the following equation

$$\frac{dx}{dt} = \beta_1(\lambda)x + G_1(x, h(x,\lambda), \lambda) \tag{5.3.54}$$

where

$$\beta_1(\lambda) = \begin{cases} < 0 \ (or \ > 0), & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0 \ (or \ < 0), & \lambda > \lambda_0 \end{cases}$$

By (5.3.52) and (5.3.53), we have the Taylor's expansion

$$G_1(x, h(x,\lambda), \lambda) = b(\lambda)x^2 + c(\lambda)x^3 + o(|x|^3) \tag{5.3.55}$$

where

$$b(\lambda) = \frac{\partial^2 G_1(0,0,\lambda)}{\partial x^2}$$

It is clear that $b(\lambda)$ continuously depends on λ and the module $\|F\|_{C^2}$. The number $b(\lambda_0)$ is the one expected by us.

In fact, when $b(\lambda_0) \neq 0$, the equation (5.3.54) bifurcates an unique branch of singular points $x_0(\lambda)$. Therefore, $x^*(\lambda) = (x_0(\lambda), h(x_0,\lambda))$ is a bifurcated branch of singular points of (5.3.51). We shall show that $x^*(\lambda)$ are non-degenerate for all $|\lambda - \lambda_0| > 0$ sufficiently small. We see that

$$DF(x^*,\lambda) = \begin{pmatrix} \beta_1(\lambda) & 0 \\ 0 & B_\lambda \end{pmatrix} + \begin{pmatrix} DG_1(x_0, h(x_0,\lambda),\lambda) \\ DG_2(x_0, h(x_0,\lambda),\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} \beta_1(\lambda) + 2b(\lambda)x_0 + o(|x_0|) & O(|x_0|) \\ O(|x_0|) & B_\lambda + O(|x_0|) \end{pmatrix} \tag{5.3.56}$$

Hence

$$detDF(x^*,\lambda) = (\beta_1 + 2bx_0)detB_\lambda + o(|x_0|)$$

Because $x_0(\lambda)$ satisfy

$$\beta_1(\lambda)x + b(\lambda)x^2 + o(|x|^2) = 0$$

one can deduce that

$$x_0(\lambda) = -\frac{1}{b(\lambda)}\beta_1(\lambda) + o(|\beta_1|)$$

Therefore we obtain

$$\begin{aligned} detDF(x^*,\lambda) &= -\beta_1(\lambda)detB_\lambda + o(|\beta_1|) \\ &\neq 0,\ for\ all\ |\lambda-\lambda_0|>0\ sufficiently\ small, \end{aligned}$$

and

$$sign\ detDF(x^*,\lambda) = \begin{cases} sign\ detB_\lambda, & \lambda<\lambda_0\ (or\ \lambda>\lambda_0) \\ -sign\ detB_\lambda, & \lambda>\lambda_0\ (or\ \lambda<\lambda_0) \end{cases} \tag{5.3.57}$$

Because the eigenvalues $\beta_i^*(\lambda)(1 \leq i \leq n)$ of $DF(x^*,\lambda)$ continuously depends on λ, and $\beta_j^*(\lambda_0) = \beta_j(\lambda_0)(2 \leq j \leq n)$, and by (5.3.49), $Re\beta_j(\lambda_0) \neq 0$, consequently the derivative operator $DF(x^*,\lambda)$ are hyperbolic for all $|\lambda-\lambda_0|>0$ sufficiently small. From (5.3.57) we get that

$$k_1 = \begin{cases} k, & as\ \lambda<\lambda_0\ (or\ \lambda>\lambda_0) \\ k+1, & as\ \lambda>\lambda_0\ (or\ \lambda<\lambda_0) \end{cases}$$

where k_1=number of the eigenvalues of $DF(x^*,\lambda)$ which have the negative real parts. Hence, by the stable manifold theorem we obtain the conclusion ii).

We are now in a position to prove the conclusion i) for $m=1$. Let λ_0 be regular and $b(\lambda_0)>0$ (for the case of $b(\lambda_0)<0$, the proof is the same). By the definition of norm $\|\cdot\|_{C^{3,1}}, \forall\epsilon>0$ there exists $\delta>0$ such that if $F_1 \in C_0^{3,1}(\Omega\times R, R^n)$ and

$$\|F_1 - F\| < \delta$$

then the eigenvalues of $DF_1(0,\lambda)$ satisfy that $\widetilde{\beta}_i \in C^1(R)$, and

$$\|\widetilde{\beta}_i - \beta_i\|_{C^1} < \epsilon, \quad \forall 1 \leq i \leq n$$

Because λ_0 is regular,

$$\beta_1(\lambda) = \alpha(\lambda-\lambda_0) + o(|\lambda-\lambda_0|), \quad \alpha \neq 0$$

It means that as $\epsilon>0$ sufficiently small, the eigenvalues $\widetilde{\beta}_1(\lambda)$ of $DF_1(0,\lambda)$ has the expansion

$$\widetilde{\beta}_1(\lambda) = \alpha_1(\lambda-\lambda_1) + o(|\lambda-\lambda_1|), \quad \alpha_1 \neq 0$$

and

$$\alpha_1 \to \alpha, \quad \lambda_1 \to \lambda_0 \quad as\ F_1 \to F\ in\ C^{3,1}-norm.$$

Hence, it follows that there is a neighborhood $O \subset C_0^{3,1}(\Omega\times R, R^n)$ of F such that for any $F_1 \in O, F_1$ has a simple and regular eigen-parameter λ_1 with $|\lambda_1-\lambda_0|<\epsilon$, and the bifurcation number of F_1 at λ_1 satisfies $b^*(\lambda_1)>0$.

We say that a linear vector field A is hyperbolic if all the eigenvalues of A have the non-zero real parts. The number of eigenvalues of A with negative real parts is called the index of A. It is well known that if A and B are hyperbolic, then A and B are topologically conjugate (therefore the flows of A and B are topologically equivalent) if and only if they have the same index (Cf.[PM]).

Let $x_1^*(\lambda)$ be the bifurcated branch of singular points of F_1 from $(0, \lambda_1)$. By the conclusion ii), $DF_1(x_1^*, \rho_1)$ and $DF(x^*, \rho_0)$ are hyperbolic respectively for $\rho_1 \neq \lambda_1$ and $\rho_0 \neq \lambda_0$. On the other hand, as (5.3.56) we have that

$$DF_1(x_1^*, \rho_1) = \begin{pmatrix} -\widetilde{\beta}_1(\rho_1) + o(|\widetilde{\beta}_1) & O(|\widetilde{\beta}_1|) \\ O(|\widetilde{\beta}_1|) & B^*_{\rho_1} \end{pmatrix} \tag{5.3.58}$$

where

$$\|B^*_\lambda - B_\lambda\| < \epsilon, \quad \epsilon > 0 \ sufficiently\ small.$$

Hence, from (5.3.56) and (5.3.58) it follows that $DF_1(x_1^*, \rho_1)$ and $DF(x^*, \rho_0)$ with $\rho_1 - \lambda_1 = \rho_0 - \lambda_0$ have the same index. By the Hartman-Grobman theorem we derive the sufficient conditions for $m = 1$ of conclusion i).

The necessary condition.

When $b(\lambda_0) = 0$ and $C(\lambda_0) \neq 0$ in (5.3.55), the topological structure of bifurcation of (5.3.54) is clearly different from the case that $b(\lambda_0) \neq 0$. If $b(\lambda_0) = C(\lambda_0) = 0$, then F can be approximated by F_1 and F_2 in $C_0^{3,1}(\Omega \times R, R^n)$, where the bifurcation numbers of F_1 are non-zero, and the numbers as defined in (5.3.55) of F_2 have $b(\lambda_2) = 0$ and $C(\lambda_2) \neq 0, \lambda_2$ the eigen-parameter of F_2. Hence λ_0 is not stable on bifurcation.

When λ_0 is not regular, i.e. $\beta_1(\lambda) = o(|\lambda - \lambda_0|)$, then F can be approximated by F_N in $C_0^{3,1}(\Omega \times R, R^n)$, where $DF_N(0, \lambda)$ have the eigenvalues passing through zero as follows

$$\beta_1^N(\lambda) = \begin{cases} 0, & \lambda \in [\lambda_0 - \epsilon_N, \lambda_0 + \epsilon_N], \quad \epsilon_N > 0, \quad \epsilon_N \to 0, \quad N \to \infty \\ \neq 0, & \lambda \bar{\in} [\lambda_0 - \epsilon_N, \lambda_0 + \epsilon_N]. \end{cases}$$

Obviously, the bifurcation structure of F_N is different from that of F at $\lambda = \lambda_0$. Thus, the conclusion i) for $m = 1$ is proven.

Next, we consider the case that λ_0 has multiplicity $m = 2$.

For the sake of simplicity, let $F(z, \lambda)$ be expressed as

$$F(z, \lambda) = \begin{pmatrix} \alpha(\lambda) & -1 & \\ 1 & \alpha(\lambda) & \\ & O & B_\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} + \begin{pmatrix} G_1(x, y, \lambda) \\ G_2(x, y, \lambda) \\ G_3(x, y, \lambda) \end{pmatrix} \tag{5.3.59}$$

where $z = (x, y) \in \Omega, x = (x_1, x_2) \in R^2 \bigcap \Omega, y \in R^{n-2} \bigcap \Omega, G_1, G_2 : \Omega \times R \to R, G_3 : \Omega \times R \to R^{n-2}$ are C^3 functions on $z \in \Omega$, and

$$G_i(x, y, \lambda) = o(|x|, |y|), \quad (i = 1, 2, 3) \tag{5.3.60}$$

while B_λ is the $(n-2)$ order matrix possessing the eigenvalues $\{\beta_j(\lambda)|3 \leq j \leq n\}$ defined as in (5.3.49)

By the center manifold theorem, the bifurcation of (5.3.59) at λ_0 is equivalent to that of the following system

$$\begin{cases} \frac{dx_1}{dt} = \alpha(\lambda)x_1 - x_2 + f_1(x_1, x_2, \lambda) \\ \frac{dx_2}{dt} = x_1 + \alpha(\lambda)x_2 + f_2(x_1, x_2, \lambda) \end{cases} \tag{5.3.61}$$

where

$$\alpha(\lambda) = \begin{cases} < 0 \ (or \ > 0), & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0 \ (or \ < 0) & \lambda > \lambda_0 \end{cases}$$

and

$$f_i(x_1, x_2, \lambda) = G_i(x_1, x_2, h(x_1, x_2, \lambda), \lambda), \quad i = 1, 2.$$

The function $y = h(x, \lambda)$ defined as in the center manifold theorem satisfies the properties (5.3.53). By the Taylor expansion, from (5.3.53) and (5.3.60) we have

$$f_i(x, \lambda) = \sum_{2 \leq p+q \leq 3} a^i_{pq} x_1^p x_2^q + \sum_{j=1}^{2} \sum_{k=1}^{n-2} b^i_{jk} x_j y_k + o(|x|^3)$$

$$y_k = h_k(x_1, x_2, \lambda), \quad h(x, \lambda) = \{h_1(x, \lambda), \cdots, h_{n-2}(x, \lambda)\} \tag{5.3.62}$$

$$h_k(x_1, x_2, \lambda) = \sum_{r+s=2} C^k_{rs} x_1^r x_2^s + o(|x|^2)$$

where

$$\begin{aligned} a^i_{pq} &= \frac{\partial^{p+q} G_i(0,0,0,\lambda)}{\partial x_1^p \partial x_2^q}, \quad (0 \leq p, q \leq 3, \quad i = 1, 2) \\ b^i_{jk} &= \frac{\partial^2 G_i(0,0,0,\lambda)}{\partial x_j \partial y_k} \\ C^k_{rs} &= \frac{\partial^2 h_k(0,0,\lambda)}{\partial x_1^r \partial x_2^s}, \quad (0 \leq r, s \leq 2) \end{aligned}$$

It is clear that the coeficients a^i_{pq}, b^i_{jk} and c^k_{rs} continuously depends on λ and the module $\|F\|_{C^{3,1}}$.

We adopt the polar coordinate system to discuss the equation (5.3.61). Let $x_1 = r\cos\theta, x_2 = r\sin\theta$. Then we have

$$\frac{dr}{dt} = \cos\theta \frac{dx_1}{dt} + \sin\theta \frac{dx_2}{dt}$$

$$r\frac{d\theta}{dt} = \cos\theta \frac{dx_2}{dt} - \sin\theta \frac{dx_1}{dt}$$

From (5.3.61) and (5.3.62) it follows that

$$\frac{dr}{d\theta} = \frac{\alpha(\lambda)r^2 + r\cos\theta f_1(r\cos\theta, r\sin\theta) + r\sin\theta f_2(r\cos\theta, r\sin\theta)}{r + \cos\theta f_2(r\cos\theta, r\sin\theta) - \sin\theta f_1(r\cos\theta, r\sin\theta)}$$

$$\begin{aligned} &= [\alpha r + u_1(\theta,\lambda)r^2 + u_2(\theta,\lambda)r^3 + o(r^3)]\times \\ &\quad \times[1 + v_1(\theta,\lambda)r + v_2(\theta,\lambda)r^2 + o(r^2)] \\ &= \alpha r + (u_1 + \alpha v_1)r^2 + (u_1v_1 + u_2 + \alpha v_2)r^3 + o(r^3) \end{aligned} \tag{5.3.63}$$

where $u_i, v_i (i = 1, 2)$ are the homogeneous functions of degree $i+2$ on $\cos\theta$ and $\sin\theta$, and continuously depend on the coeficients a^i_{pq}, b^i_{jk} and c^k_{rs} in (5.3.62). It is readily to check that

$$\int_0^{2\pi} [u_1(\theta,\lambda) + \alpha(\lambda)v_1(\theta,\lambda)]d\theta = 0 \tag{5.3.64}$$

Denote by

$$b(\lambda) = \int_0^{2\pi} [u_1(\theta,\lambda)\cdot v_1(\theta,\lambda) + u_2(\theta,\lambda) + \alpha(\lambda)v_2(\theta,\lambda)]d\theta$$

The number $b(\lambda_0)$ is defined as the bifurcation number of $F(z,\lambda)$ at λ_0. We divide into a few steps to prove the conclusions i) and iii) for $m = 2$.

Step 1. $F(z,\lambda)$ bifurcates an unique branch of periodic orbits Γ_λ for $\lambda < \lambda_0$ (or $\lambda > \lambda_0$) as $b(\lambda_0) > 0$, and for $\lambda_0 < \lambda$(or $\lambda < \lambda_0$) as $b(\lambda_0) < 0$, depending the manner that $\alpha(\lambda)$ passes through zero.

This claim above is well known. But, for the sake of completion, here we still give a proof. It suffices to only prove the case of $b(\lambda_0) > 0$. Let $r(\theta,\lambda,a)$ be the solution of (5.3.63) with the initial value $r(0,\lambda,a) = a$. We know, by the ODE theory, that $r(\theta,\lambda,a)$ is third differentiable on $a \geq 0$. Then $r(\theta,\lambda,a)$ can be expressed near $a = 0$ as

$$r(\theta,\lambda,a) = a + r_0(\theta,\lambda)O(a^2) \tag{5.3.65}$$

In the case without confusion, we denote $r(\theta,\lambda,a)$ by $r(\theta,a)$.

On the other hand, by (5.3.63) we get

$$\frac{1}{r^2}\frac{dr}{d\theta} = \frac{1}{r}\alpha(\lambda) + (u_1 + \alpha v_1) + (u_1v_1 + u_2 + \alpha v_2)r + o(r) \tag{5.3.66}$$

Inserting (5.3.65) into the right side of (5.3.66), and then integrating it, by (5.3.64) and $r(0,a) = a$, we obtain that

$$\frac{r(2\pi,a) - r(0,a)}{r(2\pi,a)} = c(a,\lambda)\alpha(\lambda) + b(\lambda)a^2 + o(a^2) \tag{5.3.67}$$

where

$$\begin{cases} c(a,\lambda) = \int_0^{2\pi}[1 + o(a)r_0(\theta,\lambda)]d\theta \\ c(0,\lambda) = 2\pi \end{cases}$$

It is easy to see that every positive solution $a > 0$ of the following algebraic equation is corresponding to a periodic orbit through the point $(x_1, x_2) = (a, 0)$ of (5.3.61):

$$c(a,\lambda)\alpha(\lambda) + b(\lambda)a^2 + o(a^2) = 0 \tag{5.3.68}$$

Obviously, when $b(\lambda_0) > 0$, the equation (5.3.68) uniquely bifurcates from $(a, \lambda) = (0, \lambda_0)$ a branch of positive solutions $a(\lambda)$ for $\lambda < \lambda_0$, or $\lambda_0 < \lambda$, depending on the signs of $\alpha(\lambda)$, and there is no bifurcated branch for $\lambda_0 < \lambda$(or $\lambda < \lambda_0$).

Step 2. The bifurcated periodic orbits Γ_λ are hyperbolic

For convenience, here we shall introduce the concepts of the poincare maps and hyperbolic periodic orbits, which can be found in some textbooks on the dynamical systems, e.g. see [PM].

Let γ be a periodic orbit of a vector field $F, x_0 \in \gamma$ a point. Let $\sum$ be a section transversal to F through the point x_0.

The orbit through x_0 returns to intersect $\sum$ at time $T > 0$, where T is the period of γ. By the continuity of the flow of F, there exists a neighborhhod $U \subset \sum$ of x_0, for each point $x \in U$, the orbit through x returns to intersect $\sum$ at some time $t > 0$. Thus, we can define a map $P : U \to \sum$ as that for every point $x \in U, P(x)$ is the first point where the orbit of x returns to intersect $\sum$. This map P is called the Poincare map associated the periodic orbit γ (see Fig. 5.13).

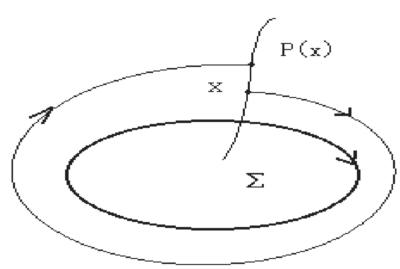


Fig 5.13.

The Poincare map $P : U \to P(U) \subset \sum$ is a homeomorphism, and each fixed point of P associates a periodic orbit of F.

We say that γ is a hyperbolic periodic orbit of F, if x_0 is a hyperbolic fixed point of the Poincare map $P : U \subset \sum \to \sum$, i.e. DP_{x_0} has no eigenvalues of modulus 1.

Without loss generality, we assume the center manifold M_λ of the vector field (5.3.59) is the x-plane, namely

$$y = h(x, \lambda) \equiv 0, \forall |\lambda - \lambda_0| \geq 0 \ sufficiently \ small \ and \ x \in R^2.$$

In fact, in the transformation of coordinate system as

$$\begin{cases} \widetilde{x} = x, \quad \widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2), \quad x = (x_1, x_2) \\ \widetilde{y} = y - h(x, \lambda), \end{cases}$$

the vector field (5.3.59) is transformed into the below form

$$\widetilde{F}(\widetilde{x}, \widetilde{y}, \lambda) = \begin{pmatrix} \alpha(\lambda) & -1 & \\ 1 & \alpha(\lambda) & 0 \\ & 0 & B_\lambda \end{pmatrix} \begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{y} \end{pmatrix} + \begin{pmatrix} H_1(\widetilde{x}, \widetilde{y}, \lambda) \\ H_2(\widetilde{x}, \widetilde{y}, \lambda) \\ H_3(\widetilde{x}, \widetilde{y}, \lambda) \end{pmatrix}$$

where

$$\begin{aligned} &H_i(\widetilde{x}, \widetilde{y}, \lambda) = G_i(\widetilde{x}, \widetilde{y} + h(\widetilde{x}, \lambda), \lambda), \quad i = 1, 2 \\ &H_3(\widetilde{x}, \widetilde{y}, \lambda) = B_\lambda h(\widetilde{x}, \lambda) - \nabla h \cdot \frac{d\widetilde{x}}{dt} + G_3(\widetilde{x}, \widetilde{y} + h, \lambda) \\ &\frac{d\widetilde{x}}{dt} = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix} \widetilde{x} + \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \end{aligned}$$

Therefore

$$H_i(\widetilde{x}, \widetilde{y}, \lambda) = o(|\widetilde{x}|, |\widetilde{y}|), \quad 1 \leq i \leq 3.$$

It is clear that the both vector fields $\widetilde{F}$ and F are topologically conjugate, and the $\widetilde{x}$-plane is a center manifold of $\widetilde{F}$.

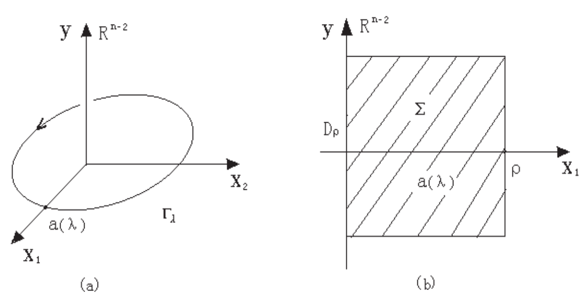


Fig. 5.14

Now we give the Poincare map associated the bifurcated periodic orbits Γ_λ of F. By the assumption, the x-pland $y=0$ is a center manifold of F, then $\Gamma_\lambda \subset R^2$ the x-plane. Let Γ_λ be through the point $(a(\lambda),0)\in R^2$ with $0<a(\lambda)<\rho(\rho$ sufficiently small). Denote by $D_\rho=\{y\in R^{n-2}|\|y\|<\rho\}$. We take the section transversal to F through the point $(a(\lambda),0)$ as (see Fig. 5.14)

$$\begin{aligned}\Sigma &= \{(x_1,y)\in R^{n-1}|0<x_1<\rho, y\in D_\rho\subset R^{n-2}\}\\ &= (0,\rho)\times D_\rho\end{aligned}$$

The Poincare map $P:\sum\to\sum$ is given by

$$P(z)=\{p_1(z),\Phi(t_z,z)\},\quad z=(x_1,y)\in\Sigma=(0,\rho)\times D_\rho$$

where $p_1:\sum\to(0,\rho),\Phi(t_z,\cdot):\sum\to D_\rho$, and

(5.3.69) $$\Phi(t,z)=e^{tB_\lambda}z+\int_0^t e^{(t-\tau)B_\lambda}G_3(x,\Phi)d\tau$$

which is the solution of the equation

$$\begin{cases}\frac{dy}{dt}=B_\lambda y+G_3(x,y,\lambda)\\ y(0)=z\end{cases}$$

and t_z is the time that the orbit through z returns to intersect $\sum$. Because the x-plane is invariant for the vector field (5.3.59), therefore the line segment $(0,\rho)\times\{0\}\subset\sum$ is invariant for the Poincare map $P(z)$. Thus, we have

(5.3.70) $$\Phi(t_z,z)|_{z=(x_1,0)}=0$$

(5.3.71) $$p_1(x_1,0)=r(2\pi,x_1)$$

$$\begin{aligned}&= \text{by (5.3.67)}\\ &= \frac{x_1}{1-c(x_1,\lambda)\alpha(\lambda)-bx_2^2+o(x_1^2)}\\ &= x_1+c(x_1,\lambda)\alpha x_1+bx_1^3+o(|\alpha|,|x_1|^3)\end{aligned}$$

For the fixed point $(a(\lambda),0)\in\sum$ of $p(z)$, from (5.3.69)-(5.3.71) we derive that

$$\begin{aligned}DP(a,0) &= \begin{pmatrix}\frac{\partial p_1(a,0)}{\partial x_1} & *\\ \frac{\partial\Phi(t_z,z)}{\partial x_1}|_{z=(a,0)} & \frac{\partial\Phi(t_z,z)}{\partial y}|_{z=(a,0)}\end{pmatrix}\\ &= \begin{pmatrix}1+c\alpha+3ba^2+o(|\alpha|,|a|^2) & *\\ 0 & e^{t_zB_\lambda}|_{z=(a,0)}\end{pmatrix}\end{aligned}$$

Since $t_z|_{z=(a,0)} = 1$, and $a(\lambda)$ is the positive solution of (5.3.68), we have that

$$a^2(\lambda) = -\frac{1}{b}c \cdot \alpha + o(|\alpha|) = -\frac{2\pi}{b}\alpha + o(|\alpha|)$$

Hence the eigenvalues of $DP(a, 0)$ are as follow

$$1 - 4\pi\alpha(\lambda) + o(|\alpha|), \quad e^{\beta_j(\lambda)}, \quad 3 \leq j \leq n.$$

Because $Re\beta_j(\lambda_0) \neq 0 (3 \leq j \leq n)$, we can see that $DP(a, 0)$ is hyperbolic for all $|\lambda - \lambda_0| > 0$ sufficiently small. Thus, the conclusion iii) is verified.

Step 3. Finally, we prove the conclusion i) for $m = 2$.

We only need to prove the sufficientness because the necessity is obvious. Let $b(\lambda_0) > 0$ and λ_0 be regular, i.e.

$$\alpha(\lambda) = \sigma(\lambda - \lambda_0) + o(|\lambda - \lambda_0|), \quad \sigma \neq 0$$

When $F_1 \in C_0^{3,1}(\Omega \times R, R^n)$ and $\|F_1 - F\|_{C^{3,1}}$ sufficiently small, there exists an eigen-parameter λ_1 of F_1, which is close to λ_0, and in a neighborhood of λ_1,

$$DF_1(0, \lambda) = \begin{pmatrix} \alpha_1(\lambda) & -\beta & \\ \beta & \alpha_1(\lambda) & \\ & 0 & B_\lambda^* \end{pmatrix}, \quad \beta \simeq 1$$

where

$$\alpha_1(\lambda) = \sigma_1(\lambda - \lambda_1) + o(|\lambda - \lambda_1|), \quad \sigma_1 \simeq \sigma$$

and B_λ^* is close to B_λ. Meanwhile, the bifurcation number $b^*(\lambda)$ of F_1 is also close to $b(\lambda_0)$, therefore $b^*(\lambda_1) > 0$. By the steps 1-2 above, F_1 bifurcates an unique branch of perioidic orbits Γ_λ^1 for $\lambda < \lambda_1$ (if $\sigma_1 < 0$, then $\lambda_1 < \lambda$), and which are hyperbolic for all $|\lambda - \lambda_1| > 0$ sufficiently small. Thus the bifurcated branchs Γ_λ^1 and Γ_λ are on the same side of their bifurcation points.

It is knwon that the hyperbolic periodic orbits of vector fields in $C^r(\Omega, R^n)(r \geq 1)$ are locally structurally stable. Because the parameterized vector fields $F \in C_0^{3,1}(\Omega \times R, R^n)$ are continuous on λ in the C^3-norm, in the bifurcated branch of hyperbolic periodic orbits Γ_λ of F, any two $F(\lambda_1, \cdot)$ and $F(\lambda_2, \cdot)$ are locally topologically equivalent at Γ_{λ_1} and Γ_{λ_2}. On the other hand, when F_1 is close to F in $C_0^{3,1}(\Omega \times R, R^n)$, for some fixed parameter $\rho, F_1(\rho, \cdot)$ is close to $F(\rho, \cdot)$ in $C^3(\Omega, R^n)$. Therefore there exists a neighborhood $U \subset C_0^{3,1}(\Omega \times R, R^n)$ of F such that for any $F_1 \in U, F_1(\rho, \cdot)$ and $F(\rho, \cdot)$ are locally topologically equivalent at Γ_ρ^1 and Γ_ρ, which implies that the bifurcation point λ_1 and λ_0 of F_1 and F have the same structure. The proof of Theorem 5.3.1 is complete.

It is not difficult to obtain directly from Theorem 5.3.21 the following global stability theorem on bifurcation

Theorem 5.3.22. A parameterized vector field $F \in C_0^{3,1}(\Omega \times R, R^n)$ is stable on bifurcation, if and only if
1). all the eigen-parameters of F are simple and regular;
2). each of the bifurcation numbers of F at eigen-parameters is nonzero.
Moreover, the set of all vector fields with stable bifurcation is open and dense in $C_0^{3,1}(\Omega \times R, R^n)$.

Obviously, if all the eigen-parameters of F are regular, then the eigen-parameters must be discrete.

Remark 5.3.23. It is worth to point that the order $k = 3$ of $C_0^{k,1}(\Omega \times R, R^n)$ in Theorem 5.3.22 can not be reduced anymore, because in $C_0^{1,1}(\Omega \times R, R^n)$ there are no vector fields with stable bifurcation, and in $C_0^{2,1}(\Omega \times R, R^n)$, the set of all vector fields with stable bifurcation in $C_0^{2,1}(\Omega \times R, R^n)$ are those whose eigent-parameters regular and simple with multiplicity $m = 1$.

5.4. Bifurcation of Invariant Sets of Infinite Dimensional Fields

In this section, we shall generalize the dynamic bifurcation theory of invariant sets of finite dimensional systems in Section 5.3 to the infinite dimensional fields in Hilbert spaces. These generalizations can be applied to the bifurcation problems of various partial differential equations, for instance, the Navier-Stokes equations, dissipative wave equations and reaction-diffusion equations, ets.

5.4.1. Locally invariant manifolds

Let H, H_1 be the Hilbert spaces, and $H_1 \hookrightarrow H$ be an dense inclusion embedding. We consider the nonlinear evolution equations given by

$$\begin{cases} \frac{du}{dt} = L_\lambda u + G(u, \lambda), \quad u \in H_1, \quad \lambda \in R \\ u(0) = \phi \end{cases} \tag{5.4.1}$$

where $G(\cdot, \lambda) : H_1 \to H$ are the parameterized C^r bounded operators $(r \geq 1)$ continuously depending on the parameter $\lambda \in R$, and

$$G(u, \lambda) = o(\|u\|_{H_1}), \quad \forall \lambda \in R \tag{5.4.2}$$

and $L_\lambda : H_1 \to H$ are the parameterized linear completely continuous fields continuously depending on $\lambda \in R$, which satisfy

$$(5.4.3) \qquad \begin{cases} L_\lambda = -A + B_\lambda \\ A : H_1 \to H \textit{ a linear homeomorphism} \\ B_\lambda : H_1 \to H \textit{ the parameterized linear compact operators} \end{cases}$$

In this subsection, we give the generalizations of the center manifold theorems of finite dimensional systems to the infinite dimensional system (5.4.1), although these generalizations have been essentially known.

We assume that the operators L_λ are generators of strongly continous semigroups of bounded linear operators, and the spaces H_1 and H can be decomposed into

$$(5.4.4) \qquad \begin{cases} H_1 = E_1^\lambda \bigoplus E_2^\lambda, \quad dim E_1^\lambda < \infty \\ H = \widetilde{E}_1^\lambda \bigoplus \widetilde{E}_2^\lambda, \quad \widetilde{E}_1^\lambda = E_1^\lambda, \quad \widetilde{E}_2^\lambda = \textit{ closure of } E_2^\lambda \textit{ in } H \end{cases}$$

where E_1^λ and E_2^λ are the invariant subspaces of L_λ, i.e. L_λ are decomposed into $L_\lambda = \mathcal{L}_1^\lambda \bigoplus \mathcal{L}_2^\lambda$, and

$$(5.4.5) \qquad \begin{cases} \mathcal{L}_1^\lambda = L_\lambda|_{E_1^\lambda} : E_1^\lambda \to \widetilde{E}_1^\lambda \\ \mathcal{L}_2^\lambda = L_\lambda|_{E_2^\lambda} : E_2^\lambda \to \widetilde{E}_2^\lambda \end{cases}$$

in where the eigenvalues of $\mathcal{L}_1^\lambda$ and $\mathcal{L}_2^\lambda$ respectively possese the non-negative real parts(this condition is not necessarily required in the center manifold theorem) and the negative real parts.

Thus, the equations (5.4.1) can be written as to read

$$(5.4.6) \qquad \begin{cases} \frac{dx}{dt} = \mathcal{L}_1^\lambda x + G_1(x, y, \lambda) \\ \frac{dy}{dt} = \mathcal{L}_2^\lambda y + G_2(x, y, \lambda) \end{cases}$$

where $u = x + y \in H_1, x \in E_1^\lambda, y \in E_2^\lambda$, and

$$\begin{cases} G_i(x, y, \lambda) = P_i G(u, \lambda) \quad (i = 1, 2) \\ P_i H \to \widetilde{E}_i \textit{ are the projective operators} \end{cases}$$

Let $S_\lambda(t) : E_2^\lambda \to \widetilde{E}_2^\lambda$ be the strongly continuous semigroups generated by $\mathcal{L}_2^\lambda$. We have the following theorem on the locally invariant manifolds for the system (5.4.1).

Theorem 5.4.1. Let $G(u, \lambda)$ be $C^r (r \geq 1)$ on $u \in H_1$. Under the hypotheses (5.4.2)-(5.4.5), if

$$(5.4.7) \qquad \|S_\lambda(t)\| \leq K_\lambda e^{-\alpha_\lambda t}, \quad \textit{for some constants } K_\lambda, \alpha_\lambda > 0$$

then there exists a number $\delta > 0$ such that as $|\lambda - \lambda_0| < \delta$, there are neighborhood $\Omega_\lambda \subset E_1^\lambda$ of $x = 0$ and C^r function $h(\cdot, \lambda) : \Omega_\lambda \to E_1^\lambda$, continuously depending on λ, which satisfy
i). $h(0, \lambda) = 0, D_x h(0, \lambda) = 0$
ii). the sets

$$M_\lambda = \{(x, y) \in H_1| \quad x \in \Omega_\lambda, y = h(x, \lambda) \in E_2^\lambda\}$$

called the local center manifolds, are locally invariant for the system (5.4.1), i.e. $\forall \phi \in M_\lambda, \exists t_\phi > 0$ such that

$$u_\lambda(t, \phi) \in M_\lambda, \quad \forall 0 \leq t < t_\phi$$

where $u_\lambda(t, \phi)$ is the solution of (5.4.1);
iii). if $(x_\lambda(t), y_\lambda(t))$ is a solution of (5.4.6), then there is a $\beta_\lambda > 0$ and $k_\lambda > 0$ with k_λ depending on $(x_\lambda(0), y_\lambda(0))$ such that

$$\|y_\lambda(t) - h(x_\lambda(t), \lambda)\|_H \leq k_\lambda e^{-\beta_\lambda t}$$

Proof. The ideas employed here follows the finite dimensional cases (Cf.[CH]). For the sake of simplicity, in all satuation without confusion, we drop the λ in the functions, for example denote $G_i(x, y) = G_i(x, y, \lambda)$, etc. Let $\epsilon > 0$, and $\rho_\epsilon : E_1 \to [0, 1]$ be C^∞ function with

$$\rho_\epsilon(x) = \begin{cases} 1, & \|x\| < \epsilon \\ 0, & \|x\| > 2\epsilon \end{cases}$$

We denote

$$C^r(E_1, E_2) = \{h : E_1 \to E_2| \quad h(0) = 0, h \ is \ r - th \ defferentiable\}$$

We only proceed for the case $r = 1$, and the proof of r-th differentiable of $h(x)$ is omitted here. For each $h \in C^1(E_1, E_2)$, there is an unique solution $x(t, h, \phi)$ of the inital value problem

$$\text{(5.4.8)} \qquad \begin{cases} \frac{dx}{dt} = \mathcal{L}_1 x + \rho_\epsilon(x) G_1(x, h(x)) \\ x(0) = \phi, \phi \in E_1 \end{cases}$$

and $x(t, h, \phi)$ is differentiable on h and ϕ. We define a mapping $T : [0, \infty) \times R \times C^1(E_1, E_2) \to C^1(E_1, E_2)$ by

$$\begin{cases} T(\epsilon, \lambda, h) = \int_{-\infty}^0 S(-\tau) \rho_\epsilon(x(\tau, h, \cdot)) G_2(x(\tau, h, \cdot), h(x(\tau, h, \cdot))) d\tau \\ T(0, \lambda, h) = 0 \end{cases}$$

If this definition makes sense and T has a fixed point $h \in C^1(E_1, E_2)$

$$h(\cdot) = \int_{-\infty}^{0} S(-\tau)\rho_\epsilon(x(\tau, h, \cdot))G_2(x(\tau, h, \cdot), h(x(\tau, h, \cdot)))d\tau$$

then it is clear that for the solution $x(t, h, \phi)$ of (5.4.8), simply denoted by $x(t, \phi)$, the function

$$\begin{aligned} y(t, h(\phi)) &= h(x(t, \phi)) \\ &= \int_{-\infty}^{0} s(-\tau)\rho_\epsilon(x(\tau, h, x(t, \phi)))G_2(x(\tau, h, x(t, \phi)), h)d\tau \\ &= \int_{-\infty}^{0} s(-\tau)\rho_\epsilon(x(\tau + t, h, \phi))G_2(x(t + \tau, h, \phi), h)d\tau \\ &= \int_{-\infty}^{t} s(t - s)\rho_\epsilon(x(s, \phi))G_2(x(s, \phi), h(x(s, \phi)))d\tau \end{aligned}$$

satisfies the equation

$$\begin{cases} \frac{dy}{dt} = \mathcal{L}_2 y + \rho_\epsilon(x(t, \phi))G_2(x(t, \phi), y) \\ y(0) = h(\phi) \end{cases}$$

Thus, $\{x(t, \phi), h(x(t, \phi))\}$ is a solution of the below problem

$$\begin{cases} \frac{dx}{dt} = \mathcal{L}_1 x + \rho_\epsilon(x)G_1(x, y) \\ \frac{dy}{dt} = \mathcal{L}_2 y + \rho_\epsilon(x)G_2(x, y) \\ x(0) = \phi, y(0) = h(\phi) \end{cases}$$

which means that the set $M = \{(x, y) \in H_1 | x \in B_\epsilon, y = h(x)\}$ is a locally invariant manifold of (5.4.1), where $B_\epsilon = \{x \in E_1 | \|x\|_{H_1} < \epsilon\}$.

In the following we divide a few steps to prove this theorem.

Step 1. The mapping T makes sense.

Because $G_2(x, y), h(x)$ are C^1 on their arguments, and the solution $x(t, h, \phi)$ of (5.4.8) is C^1 on $\phi \in E_1$, if the integral of T exists, then $T(\epsilon, \lambda, h)(\phi)$ is differentiable on $\phi \in H_1$. On the other hand, by (5.4.2) and $h(0) = 0, x(t, h, 0) = 0$, we have that $T(\epsilon, \lambda, h)(0) = 0$. Hence, we only need to show that the integral of T makes sense, namely, the following limit exists

(5.4.9)
$$\lim_{t \to \infty} \int_{0}^{t} s(\tau)\rho_\epsilon(x(-\tau, h, \phi))G_2(x(-\tau, h, \phi), h(x(-\tau, h, \phi)))d\tau, \quad \forall \phi \in E_1$$

By the theory of semigroups of linear operators, we know that for any differentiable curve $f \in C^1([0, \infty), H)$ and a strongly continuous semigroup $T(t)$:

$H \to H$ generated by $L : H_1 \to H$, which is invertible, we have

$$\int_{t_1}^{t_2} T(\tau) f(\tau) d\tau \in H_1, \quad \forall t_2 > t_1 \geq 0$$

and

$$\| \int_{t_1}^{t_2} T(\tau) f(\tau) d\tau \|_{H_1} = \| L \int_{t_1}^{t_2} T(\tau) f(\tau) d\tau \|_H$$

$$= \| T(t_2) f(t_2) - T(t_1) f(t_1) + \int_{t_1}^{t_2} T(\tau) \frac{df}{dt}(\tau) d\tau \|_H$$

Thus we obtain

$$\| \int_{t_1}^{t_2} s(\tau) \rho_\epsilon G_2 d\tau \|_{H_1} \leq \| s(t_2) \rho_\epsilon G_2 \|_H + \| s(t_1) \rho_\epsilon G_2 \|_H$$

$$+ \int_{t_1}^{t_2} \| s(\tau) \| \cdot \| \frac{d}{dt} (\rho_\epsilon G_2) \|_H d\tau$$

By (5.4.7), from the boundedness of $\|\rho_\epsilon G_2\|_H$ and $\|\frac{d}{dt}(\rho_\epsilon G_2)\|_H$ we can derive that

$$\lim_{t_1 \to \infty t_2 \to \infty} \| \int_{t_1}^{t_2} s(\tau) \rho_\epsilon (x(-\tau, h, \phi)) G_2(x(-\tau, h, \phi), h) d\tau \|_{H_1} = 0$$

$\forall \phi \in E_1$. It implies that the limit (5.4.9) exists.

Step 2. The mapping T has a fixed point.

Obviously, $T(\epsilon, \lambda, h)$ is continuous on the arguments $\epsilon > 0$ and $\lambda \in R$. We shall prove that

$$\lim_{\epsilon \to 0 \lambda \to \lambda_0} T(\epsilon, \lambda, h) = 0 \quad in\ C^1(E_1, E_2)$$

It suffices to show that

$$\lim_{\epsilon \to 0 \lambda \to \lambda_0} \| D_\phi T(\epsilon, \lambda, h)(\phi) \| = 0, \quad \forall h \in C^1(E_1, E_2)\ and\ \phi \in E_1$$

For a given $h \in C^1(E_1, E_2)$, we see that

$$D_\phi T(\epsilon, \lambda, h) = \int_0^\infty s(\tau) [D_x \rho_\epsilon \cdot D_\phi x G_2(x_h, h) +$$

$$+ \rho_\epsilon D_x G_2(x_h, h) D_\phi x + \rho_\epsilon D_y G_2(x_h, h) \cdot D_x h \cdot D_\phi x] d\tau$$

where $x_h = x_h(-t, \phi)$ satisfies

$$\text{(5.4.11)} \qquad \begin{cases} \frac{dx}{dt} = -\mathcal{L}_1 x - \rho_\epsilon(x) G_1(x, h) \\ x(0) = \phi \end{cases}$$

and $z = D_\phi x(-t)$ satisfies

$$\text{(5.4.12)} \qquad \begin{cases} \frac{dx}{dt} = -\mathcal{L}_1 z - \rho_\epsilon(x_h) D_x G_1(x_h, h) z - D_x \rho_\epsilon(x_h) G_1(x_h, h) z \\ -\rho_\epsilon(x_h) D_y G_1(x_h, h) \cdot D_x h \cdot z \\ z(0) = 1 \end{cases}$$

Due to the action of the cut-off function ρ_ϵ, by (5.4.2), for any $\phi \in E_1$ we can get the estimates

$$\|G_i(x_h, h)\| \leq c(\epsilon) \cdot \epsilon, \quad \forall x_h \in B_{2\epsilon} \quad (i = 1, 2)$$

$$\|D_x G_2(x_h, h)\| + \|D_y G_2(x_h, h)\| \leq c(\epsilon), \quad \forall x_h \in B_{2\epsilon}$$

where $c(\epsilon) \to 0$ as $\epsilon \to 0$. By the definition, we have

$$\|D_x \rho_\epsilon(x_h)\| \leq c \cdot \epsilon^{-1}, \quad c > 0 \textit{ a constant.}$$

On the other hand, because $-\mathcal{L}_1^\lambda$ has no eigenvalues with the positive real parts at $\lambda = \lambda_0$, by (5.4.2), from (5.4.11) and (5.4.12) we can obtain the estimates

$$\|x_h(-t, \phi)\| \leq c e^{t\theta_1(\epsilon, \lambda)}, \quad \theta_1(\epsilon, \lambda) \to 0 \textit{ as } \epsilon \to 0 \textit{ and } \lambda \to \lambda_0$$

$$\|D_\phi x(-t)\| \leq c e^{t\theta_2(\epsilon, \lambda)}, \quad \theta_2(\epsilon, \lambda) \to 0 \textit{ as } \epsilon \to 0 \textit{ and } \lambda \to \lambda_0$$

By (5.4.7), from the estimates above the equality (5.4.10) follows.

We now consider the mapping

$$K(\epsilon, \lambda, \cdot) = id - T(\epsilon, \lambda, \cdot) : C^1(E_1, E_2) \to C^1(E_1, E_2)$$

By (5.4.10) we see that $K(\epsilon, \lambda, h)$ is continuous on (ϵ, λ) in a neighborhood of $(\epsilon, \lambda, h) = (0, \lambda_0, 0)$. We shall use the implicity function theorem to prove this claim. To this end, we need to show that T is differentiable on h at $(0, \lambda_0, 0)$. We see that

$$D_h T(\epsilon, \lambda_0, 0) = \int_0^\infty S(\tau)[D_x \rho_\epsilon(\widetilde{x}) D_h x(-\tau) G_2(\widetilde{x}, 0) +$$

$$+ \rho_\epsilon(\widetilde{x}) D_x G_2(\widetilde{x}, 0) D_h x + \rho_\epsilon(\widetilde{x}) D_y G_2(\widetilde{x}, 0)] d\tau$$

where $\widetilde{x} = \widetilde{x}(-t, \phi)$ satisfies

$$\begin{cases} \frac{d\widetilde{x}}{dt} = -\mathcal{L}_1 \widetilde{x} - \rho_\epsilon(\widetilde{x}) G_1(\widetilde{x}, 0) \\ \widetilde{x}(0) = \phi \end{cases}$$

and $Z = D_h x(-t)$ satisfies

$$\begin{cases} \frac{dz}{dt} = -\mathcal{L}_1 z - \rho_\epsilon D_x G_1(\widetilde{x}, 0)z - D_x\rho_\epsilon G_1(\widetilde{x}, 0)z - \rho_\epsilon D_y G_1(\widetilde{x}, 0) \\ z(0) = 0 \end{cases}$$

In the same fashion as above we can derive that the derivative $D_h T(\epsilon, \lambda_0, 0)$ exists, and

$$\lim_{\epsilon \to 0} D_h T(\epsilon, \lambda_0, 0) = 0$$

Hence we obtain

$$D_h K(0, \lambda_0, 0) = id - D_h T(0, \lambda_0, 0) = id$$

By the implicity function theorem we get that there is a number $\delta > 0$ such that as $|\lambda - \lambda_0| < \delta$, there exists $\epsilon_\lambda > 0$ and $h_\lambda \in C^r(E_1, E_2)$ which satisfy

$$h_\lambda = T(\epsilon_\lambda, \lambda, h_\lambda)$$

moreover, h_λ continuously depends on λ. By (5.4.2), it is easy to see that $D_x h(0) = 0$. Thus the conclusion i) and ii) are verified.

Step 3. Finally we show the conclusion iii).
Denote by

$$C^1(R^+ \times H_1, E_2) = \{J(t, x, y) \in E_2| \quad t \in R^+, x \in E_1, y \in E_2, J(t, 0, 0) = 0\}$$

We define a mapping

$$T : R^+ \times C^1(R^+ \times H_1, E_2) \to C^1(R^+ \times H_1, E_2)$$

as that, $\forall \widetilde{J} \in C^1(R^+ \times H_1, E_2)$ and $\epsilon \in H^+$

$$T(\epsilon, \widetilde{J}) = \int_0^t S(t - \tau)\rho_\epsilon \cdot G_2 d\tau$$

where $\rho_\epsilon : H_1 \to [0, 1]$ is the C^∞ cut-off function defined on H_1, and

$$\begin{aligned} &\rho_\epsilon = \rho_\epsilon(x(\tau - t, x_0, y_0, J), J(\tau, x(\tau - t, x_0, y_0, J), y_0)) \\ &G_2 = G_2(x(\tau - t, x_0, y_0, J), J(\tau, x(\tau - t, x_0, y_0, J), y_0)) \\ &J(t, x_0, y_0) = S(t)y_0 + \widetilde{J}(t, x_0, y_0) \end{aligned}$$

and $x(t, x_0, y_0, J)$ is the unique solution of the problem

$$\begin{cases} \frac{dx}{dt} = \mathcal{L}_1 x + \rho_\epsilon(x, J(t, x, y_0))G_1(x, J(t, x, y_0)) \\ x(0) = x_0 \end{cases}$$

Consider the mapping

$$K(\epsilon,\widetilde{J})=\widetilde{J}-T(\epsilon,\widetilde{J})$$

Obviously, $K(0,0)=0$. In the same manner as above, we can get that

$$D_JK(0,0)=id-D_JT(0,0)=id$$

Therefore T has a fixed point $\widetilde{J}$ as $\epsilon>0$ sufficiently small, namely for $J=\widetilde{J}+S(t)y_0$, we have

$$J(t,x_0,y_0)=S(t)y_0+\int_0^t S(t-\tau)\rho_\epsilon\cdot G_2d\tau \tag{5.4.13}$$

It is clear that if $(x(t),y(t))$ is a solution of the equatin

$$\begin{cases}\frac{dx}{dt}=\mathcal{L}_1x+\rho_\epsilon(x,y)G_1(x,y)\\ \frac{dy}{dt}=\mathcal{L}_2y+\rho_\epsilon(x,y)G_2(x,y)\end{cases} \tag{5.4.14}$$

then

$$y(t)=J(t,x(t),y(0)) \tag{5.4.15}$$

From (5.4.2)(5.4.7) and (5.4.13) we can deduce that for any $x\in E_1,y_1,y_2\in E_2$ with $\|x\|,\|y_1\|$ and $\|y_2\|$ sufficiently small, there exist $K,\beta>0$ such that

$$\|J(t,x,y_1)-J(t,x,y_2)\|_H\leq Ke^{-\beta t} \tag{5.4.16}$$

Let $\{x(t,\xi,\eta),y(t,\xi,\eta)\}$ denote the solution of (5.4.14) with $x(0)=\xi,y(0)=\eta$. For $\{x(t),y(t)\}=\{x(t,x_0,y_0),y(t,x_0,y_0)\}$, $\|x_0\|_{H_1}+\|y_0\|_{H_1}<\epsilon$, and any $t_0\geq 0$ we define

$$\xi_1=x(t_0,x_0,y_0),\quad \eta_1=h(\xi_1)$$

Let M be the center manifold. For $(\xi_1,\eta_1)\in M$, there exists $(\xi_0,\eta_0)\in M$ such that

$$x(t_0,\xi_0,\eta_0)=\xi_1,\quad y(t_0,\xi_0,\eta_0)=\eta_1$$

Hence, by (5.4.15)

$$y(t_0,x_0,y_0)=J(t_0,\xi_1,y_0)$$

$$\begin{aligned}y(t_0,\xi_0,\eta_0)&=J(t_0,x(t_0,\xi_0,\eta_0),\eta_0)\\&=J(t_0,\xi_1,\eta_0)\\&=\ by\ the\ invariance\ of\ center\ manifold\ M\\&=h(x(t_0,\xi_0,\eta_0))\end{aligned}$$

By (5.4.16) we get

$$\begin{aligned}\|y(t_0) - y(t_0, \xi_0, \eta_0)\|_H &= \|y(t_0) - h(x(t_0))\|_H \\ &\leq Ke^{-\beta t_0}, \ for \ t_0 \geq 0\end{aligned}$$

The proof is complete.

Now, we consider a class of special linear completly continuous fields $L_\lambda = -A + B_\lambda$ in (5.4.1), which are the generators of the analytic semigroups. Suppose that $H_1 \hookrightarrow H$ is compact, and there exist real eigenvalue sequence $\{\rho_K\} \subset R$ and eigenvector sequence $\{\phi_k\} \subset H_1$ of A, with

$$(5.4.17) \qquad \begin{cases} A\phi_k = \rho_k \phi_k \\ 0 < \rho_1 \leq \rho_2 \leq \cdots; \quad \rho_k \to +\infty (k \to \infty) \end{cases}$$

such that $\{\phi_k\}$ is a common orthogonal base of H_1 and H. This condition means that A is symmetric. As defined in Subsection 3.1.3, we can define the Hilbert spaces as follows

(5.4.18)

$$\begin{cases} H_\alpha = \{x \in H| \quad x = \sum_{i=1}^\infty x_i \phi_i, \quad \sum_{i=1}^\infty \rho_i^{2\alpha} x_i^2 < \infty\}, \quad 0 \leq \alpha < \infty \\ < x, y >_{H_\alpha} = < A^\alpha x, A^\alpha y >_H = \sum_{k=1}^\infty \rho_k^{2\alpha} x_k y_k \end{cases}$$

and $H_\beta \hookrightarrow H_\alpha (H_0 = H)$ is compact $\forall 0 \leq \alpha < \beta$. We also assume that there is a constant $0 < \theta < 1$, such that

$$(5.4.19) \qquad B_\lambda : H_\theta \to H \ bounded.$$

Theorem 5.4.2. Let $G(u, \lambda)$ be $C^r (r \geq 1)$ on $u \in H_1$. Under the conditions (5.4.17) and (5.4.19), for the system (5.4.1) the conclusions of Theorem 5.4.1 are valid.

Proof. By Theorem 3.1.4, the eigenvalues of L_λ with nonnegative real parts are finite. Hence the conditions (5.4.4)-(5.4.6) are valid.

On the other hand, by Theorem 3.1.6, the semigroups $T_\lambda(t)$ generated by L_λ are analytic. Hence the semigroups $S_\lambda(t) : \widetilde{E}_2 \to \widetilde{E}_2$ generated by $\mathcal{L}_2^\lambda : E_2 \to \widetilde{E}_2$ are also analytic. Because the operators $\mathcal{L}_2^\lambda$ have no the eigenvalues with real parts ≥ 0, thus we obtain the condition (5.4.7). By Theorem 5.4.1, this theorem is proven.

If we only consider the existence of the local invariant manifold, then the condition that the eigenvalues of the operators $\mathcal{L}_1^\lambda$ and $\mathcal{L}_2^\lambda$ in (5.4.5) respectively have the nonnegative real parts and negative real parts can be relaxed as follows

$$(5.4.5)' \qquad \begin{cases} \mathcal{L}_1^\lambda = L_\lambda|_{E_1^\lambda} : E_1^\lambda \to \widetilde{E}_1^\lambda \\ \mathcal{L}_2^\lambda = L_\lambda|_{E_2^\lambda} : E_2^\lambda \to \widetilde{E}_2^\lambda \end{cases}$$

in where $\mathcal{L}_2^\lambda$ has a finite number of eigenvalues possessing positive real parts, and $E_2^\lambda, \widehat{E}_2^\lambda, \mathcal{L}_2^\lambda$ can be decomposed into

$$\mathcal{L}_2^\lambda = \mathcal{L}_{21}^\lambda \bigoplus \mathcal{L}_{22}^\lambda, E_2^\lambda = E_{21}^\lambda \bigoplus E_{22}^\lambda, \widetilde{E}_2^\lambda = \widetilde{E}_{21}^\lambda \bigoplus \widetilde{E}_{22}^\lambda$$

$$\mathcal{L}_{2i}^\lambda : E_{2i}^\lambda \to \widetilde{E}_{2i}^\lambda \quad (i = 1, 2)$$

such that dim $E_{21}^\lambda < \infty$, and $\mathcal{L}_{22}^\lambda$ has no eigenvalues possessing positive real parts.

Thus, Theorem 5.4.1 can be rewritten as follows.

Theorem 5.4.1(b): Under the conditions (5.4.2)-(5.4.4) and (5.4.5)$'$, if the operator $\mathcal{L}_{22}^\lambda$ has the semigroup $S_\lambda(t)$ satisfying (5.4.7), then the conclusions i) and ii) in Theorem 5.4.1 still hold true.

5.4.2. Equations of the first order in time

We consider the bifurcation of invariant sets of the following nonlinear evolution equations

$$(5.4.20) \qquad \begin{cases} \frac{du}{dt} = L_\lambda u + G(u, \lambda) \\ u(0) = \phi \end{cases}$$

where $G(u, \lambda)$ satisfy (5.4.2) and $L_\lambda = -A + B_\lambda$ satisfy (5.4.3),(5.4.17) and (5.4.19). Denote the eigenvalues (counting multiplicities) of L_λ by

$$\beta_1(\lambda), \beta_2(\lambda), \cdots; \qquad \beta_k(\lambda) \in C \textit{ the complex space.}$$

Suppose that

$$(5.4.21) \qquad \begin{cases} Re\beta_i(\lambda) = \begin{cases} < 0, & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0 \end{cases} \quad (1 \le i \le m) \\ Re\beta_j(\lambda_0) < 0, \forall m + 1 \le j \end{cases}$$

As in Section 5.3, for the system (5.4.20), we have the following attractor bifurcation theorems.

Theorem 5.4.3. Let $m = 1$ in (5.4.21), and $e_0, e_0^* \in H_1$ be respectively the eigenvectors of L_{λ_0} and $L_{\lambda_0}^*$ corresponding to $\beta_1(\lambda_0) = 0$. If for given

$u_0 \in H_1(u_0 \neq 0)$ and $\beta \in R, \|G(\beta u_0, \lambda_0)\|_H \sim \beta^k$(as $\beta \to 0$) for some $k > 1$, and

(5.4.22) $$< G(xe_0, \lambda_0), e_0^* >_H = \alpha|x|^{k-1}x + o(|x|^k), \quad \alpha < 0$$

then there exists an neighborhhod $U \subset H_1$ of $u = 0$, such that as $\lambda > \lambda_0$ the system (5.4.20) bifurcates from $u = 0$ exactly two equilibrum points u_1^λ and $u_2^\lambda \in U$, and the open set U is decomposed into two open sets U_1^λ and U_2^λ,

$$\bar{U}^\lambda = \bar{U}_1^\lambda + \bar{U}_2^\lambda, \quad U_1^\lambda \bigcap U_2^\lambda = \phi, \; and \; 0 \in \bar{U}_1^\lambda \bigcap \bar{U}_2^\lambda.$$

with $u_i^\lambda \in U_i^\lambda (i = 1, 2)$, such that

$$\lim_{t\to\infty} \|u_\lambda(t, \phi)\|_H = u_i^\lambda, \qquad as \; \phi \in U_i^\lambda (i = 1, 2)$$

where $u_\lambda(t, \phi)$ are the solutions of (5.4.20).

Proof. Let L_λ^* be the conjugate operator of L_λ, and e_λ and $e_\lambda^* \in H_1$ respectively the eigenvectors of L_λ and L_λ^* corresponding to $\beta_1(\lambda)(e_0 = e_{\lambda_0}, e_0^* = e_{\lambda_0}^*)$. We know that the system (5.4.20) can be decomposed into

$$\begin{cases} \frac{dx}{dt} = \beta_1(\lambda)x + < G(u, \lambda), e_\lambda^* >_h, \\ \frac{dy}{dt} = \mathcal{L}^\lambda y + P_2 G(u, \lambda) \end{cases}$$

where

$$H_1 = E_1^\lambda \bigoplus E_2^\lambda$$

$$E_1^\lambda = span\{e_\lambda\}, \quad E_2^\lambda = \{y \in H_1 | < y, e_\lambda^* > = 0\}$$

and $u = xe_\lambda + y, x \in R^1, y \in E_2^\lambda, P_2 : H \to \widetilde{E}_2^\lambda$ the projective, $\mathcal{L}^\lambda : E_2^\lambda \to \widetilde{E}_2^\lambda$ the linear operator possessing the eigenvalues $\beta_j(\lambda)(2 \leq j)$. Let $h_\lambda : (\lambda_0 - \delta, \lambda_0 + \delta) \to E_2^\lambda$ be the function of center manifold, $\delta > 0$. By Theorem 5.4.2, it suffices to only consider the bifurcation equation of dimension one

$$\frac{dx}{dt} = \beta_1(\lambda)x + < G(xe_\lambda + h(x, \lambda)), e_\lambda^* >_H .$$

At $\lambda = \lambda_0$ and $u = xe_0$, we have the Taylor expansion

$$< G(xe_0 + h(x)), e_0^* >_H = < G(xe_0), e_0^* >_H +$$

$$+ \sum_{p=1}^{r} < D^p G(xe_0) h^p(x), e_0^* >_H + o(|h|^r)$$

Because $G(xe_0) \sim x^p D^p G(xe_0)$ as $x \to 0$, by (5.4.22) and $h(x) = o(|x|)$, we have

$$< G(xe_0 + h(x)), e_0^* >_H = \alpha|x|^{k-1}x + o(|x|^k), \alpha < 0.$$

The remander proof is the same as that of Theorem 5.3.1. This proof is complete.

Theorem 5.4.4. Under the condition (5.4.21), if $u = 0$ is asymptotically stable for (5.4.20) at $\lambda = \lambda_0$, then the following assertions hold true for $\lambda - \lambda_0 > 0$ sufficiently small.

i). The system (5.4.20) bifurcates an attractor $\sum_\lambda$ with $m-1 \leq dim \sum_\lambda \leq m$, which is connected as $m > 1$.

ii). $\sum_\lambda$ is a limit of a family of m-manifolds $M_\tau (0 \leq \tau < \infty)$ with boundary, which have the homotopy type of m-annulus:
$\Sigma_\lambda = \bigcap_{\tau \geq 0} M_\tau, \quad M_{\tau_2} \subset M_{\tau_1} \ as \ \tau_2 \geq \tau_1.$

iii) If $\sum_\lambda$ is a finite simplicial complex, then $\sum_\lambda$ is a deformation retract of a m-manifold with boundary having the homotopy type of m-annulus, hence $\sum_\lambda$ has the homotopy type of S^{m-1}.

iv). If the mapping $G(\cdot, \lambda) : H_1 \to H$ are compact, and the equilibrum points of (5.4.20) in $\sum_\lambda$ are finite, then we have the index formula

$$\sum_{u_i \in \sum_\lambda} ind[-(L_\lambda + G(\cdot, \lambda)), u_i] = \begin{cases} 2, & m = odd \\ 0, & m = even \end{cases}$$

This theorem can be directly obtained from Theorem 5.4.2 and Theorem 5.3.4. For the more general cases, we have the following results.

Theorem 5.4.5. Under the condition (5.4.21), if $u = 0$ is unstable for (5.4.20) at $\lambda = \lambda_0$, then (5.4.20) must bifurcate from $(0, \lambda_0)$ an invariant set $\sum_\lambda$ with $0 \leq dim \sum_\lambda \leq m$.

We now replace the condition (5.4.21) by

$$(5.4.23) \qquad \begin{cases} Re\beta_i(\lambda) = \begin{cases} < 0 \ (or \ > 0), & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0 \ (or \ < 0), & \lambda > \lambda_0 \end{cases} \quad (1 \leq i \leq m) \\ Re\beta_j(\lambda_0) \neq 0, \forall m+1 \leq j \end{cases}$$

Then we have

Theorem 5.4.6. Let $m = 2$ in (5.4.23). If the system (5.4.20) has no bifurcation of the equilibrum points from $(0, \lambda_0)$, then it must bifurcate a periodic orbit.

Remark 5.4.7. It is interesting to know whether the following conclusion is valid or not:

Under the condition (5.4.23), the system (5.4.20) must bifurcate from $(0, \lambda_0)$ a k-dimensional invariant manifold with $0 \le k \le m-1$.

Remark 5.4.8. If the conditions (5.4.21) and (5.4.23) are relaxed as to read

$$\begin{aligned} &Re\beta_i(\lambda) = \begin{cases} < 0, & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \end{cases} \qquad (1 \le i \le N) \\ &Re\beta_k(\lambda) > 0, \lambda > \lambda_0 \ for\ 1 \le k \le m \\ &Re\beta_l(\lambda) < 0, \lambda > \lambda_0 \ for\ m+1 \le l \le N \\ &Re\beta_j(\lambda_0) < 0 \ (or\ \ne 0), \forall N+1 \le j \end{aligned}$$

then the theorems above still hold true.

5.4.3. Equations of the second order in time

This subsection is devoted to the attractor bifurcation for a class of abstract nonlinear evolution equations of the second order in time with some damping terms. We first consider the system given by

$$\begin{cases} \frac{d^2u}{dt^2} + 2\alpha\frac{du}{dt} = L_\lambda u + G(u, \lambda), & \alpha > 0 \\ u(0) = \phi, u_t(0) = \psi \end{cases} \tag{5.4.24}$$

We always assume that the operators G and $L_\lambda = -A + B_\lambda$ satisfy the conditions (5.4.2)(5.4.3) and (5.4.17).

For the system (5.4.24), when the operators B_λ are symmetric for all $\lambda \in R$, namely

$$< B_\lambda u, v >_H = < u, B_\lambda v >_H, \quad \forall u, v \in H_1 \tag{5.4.25}$$

then, by Theorem 3.1.4, the operators L_λ, which are also symmetric, have the complete real eigenvalues

$$\beta_1(\lambda) \ge \beta_2(\lambda) \ge \cdots; \quad \beta_k(\lambda) \to -\infty (k \to \infty)$$

and the eigenvector sequences $\{e_k(\lambda)\} \subset H_1$ consist of the orthogonal base of H.

Theorem 5.4.9. Let the hypotheses (5.4.21) and (5.4.25) hold, and $m = 1$ in (5.4.21). If for give $u_0 \in H_1 (u_0 \ne 0)$ and $\beta \in R, \|G(\beta u_0, \lambda_0)\|_H \sim \beta^k$(as $\beta \to 0$) for some $k > 1$, and

$$< G(xe_1, \lambda_0), e_1 >_H = -\sigma|x|^{k-1}x + o(|x|^k,), \sigma > 0$$

where $e_1 = e_1(\lambda_0)$, then there exists an open set $U \subset H_1 \times H_{\frac{1}{2}}, (0,0) \in U$, such that when $\lambda > \lambda_0$ the system (5.4.24) bifurcates from $(u, u_t) = (0,0)$ exactly two equilibrum points $(u_1^\lambda, 0)$ and $(u_2^\lambda, 0) \in U$, and U is decomposed into two open sets U_1^λ, U_2^λ:

$$\bar{U} = \bar{U}_1^\lambda + \bar{U}_2^\lambda, \quad U_1^\lambda \bigcap U_2^\lambda = \phi, \ and \ (0,0) \in \bar{U}_1^\lambda \bigcap \bar{U}_2^\lambda$$

with $(u_i^\lambda, 0) \in U_i^\lambda (i = 1,2)$ such that

$$\lim_{t\to\infty} \|u(t,\lambda,\phi,\psi)\|_{H_{\frac{1}{2}}} = u_i^\lambda \ as \ (\phi,\psi) \in U_i^\lambda$$

$$\lim_{t\to\infty} \|\frac{d}{dt} u(t,\lambda,\phi,\psi)\|_H = 0,$$

where $u(t,\lambda,\phi,\psi)$ are the solutions of (5.4.24).

Proof. The system (5.4.24) is equivalent to

$$\begin{cases} \frac{du}{dt} = -\alpha u + v \\ \frac{dv}{dt} = L_\lambda u + \alpha^2 u - \alpha v + G(u,\lambda) \\ u(0) = \phi_0, \quad v(0) = \psi_0 \quad (\phi_0 = \phi, \psi_0 = \psi + \alpha\phi) \end{cases} \tag{5.4.26}$$

In order to applied Theorem 5.4.2 and Theorem 5.3.1 to the system (5.4.24) we need to reduce the equation (5.4.26) to the form (5.4.1). To this end, we define the Hilbenrt spaces as follows

$$\mathcal{H}_1 = H_1 \times H_{\frac{1}{2}}, \quad \mathcal{H} = H_{\frac{1}{2}} \times H$$

respectively with the inner products

$$< (u_1,v_1),(u_2,v_2) >_{\mathcal{H}_1} = < u_1,u_2 >_{H_1} + < v_1,v_2 >_{H_{\frac{1}{2}}}; \quad \forall (u_i,v_i) \in \mathcal{H}_1$$

$$< (u_1,v_1),(u_2,v_2) >_{\mathcal{H}} = < u_1,u_2 >_{H_{\frac{1}{2}}} + < v_1,v_2 >_H; \quad \forall (u_i,v_i) \in \mathcal{H}, i = 1,2.$$

Define the mapping $\widetilde{L}_\lambda : \mathcal{H}_1 \to \mathcal{H}$ by

$$\begin{aligned} &\widetilde{L}_\lambda = -\widetilde{A} + \widetilde{B}_\lambda \\ &\widetilde{A}(u,v) = \begin{pmatrix} \alpha I & -I \\ A & \alpha I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (\alpha u - v, Au + \alpha v) \\ &\widetilde{B}_\lambda(u,v) = \begin{pmatrix} 0 & 0 \\ \alpha^2 I + B_\lambda & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (0, \alpha^2 u + B_\lambda u), \end{aligned}$$

and $\widetilde{G}(\cdot,\lambda) : \mathcal{H}_1 \to \mathcal{H}$ by

$$\widetilde{G}(u,v,\lambda) = (0, G(u,\lambda))$$

$\forall (u,v)\in \mathcal{H}_1$. Thus, the equation (5.4.26) is written as to read

$$\frac{dv}{dt}=\widetilde{L}_\lambda v+\widetilde{G}(v,\lambda),\quad V=(u,v)\in \mathcal{H}_1 \tag{5.4.27}$$

It is clear that the operators $\widetilde{L}_\lambda$ and $\widetilde{G}_\lambda$ satisfy (5.4.2) and (5.4.3).

It is known that if the eigenvalues $\beta_k(\lambda)\neq 0 (1\leq k<\infty)$ of L_λ, then the norm $\|u\|^*_\theta=\|L^\theta_\lambda u\|_H$ is equivalent to the H_θ-norm defined as in (5.4.18), and the eigenvectors $\{e_k(\lambda)\}$ of L_λ are the comon orthogonal base of $H_\theta(0\leq\theta<\infty)$ under the equivalent norms $\|\cdot\|^*_\theta$. If there are some $\beta_k(\lambda)=0$, then we take $L_\lambda+\rho I$ instead of L_λ, which still remain $\{e_k(\lambda)\}$ as their eigenvectors, and the eigenvalues $\beta_k(\lambda)+\rho\neq 0\quad \forall 1\leq k<\infty$. Thus we can take some proper equivalent norms of $H_\theta(0\leq\theta<\infty)$ such that $\{e_k(\lambda)\}$ are the common orthogonal bases of H_θ.

Under the bases $\{e_k(\lambda)\}$, the equation (5.4.27) can be decomposed into the following form

$$\begin{aligned}
&\frac{dx}{dt}=-\alpha x+y\\
&\frac{dy}{dt}=\beta_1(\lambda)x+\alpha^2x-\alpha y+<G(u,\lambda),e_1(\lambda)>_H \\
&\frac{dv_1}{dt}=\widetilde{\mathcal{L}}_\lambda V_1+P_2\widetilde{G}(V,\lambda),V_1=(u_1,v_1)\in E_2
\end{aligned}\tag{5.4.28}$$

where

$$\begin{aligned}
&\mathcal{H}_1=E^\lambda_1\bigoplus E^\lambda_2,\quad \mathcal{H}=\widetilde{E}^\lambda_1\bigoplus \widetilde{E}^\lambda_2\\
&E^\lambda_1=\{(xe_1,ye_1)|\quad x,y\in R^1\}\\
&E^\lambda_2=\{(u_1,v_1)\in\mathcal{H}_1|<u_1,e_1(\lambda)>_H=0,<v_1,e_1(\lambda)>_H=0\}
\end{aligned}$$

Hence, for the system (5.4.27), the conditions (5.4.4) and (5.4.5) are fulfiled, and the decomposition (5.4.28) is corresponding to (5.4.6). We are now in a position to show that the operators $\widetilde{\mathcal{L}}_\lambda:E_2\to\widetilde{E}_2$ generate the semigroups $S_\lambda(t)$ which satisfy the condition (5.4.7).

We denote

$$H^*_\theta=\{\sum_{k=2}^\infty x_ke_k(\lambda)|\sum_{k=2}^\infty(-\beta_k(\lambda))^{2\theta}x_k^2<\infty\}$$

Then $E_2=H^*_1\times H^*_{\frac12},\widetilde{E}_2=H^*_{\frac12}\times H^*$, and $\widetilde{\mathcal{L}}_\lambda:E_2\to\widetilde{E}_2$ can be expressed as

$$\begin{aligned}
\widetilde{\mathcal{L}}_\lambda(u,v)&=\begin{pmatrix}-\alpha I & I\\ -A+B_\lambda+\alpha^2I & -\alpha I\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}\\
&=(-\alpha u+v,L^*_\lambda u+\alpha^2u-\alpha v),\forall(u,v)\in E_2
\end{aligned}$$

where $L_\lambda^* = L_\lambda|_{H_1^*}$. By the hypothese (5.4.21), in a neighborhood of $\lambda = \lambda_0$, the eigenvalues $\beta_j(\lambda)(2 \leq j)$ of L_λ^* are negative, therefore the eigenvalues of $\widetilde{\mathcal{L}}_\lambda$ have the negative real parts, which are as follows

$$-\alpha \pm \sqrt{\alpha^2 + \beta_k(\lambda)}, \quad \beta_k(\lambda_0) < 0, \quad k = 2, 3, \cdots.$$

On the other hand, it is readily to check that the semigroups $S_\lambda(t) : \widetilde{E}_2 \to \widetilde{E}_2$ generated by $\widetilde{\mathcal{L}}_\lambda$ are as to read

$$\begin{aligned} S_\lambda(t)(\phi, \psi) &= \frac{e^{-\alpha t}}{2} \begin{pmatrix} \Phi_1(t) & -L^{-1}\Phi_2(t) \\ -L\Phi_2(t) & \Phi_1(t) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \\ &= \{\frac{1}{2}e^{-\alpha t}(\Phi_1\phi - L^{-1}\Phi_2\psi), \frac{1}{2}e^{-\alpha t}(-L\Phi_2\phi + \Phi_1\psi)\} \end{aligned}$$

$\forall (\phi, \psi) \in \widetilde{E}_2$, where

$$\begin{aligned} &L = (L_\lambda^* + \alpha^2 I)^{\frac{1}{2}} \\ &\Phi_1(t) = e^{-tL} + e^{tL} \\ &\Phi_2(t) = e^{-tL} - e^{tL} \end{aligned}$$

namely, for $(\phi, \psi) \in \widetilde{E}_2, \phi = \sum_{k=2}^\infty \phi_k e_k(\lambda), \psi = \sum_{k=2}^\infty \psi_k e_k(\lambda)$,

$$\begin{aligned} \Phi_1(t)\phi &= \sum_{k=2}^\infty \phi_k (e^{-tL} + e^{tL}) e_k(\lambda) \\ &= \sum_{k=2}^N \phi_k (e^{-t\sqrt{\alpha^2 - |\beta_k(\lambda)|}} + e^{t\sqrt{\alpha^2 - |\beta_k(\lambda)|}}) e_k(\lambda) \\ &\quad +2 \sum_{k=N+1}^\infty \phi_k \cos t\sqrt{|\beta_k(\lambda)| - \alpha^2} e_k(\lambda) \end{aligned}$$

$$\begin{aligned} \Phi_2(t)\psi &= \sum_{k=2}^\infty \psi_k (e^{-tL} - e^{tL}) e_k(\lambda) \\ &= \sum_{k=2}^N \psi_k (e^{-t\sqrt{\alpha^2 - |\beta_k(\lambda)|}} - e^{t\sqrt{\alpha^2 - |\beta_k(\lambda)|}}) e_k(\lambda) \\ &\quad -2i \sum_{k=N+1}^\infty \psi_k \sin t\sqrt{|\beta_k(\lambda)| - \alpha^2} e_k(\lambda) \end{aligned}$$

where the natural number N satisfies

$$\alpha^2 + \beta_k(\lambda) = \begin{cases} \geq 0 & as\ k \leq N \\ < 0 & as\ k \geq N+1 \end{cases}$$

By a direct caculation we can derive that there exists a constant $k > 0$ such that

$$\begin{aligned}\|S_\lambda(t)\| &= \sup_{\|\phi\|_{\frac{1}{2}}=1,\|\psi\|_0=1}[\frac{e^{-\alpha t}}{2}\|\Phi_1(t)\phi - L^{-1}\Phi_2(t)\psi\|_{H_{\frac{1}{2}}} \\ &\quad +\frac{e^{-\alpha t}}{2}\| - L\Phi_2\phi + \Phi_1(t)\psi\|_{H_0}] \\ &\le Ke^{-\alpha_1 t},\end{aligned}$$

where

$$0 < \alpha_1 = \begin{cases} \alpha - \sqrt{\alpha^2 - |\beta_2(\lambda)|} \ as \ \alpha^2 > |\beta_2(\lambda)| \\ \alpha, \ as \ \alpha^2 \le |\beta_2(\lambda_0)| \end{cases}$$

Hence the condition (5.4.7) is checked.

By Theorem 5.4.1, we reduce the bifurcation problem of (5.4.26) to that of the below equation

$$\begin{cases} \frac{dx}{dt} = -\alpha x + y \\ \frac{dy}{dt} = \beta_1(\lambda)x + \alpha^2 x - \alpha y + < G(xe_1(\lambda) + h(x,\lambda)), e_1(\lambda) >_H \end{cases} \tag{5.4.29}$$

By (5.4.21), the eigenvalues of (5.4.29) are as follows

$$\begin{cases} \sigma_1(\lambda) = -\alpha + \sqrt{\alpha^2 + \beta_1(\lambda)} = \begin{cases} < 0, \lambda < \lambda_0 \\ = 0, \lambda = \lambda_0 \\ > 0, \lambda > \lambda_0 \end{cases} \\ \sigma_2(\lambda_0) = -2\alpha \end{cases}$$

In the same fashion as in Theorem 5.4.3, by the hypotheses we can get

$$< G(xe_1 + h(x,\lambda_0),\lambda_0), e_1 >_H = -\sigma|x|^{k-1}x + o(|x|^k), \quad \sigma > 0$$

If we make the change $x = \widetilde{x}, y = \widetilde{y} + \alpha\widetilde{x}$, then the equation (5.4.29) with $\lambda = \lambda_0$ is transformed into

$$\begin{cases} \frac{d\widetilde{x}}{dt} = \widetilde{y} \\ \frac{d\widetilde{y}}{dt} = -2\alpha\widetilde{y} - \sigma|\widetilde{x}|^{k-1}\widetilde{x} + o(|\widetilde{x}|^k), \sigma > 0 \end{cases}$$

Applying Theorem 5.1.2 it is easy to show that $(\widetilde{x},\widetilde{y}) = (0,0)$ is asymptotically stable. Therefore $(x,y) = (0,0)$ is also asymptotically stable for (5.4.29) at $\lambda = \lambda_0$. Thus, by using Theorem 5.3.1 we can obtain the desired conclusion. This proof is complete.

Likewise, for the higher dimensional cases $m \ge 1$, we have the following result.

Theorem 5.4.10. Under the conditions (5.4.21), if $u = 0$ is asymptotically stable for (5.4.24) at $\lambda = \lambda_0$, then the system (5.4.24) bifurcates an attractor $\sum_\lambda$ with $m - 1 \leq dim \sum_\lambda \leq m$.

Next, we investigate the attractor bifurcation of the system given by

$$\left\{ \begin{array}{l} \frac{d^2u}{dt^2} + 2\alpha\frac{d}{dt}(A^\theta - \lambda I)u = -Au + G(u,\lambda) \\ u(0) = \phi, u_t(0) = \psi \end{array} \right. \tag{5.4.30}$$

where $0 < \alpha$ and $0 < \theta < 1$.

Theorem 5.4.11. Let the first eigenvalue ρ_1 of A in (5.4.17) have the multiplicity $m \geq 1$, then the following assertions hold.
i). When $\lambda < \rho_1^\theta, u = 0$ is asymptotically stable for (5.4.30) in $H_{\frac{1}{2}} \times H$.
ii). If $u = 0$ is asymptotically stable for (5.4.30) at $\lambda = \rho_1^\theta$, then when $\lambda > \rho_1^\theta$ (5.4.30) bifurcates a attractor $\sum_\lambda$ with $2m-1 \leq dim \sum_\lambda \leq 2m$, which has the homotopy type of S^{2m-1}. Especially, if $m = 1$, then $\sum_\lambda$ is a periodic orbit.

Proof. The equation (5.4.30) can be decomposed into

$$\left\{ \begin{array}{l} \frac{dx_i}{dt} = -\alpha(\rho_1^\theta - \lambda)x_i + y_i \quad (1 \leq i \leq m) \\ \frac{dy_i}{dt} = -\rho_1 x_i + \alpha^2(\rho_1^\theta - \lambda)^2 x_i - \alpha(\rho_1^\theta - \lambda)y_i + < G(u,\lambda), \phi_i > \end{array} \right. \tag{5.4.31}$$

$$\left\{ \begin{array}{l} \frac{d\widetilde{u}}{dt} = -\alpha(A^\theta - \lambda I)\widetilde{u} + \widetilde{v} \\ \frac{d\widetilde{v}}{dt} = -A\widetilde{u} + \alpha^2(A^\theta - \lambda I)^2\widetilde{u} - \alpha(A^\theta - \lambda I)\widetilde{v} + PG(u,\lambda) \end{array} \right. \tag{5.4.32}$$

where $P : H \to \widetilde{E}_2 = \{\sum_{k=m+1}^\infty x_k\phi_k \in h | \sum_{k=m+1}^\infty x_k^2 < \infty\}$ is the projection, and $\widetilde{u} \in H_1 \bigcap \widetilde{E}_2, \widetilde{v} \in H_{\frac{1}{2}} \bigcap \widetilde{E}_2$.

We can find that the equation (5.4.31) has the eigenvalues

$$\beta_i(\lambda) = -\alpha(\rho_1^\theta - \lambda) \pm i\sqrt{\rho_1 - \alpha^2(\rho_1^\theta - \lambda)^2}, \quad 1 \leq i \leq m$$

and the equation (5.4.32) has

$$\beta_j(\lambda) = -\alpha(\rho_j^\theta - \lambda) \pm \sqrt{\alpha^2(\rho_j^\theta - \lambda)^2 - \rho_j}, \quad m + 1 \leq j$$

Obviously the eigenvalues of (5.4.31) and (5.4.32) in a neighborhood of $\lambda_0 = \rho_1^\theta$ satisfy the condition (5.4.21). Thus, by using the same method of proof as the above theorem, one can derive the desired conclusions. This proof is complete.

If the first eigenvalue ρ_1 of A has multiplicity $m = 1$, then we obtain the Hopf bifurcation for the equation (5.4.30)

Theorem 5.4.12. Let the first eigenvalue ρ_1 of A in (5.4.17) have multiplicity $m = 1$, then, at $\lambda = \rho_1^\theta$ the equation (5.4.30) will occur the Hopf bifurcation.

Remark 5.4.13. In fact, for any eigenvalue ρ_j of A, if ρ_j has multiplicity $m = 1$, then the equation (5.4.30) will occur the Hopf bifurcation at $\lambda = \rho_j$.

5.5. Dissipative Partial Differential Equations in Mechanics and Physics

5.5.1. Nonlinear wave equations with a damping term

First, we shall apply Theorem 5.4.9 to discuss the equilibrum attractor bifurcation of the following nonlinear wave equation

$$(5.5.1)\qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} = \Delta u + \lambda u + g(x, u, \nabla u, D^2u, \lambda), x \in \Omega \subset R^n \\ u|_{\partial\Omega} = 0 \\ u(x, 0) = \phi(x) \end{cases}$$

where $\alpha > 0$ is a constant, $g(x, z, \xi, \eta, \lambda)$ is C^1 on its arguments, and

$$(5.5.2)\qquad |g(x, z, \xi, \eta, \lambda)| \leq \mu_1(z) + \mu_2(\xi) + c|\eta|$$

where

$$\mu_1(z) = \begin{cases} \mu_1 \in C^1(R^1),\ as\ 1 \leq n < 4 \\ c(|z|^p + 1), \quad p < \infty,\ as\ n = 4 \\ c(|z|^{\frac{n}{n-4}} + 1),\ as\ n > 4 \end{cases}$$

$$\mu_2(\xi) = \begin{cases} \mu_2 \in C^1(R^1),\ as\ n = 1 \\ c(|\xi|^p + 1), \quad p < \infty,\ as\ n = 2 \\ c(|\xi|^{\frac{n}{n-2}} + 1),\ as\ n > 2 \end{cases}$$

For any given $z > 0$ and $\xi, \eta, \lambda \neq 0$, we assume that

$$(5.5.3)\qquad g(x, \beta z, \beta\xi, \beta\eta, \lambda) = -\sigma|\beta|^{k-1}\beta + o(|\beta|^k)$$

where $k > 1$ is some constant, $\delta > 0$ depends on $z > 0, \xi, \eta, \lambda \neq 0$.

Let λ_1 and $u_1(x)$ be respectively the first eigenvalues and eigenfunction of the Laplace operator:

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 \\ u_1|_{\partial\Omega} = 0, u_1 > 0\ in\ \Omega \end{cases}$$

We have the following result.

Theorem 5.5.1. Under the conditions (5.5.2) and (5.5.3), as $\lambda \leq \lambda_1, u = 0$ is an asymptotically stable equilibrum point of (5.5.1), and as $\lambda > \lambda_1$, there exists an open set $U \subset H^2(\Omega) \bigcap H_0^1(\Omega) \times H_0^1(\Omega), o \in U$, such that the equation (5.5.1) bifurcates from $(u, u_t) = 0$ exactly two equilibrum point $(u_1^\lambda, 0)$ and $(u_2^\lambda, 0) \in U$, and U is decomposed into two open sets $U_i^\lambda (i = 1, 2)$:

$$\bar{U} = \bar{U}_1^\lambda + \bar{U}_2^\lambda, \quad U_1^\lambda \bigcap U_2^\lambda = \phi, \quad o \in \bar{U}_1^\lambda \bigcap \bar{U}_2^\lambda$$

with $(u_i^\lambda, 0) \in U_i^\lambda (i = 1, 2)$ such that

$$\lim_{t \to \infty} \|u(t, \lambda, \phi, \psi)\|_{H^1(\Omega)} = u_i^\lambda, \qquad as\ (\phi, \psi) \in U_i^\lambda$$

$$\lim_{t \to \infty} \|u_t(t, \lambda, \phi, \psi)\|_{L^2(\Omega)} = 0$$

where $u(t, \lambda, \phi, \psi)$ are the solutions of (5.5.1)

Proof. Let $H_1 = H^2(\Omega) \bigcap H_0^1(\Omega), H = L^2(\Omega)$, and $L_\lambda = -A + B, G : H_1 \to H$ defined by that

$$\begin{aligned} &-Au = \Delta u \in H, u \in H_1 \\ &B_\lambda u = \lambda u \in H, u \in H_1 \\ &Gu = g(x, u, Du, D^2 u, \lambda) \in H, u \in H_1 \end{aligned}$$

Obviously, A and B_λ satisfy (5.4.3)(5.4.17)(5.4.19)(5.4.21) and (5.4.25). By the conditions (5.5.2) and (5.5.3) it is easy to see that $G : H_1 \to H$ is continuous and bounded, which satisfies (5.4.2).

From (5.5.3) if follows that

$$\begin{aligned} < G(\beta u_1), u_1 >_H &= \int_\Omega g(x, \beta u_1, \beta D u_1, \beta D^2 u_1, \lambda) u_1 dx \\ &= -\sigma_1 |\beta|^{k-1} \beta + o(|\beta|^k) \end{aligned}$$

$$\sigma_1 = \int_\Omega \sigma \cdot u_1 dx > 0 \qquad (by\ u_1 > 0\ in\ \Omega)$$

Thus, this theorem follows from Theorem 5.4.9. The proof is complete.

Remark 5.5.2. In Theorem 5.5.1 and the later theorems, the basic condition of the existence of global strong solutions for all small initial values is required, and which can be ensured by the center manifold theorem provided that the conditions in Theorem 5.4.1 (or in Theorem 5.4.3-Theorem 5.4.12) are satisfied.

As a corollary of Theorem 5.5.1, we immediately obtain the equilibrum attractor bifurcation for the following Sine-Gordan equation

$$
\begin{cases} \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} = \Delta u + \lambda \sin u, x \in \Omega \subset R^n \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi, u_t(x,0) = \psi \end{cases} \tag{5.5.4}
$$

Corollary 5.5.3. When $\lambda > \lambda_1$, the Sine-Gordan equation (5.5.4) will have the equilibrum attracter bifurcation from $(u, \lambda) = (0, \lambda_1)$.

Proof. By the Taylor expansion

$$
\sin u = u - \frac{1}{3!} u^3 + o(|u|^3)
$$

and $g(x, u, \lambda) = \lambda \sin u - \lambda u$ satisfies (5.5.2) and (5.5.3). Thus, we obtain the corollary. The proof is complete.

Next, we consider the Hopf bifurcation for the vibrating equations with strong damping given by

$$
\begin{cases} \frac{\partial^2 u}{\partial t^2} - \alpha \Delta u_t - \lambda u_t = -\Delta^2 u + g(x, u, Du, D^2 u), x \in \Omega \subset R^n \\ u|_{\partial\Omega} = 0, \Delta u|_{\partial\Omega} = 0 \\ u(x,0) = \phi, u_t(x,0) = \psi \end{cases} \tag{5.5.5}
$$

where $\alpha > 0, g \in C^1(\Omega \times R^1 \times R^n \times R^{n^2}), 1 \leq n \leq 3$, and

$$
g(x, \beta z, \beta \xi, \beta \eta) = o(|\beta|) \tag{5.5.6}
$$

Applying Theorem 5.4.11 and Remark 5.4.13, we can obtain the following result.

Theorem 5.5.4. Under the condition (5.5.6), we have the following assertions.

i). For any simple eigenvalue λ_i of the Laplace operator $-\Delta$, the equation (5.5.5) will occur the Hopf bifurcation from $(0, \alpha\lambda_i)$.

ii). If $u = 0$ is asymptotically stable for (5.5.5) at $\lambda = \alpha\lambda_1$, then the equation (5.5.5) bifurcates a stable periodic orbit for $\lambda > \alpha\lambda_1$ from $(u, \lambda) = (0, \alpha\lambda_1)$.

Proof. We take the spaces as to read

$$
\begin{aligned} &H_1 = \{u \in H^4(\Omega) | u|_{\partial\Omega} = 0, \Delta u|_{\partial\Omega} = 0.\} \\ &H = L^2(\Omega) \end{aligned}
$$

Then we have that $H_{\frac{1}{2}} = H^2(\Omega) \bigcap H_0^1(\Omega)$. Let A and $G : H_1 \to H$ be defined by

$$Au = \Delta^2 u \in H, \quad u \in H_1$$
$$Gu = g(x, u, Du, D^2 u) \in H, \quad u \in H_1$$

Thus, the operator $A^{\frac{1}{2}} : H_{\frac{1}{2}} \to H$ is as follows

$$A^{\frac{1}{2}} u = -\Delta u \in H, \quad u \in H_{\frac{1}{2}}$$

Obviously, the conditions in Theorem 5.4.11 are satisfied. Thus, this theorem is proven.

Remark 5.5.5. Under centain restrictions on the exponent growth the function $g(x, u, Du, D^2u)$ in (5.5.5) can be relaxed by $g(x, D^\alpha u), 0 \leq \alpha \leq 4$, and $\Omega \subset R^n$ for any $n \geq 1$.

5.5.2. Ginzburg-Landau equations

This subsection is devoted to the study of cycle attractor bifurcation and invariant set bifurcation for the Ginzburg-Landau equation. The Dirichlet and periodic boundary conditions will be considered. The equation is given by

$$\frac{\partial u}{\partial t} - (\alpha + i\beta)\Delta u + (\sigma + i\rho)|u|^2 u - \lambda u = 0 \tag{5.5.7}$$

where the unknown u is a complex-valued function defined on $\Omega \times R^+, \Omega \subset R^n, 1 \leq n \leq 6$. The parameters $\alpha, \beta, \rho, \sigma, \lambda$ are real numbers, for them we make the following assumption:

$$\alpha > 0, \quad \sigma > 0 \tag{5.5.8}$$

The equation will be supplemented with on of the following boundary conditions:

the periodic boundary condition, in which case

$$\Omega = (0, 2\pi)^n \textit{ and } u \textit{ is } \Omega - \textit{periodic} \tag{5.5.9}$$

the Dirichlet boundary condition

$$u|_{\partial\Omega} = 0 \tag{5.5.10}$$

For the equation (5.5.7) we provide the initial value of u:

$$u(x, 0) = \phi + i\psi \tag{5.5.11}$$

Let $u = u_1 + iu_2$. Then the initial value problem of (5.5.7) is equivalent to the following form

$$\begin{cases} \frac{\partial u_1}{\partial t} = \alpha\Delta u_1 - \beta\Delta u_2 + \lambda u_1 \sigma |u|^2 u_1 + \rho|u|^2 u_2 \\ \frac{\partial u_2}{\partial t} = \beta\Delta u_1 + \alpha\Delta u_2 + \lambda u_2 - \sigma|u|^2 u_2 - \rho|u|^2 u_1 \\ u_1(x,0) = \phi(x), u_2(x,0) = \psi(x) \end{cases} \tag{5.5.12}$$

The following is the S^1-attractor bifurcation theorem for the equation (5.5.7) with the boundary condition (5.5.9), or (5.5.10).

Theorem 5.5.6. Suppose that (5.5.8) holds. Then we have the following conclusions:

i). As $\lambda \leq 0, u = 0$ is an asymptotically stable equilibrum point of the problem (5.5.7)(5.5.9), and as $\lambda > 0$ the problem (5.5.7)(5.5.9) will bifurcate from $(u, \lambda) = (0, 0)$ an unique S^1-attractor. If $\rho \neq 0$, then the S^1-attractor is a periodic orbit.

ii). As $\lambda \leq \alpha\lambda_1 (\lambda_1$ is the first eigenvalue of $-\Delta$ with the boundary condition (5.5.10)), $u = 0$ is asymptotically stable for the problem (5.5.7)(5.5.10), and as $\lambda > \alpha\lambda_1$ the problem (5.5.7)(5.5.10) bifurcates from $(u, \lambda) = (0, \alpha\lambda_1)$ an unique S^1-attractor. If $|\rho| + |\beta| \neq 0$, then this attractor is a periodic orbit, especically as $\beta \neq 0$, it is the Hopf bifurcation.

Proof. Here, we only prove the conclusion i), because the proof of conclusion ii) proceeds in the same fashion.

Let the spaces be taken as follows

$$H_1 = H^2_{per}(\Omega) \times H^2_{per}(\Omega); \quad H = L^2(\Omega) \times L^2(\Omega)$$

where $\Omega = (0, 2\pi)^n$, and

$$H^2_{per}(\Omega) = \{u \in H^2(\Omega) | u(x + 2k\pi) = u(x), K = \{k_1, \cdots, k_n\}, k_i \textit{ the integers}\}$$

The mapping $L_\lambda = -A + B_\lambda$ and $G : H_1 \to H$ are defined by

$$-Au = \begin{pmatrix} \alpha(\Delta u_1 - u_1) - \beta\Delta u_2 \\ \beta\Delta u_1 + \alpha(\Delta u_2 - u_2) \end{pmatrix}$$

$$B_\lambda u = \begin{pmatrix} (\lambda + \alpha)u_1 \\ (\lambda + \alpha)u_2 \end{pmatrix},$$

$$Gu = \begin{pmatrix} -\sigma|u|^2 u_1 + \rho|u|^2 u_2 \\ -\sigma|u|^2 u_2 - \rho|u|^2 u_1 \end{pmatrix}$$

By the Sobolev embedding theorems and $1 \leq n \leq 6$, the mapping $G : H_1 \to H$ is C^∞ and bounded. It is clear that the conditions (5.4.2) and (5.4.3) are satisfied for the operators L_λ and G.

Let $\{\lambda_k\}$ and $\{e_k\} \subset H^2_{per}(\Omega)$ be the eigenvalues and eigen-functions of the Laplacian $-\Delta$ with the periodic boundary condition

$$(5.5.13) \qquad \begin{cases} -\Delta e_k = \lambda_k e_k \\ e_k(x + 2k\pi) = e_k(x) \end{cases}$$

We know that

$$0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots; \quad \lambda_k \to \infty \ as\ k \to \infty.$$

and $\{e_k\}$ constitutes a common orthogonal base of $H^2_{per}(\Omega)$ and $L^2(\Omega)$. Therefore $\{e_k\} \times \{e_j\}$ is a common orthogonal base of H_1 and H. With this bese, the problem (5.5.12) can be decomposed into the below infinite dimensional systems

$$(5.5.14) \qquad \begin{cases} \frac{dx_k}{dt} = (\lambda - \alpha\lambda_k)x_k + \beta\lambda_k y_k + P_k G_1(u) \\ \frac{dy_k}{dt} = -\beta\lambda_k x_k + (\lambda - \alpha\lambda_k)y_k + P_k G_2(u) \\ x_k(0) = \phi_k, y_k(0) = \psi_k \end{cases}$$

where

$$u_1 = \sum_{k=1}^{\infty} x_k(t)e_l; \quad u_2 = \sum_{k=1}^{\infty} y_k(t)e_k$$

$$(5.5.15) \qquad P_k G_1(u) = \int_\Omega [-\sigma|u|^2 u_1 + \rho|u|^2 u_2]e_k dx$$

$$P_k G_2(u) = \int_\Omega [-\sigma|u|^2 u_2 - \rho|u|^2 u_1]e_k dx$$

and $\phi = \sum_{k=1}^\infty \phi_k e_k, \psi = \sum_{k=1}^\infty \psi_k e_k$.

On the other hand, from (5.5.14) it is easy to see that the eigenvalues of the operator $L_\lambda = -A + B_\lambda$ are as follows

$$(5.5.16) \qquad (\lambda - \alpha\lambda_k) \pm i\beta\lambda_k, \quad k = 1, 2, \cdots.$$

From (5.5.14)-(5.5.16) it is easy to see that the conditions (5.4.4)-(5.4.6) are satisfied for the operator $L_\lambda + G$. In order to apply the center manifold theorem (Theorem 5.4.1) to this proof, we need to check the condition (5.4.7).

We see that the first and second eigenvalues of (5.5.13) are

$$\lambda_1 = 0, \quad \lambda_2 = 1$$

Hence, the spaces H_1 and H can be decomposed into

$$H_1 = E_1 \bigoplus E_2, H = \widetilde{E}_1 \bigoplus \widetilde{E}_2$$

$$E_1=\widetilde{E}_1=\{xe_1| \quad x\in R^1\}\times\{ye_1| \quad y\in R^1\}$$
$$E_2=\{\sum_{k=2}^{\infty}x_ke_k| \quad \sum_{k=2}^{\infty}\lambda_k^2x_k^2<\infty\}\times\{\sum_{k=2}^{\infty}y_ke_k| \quad \sum_{k=2}^{\infty}\lambda_k^2y_k^2<\infty\}$$
$$\widetilde{E}_2=\{\sum_{k=2}^{\infty}x_ke_k| \quad \sum_{k=2}^{\infty}x_k^2<\infty\}\times\{\sum_{k=2}^{\infty}y_ke_k| \quad \sum_{k=2}^{\infty}y_k^2<\infty\} \tag{1}$$

When $\lambda<\alpha$, the operator

$$\mathcal{L}_2^{\lambda}=L_{\lambda}|_{E_2}=-A+B_{\lambda}|_{E_2}:E_2\to\widetilde{E}_2$$

has the eigenvalues (5.5.6) with $k\geq 2$, which possesse the negative real parts. The semigroup $S_{\lambda}(t)$ generated by $\mathcal{L}_2^{\lambda}$ is as follows

(5.5.17)
$$S_{\lambda}(t)v=e^{t\mathcal{L}_2^{\lambda}}\cdot v$$
$$=\begin{cases}\sum_{k=2}^{\infty}e^{-(\alpha\lambda_k-\lambda)t}(v_k^1\cos\beta\lambda_kt+v_k^2\sin\beta\lambda_kt)e_k\\ \sum_{k=2}^{\infty}e^{-(\alpha\lambda_k-\lambda)t}(-v_k^1\sin\beta\lambda_kt+v_k^2\cos\beta\lambda_kt)e_k\end{cases}$$

where
$$v=\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{cases}\sum_{k=2}^{\infty}v_k^1e_k\\ \sum_{k=2}^{\infty}v_k^2e_k\end{cases}$$

From (5.5.17) it follows that

$$\|S_{\lambda}(t)\|\leq Ce^{-(\alpha\lambda_2-\lambda)}=ce^{-(\alpha-\lambda)}$$

Thus, the condition (5.4.7) is verified.

By the center manifold theorem, the bifurcation problem of (5.5.12) from $(u,\lambda)=(0,0)$ is equivalent to that of the bifurcation equation of (5.5.14) as follows
$$\frac{dx_1}{dt}=\lambda x_1+P_1G_1(x_1e_1+h_1(x_1,y_1),y_1e_1+h_2(x_1,y_1))$$
$$\frac{dy_1}{dt}=\lambda y_1+P_1G_2(x_1e_1+h_1(x_1,y_1),y_1e_1+h_2(x_1,y_1))$$
where $h=(h_1,h_2):E_1\to E_2$ is the center manifold function, and $P_1G_i(i=1,2)$ defined as in (5.5.15)

We notice that the eigenfunction corresponding to $\lambda_1=0$ is a constant

$$e_1=\ constant\ (\neq 0)$$

and
$$G(x_1e_1,y_1e_1)=\begin{pmatrix}-\sigma(x_1^2+y_1^2)x_1+\rho(x_1^2+y_1^2)y_1\\ -\sigma(x_1^2+y_1^2)y_1-\rho(x_1^2+y_1^2)x_1\end{pmatrix}\in E_1$$

namely $P_kG(x_1e_1, y_1e_1) = 0, \forall k \geq 2$. Hence it follows that the center manifold function $h \equiv 0$:

$$h(x_1, y_1) = (h_1(x_1, y_1), h_2(x_1, y_1)) \equiv (0, 0)$$

Thus, the bifurcation equation (5.5.18) reads

$$\begin{cases} \frac{dx_1}{dt} = \lambda x_1 - \sigma(x_1^2 + y_1^2)x_1 + \rho(x_1^2 + y_1^2)y_1 \\ \frac{dy_1}{dt} = \lambda y_1 - \sigma(x_1^2 + y_1^2)y_1 - \rho(x_1^2 + y_1^2)x_1 \end{cases} \tag{5.5.19}$$

Obviously, by (5.5.8) it is clear that as $\lambda \leq 0, (x_1, y_1) = 0$ is asymptotically stable for (5.5.19). By Theorem 5.3.9, the equation (5.5.19) bifurcates from $(u, \lambda) = (0, 0)$ an attractor for $\lambda > 0$. When $\rho = 0$, it is easy to see that the attractor is the cycle which is unique, and which consists of equilibrum points of (5.5.19):

$$x_1^2 + y_1^2 = \frac{\alpha}{\sigma}\lambda \quad (\lambda > 0).$$

When $\rho \neq 0$, the equation (5.5.19) has no equilibrum points, otherwise one can obtain that $\rho(x_1^2 + y_1^2)^2 = 0$ from the following equation

$$\lambda x_1 y_1 - \sigma(x_1^2 + y_1^2)x_1 y_1 + \rho(x_1^2 + y_1^2)y_1^2 = 0$$

$$\lambda y_1 x_1 - \sigma(x_1^2 + y_1^2)y_1 x_1 - \rho(x_1^2 + y_1^2)x_1^2 = 0$$

Therefore, by the claim ii) of Theorem 5.3.9, the attractor is either a periodic orbit or an anulus.

In the following, we shall show that the attractor is the periodic orbit. We take the polar coordinate system

$$x_1 = r\cos\theta, \quad y_1 = r\sin\theta$$

Then the equation (5.5.19) is changed to

$$\begin{cases} \frac{dr}{d\theta} = \frac{\lambda - \sigma r^2}{\rho r} \quad (\lambda > 0) \\ r(0) = a \end{cases} \tag{5.5.20}$$

From (5.5.20) it follows that

$$\frac{\rho}{2}(r^2(2\pi) - r^2(0)) = \int_0^{2\pi} (\lambda - \sigma r^2)d\theta$$

Because $r^2 = r^2(\theta, a)$ is C^∞ on $a \geq 0$, we have the Taylor expansion

$$r^2(\theta, a) = a + R(\theta) \cdot o(|a|); \quad R(0) = 0$$

Thus, we get that

$$\frac{\rho}{2}(r^2(2\pi) - r^2(0)) = 2\pi\lambda - 2\pi\sigma a + o(|a|)$$

Obviously, the initial values $a > 0$ in (5.5.20) satisfying

$$2\pi\lambda - 2\pi\sigma a + o(|a|) = 0 \quad (\sigma, \lambda > 0) \tag{5.5.21}$$

are corresponding to the periodic orbits of (5.5.20). It is easy to see that the solution $a > 0$ of (5.5.21) near $a = 0$ is unique. Thus we derive this theorem. The proof is complete.

More generally, for the Ginzburg-Landau equation we have the bifurcation theorem of the homotopical sphere $S^k (k \geq 1)$ at any eigenvalue of the Laplacian $-\Delta$.

Theorem 5.5.7. Let λ_m be an eigenvalue of $-\Delta$ with the boundary condition (5.5.9)(or (5.5.10)), which has multiplicity $m \geq 1$. Then, under the condition (5.5.8), as $\lambda > \alpha\lambda_m$ the problem (5.5.7)(5.5.9)(or (5.5.7)(5.5.10)) bifurcates from $(u, \lambda) = (0, \alpha\lambda_m)$ an invariant set $\sum$ which has the homotopy type of S^{2m-1}. If $|\beta| + |\rho| \neq 0$, then there are no equilibrum points of (5.5.7)(5.5.9)(or (5.5.7) and (5.5.10)) in $\sum$.

Proof. We still proceed only for the case of periodic boundary condition. We denote the eigenfunctions corresponding to λ_m by

$$\{e_1^*, \cdots, e_m^*\}$$

Thus, the spaces H_1 and H defined in Theorem 5.5.6 can be decomposed into

$$\begin{aligned}
&H_1 = E_m \bigoplus E_m^{\perp}, \quad H = \widetilde{E}_m \bigoplus \widetilde{E}_m^{\perp} \\
&E_m = span\{e_1^*, \cdots, e_m^*\} \times span\{e_1^*, \cdots, e_m^*\} \\
&E_m^{\perp} = \{u \in H_1 | < u, v >_{H_1} = 0, \forall v \in E_m\} \\
&\widetilde{E}_m = E_m \\
&\widetilde{E}_m^{\perp} = \{u \in H | < u, v >_H = 0, \forall v \in \widetilde{E}_m\}
\end{aligned}$$

By the center manifold theorem (Theorem 5.4.1 (b)), the bifurcation of (5.5.12) at $\lambda = \lambda_m$ is equivalent to that of the below equation

$$\begin{cases} \frac{\partial v_1}{\partial t} = \alpha\Delta v_1 - \beta\Delta v_2 + \lambda v_1 + PG_1(v + h(v)) \\ \frac{\partial v_2}{\partial t} = \beta\Delta v_1 + \alpha\Delta v_2 + \lambda v_2 + PG_2(v + h(v)) \end{cases} \tag{5.5.22}$$

where λ is near $\lambda_m, v = (v_1, v_2) \in E_m, h : E_m \to E_m^{\perp}$ is the center manifold function, $G = (G_1, G_2) : H_1 \to H$ defined as in Theorem 5.5.6, and $P : H \to \widetilde{E}_m$ the projection.

The equation (5.5.22) is a system of ordinary differential equation with order $2m$, and the eigenvalues of whose linearized operator are the same in the multiplicity, which reads

$$(\lambda - \alpha\lambda_m) \pm i\beta\lambda_m.$$

By Theorem 5.3.4, it suffices to prove that $v = 0$ is asymptotically stable for (5.5.22) at $\lambda = \alpha\lambda_m$. From (5.5.22), for $\lambda = \alpha\lambda_m$ we can obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [|v_1|^2 + |v_2|^2]dx = \int_\Omega G(v + h(v))vdx$$

We notice that

$$h(v) = o(|v|)$$

Hence we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega |v|^2 dx &= 2\int_\Omega [G(v)\cdot v + o(|G(v)\cdot v|)]dx \\ &= -2\sigma\int_\Omega [|v|^4 + o(|v|^4)]dx \end{aligned}$$

which implies that $v = 0$ is asymptotically stable for the system (5.5.22). The proof is complete.

Remark 5.5.8. For the periodic boundary condition, the multiplicities m of eigenvalues λ_k with $k \geq 2$ of $-\Delta$ are greater than one, i.e. $m > 1$. For example, the multiplicity m of $\lambda_2 = 1$ is $m = 2n$, and the eigenfunctions corresponding to $\lambda_2 = 1$ are as to read:$\{\sin x_i, \cos x_i | 1 \leq i \leq n, (x_1, \cdots, x_n) \in \Omega = (0, 2\pi)^n\}$. Hence the problem (5.5.7)(5.5.9) will bifurcate from $(u, \lambda) = (0, \alpha)$ a $(4n-1)$-dimensional homotopy sphere S^{4n-1}.

5.5.3. Pattern formation equations

The equations under study in this subsection are related to various pattern formation plenomena. First, we shall consider the bifurcation of attractors and invariant sets of the homotopy sphere S^k for the Cahn-Hilliard equation which models pattern formation in phase transition. Then we shall investigate the bifurcation of S^1-attractors and periodic orbits for the Kuramoto-Sivashinsky equation related to turbulence phenomena in chemistry and combustion.

The Cahn-Hilliard equation reads:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta K(u), \quad x \in \Omega \subset R^n (1 \leq n \leq 3) \\ K(u) = -\alpha\Delta u + f(u), \quad \alpha > 0 \end{cases} \tag{5.5.23}$$

where f is a polynomial of order $2k+1$

$$f(u) = -\lambda u + \sum_{p=2}^{2k+1} a_p u^p \tag{5.5.24}$$

This equation is added an initial condition

$$u(x,0)=\phi(x) \tag{5.5.25}$$

For the sake of convenience, here we only consider the case that Ω is a cube: $\Omega=(0,2\pi)^n$.

The equation (5.5.23) is supplemented with one of the below two types of boundary conditions

The Neumann boundary conditions

$$\frac{\partial u}{\partial n}=\frac{\partial \Delta u}{\partial n}=0 \ on \ \partial\Omega \tag{5.5.26}$$

n the unit outward normal on $\partial\Omega$.

The periodic boundary condition

$$u(x+2k\pi,t)=u(x,t) \tag{5.5.27}$$

$K=\{k_1,\cdots,k_n\},k_i$ the integers.

For the coificients in (5.5.24), we assume that

$$a_2=0, \quad a_3=\beta>0 \tag{5.5.28}$$

We notice a particular aspect of the problem (5.5.23)(5.5.25) with the boundary condition (5.5.26) or (5.5.27) is that the average of u is conserved, which means the non-existence of bifurcation of attractors and invariant sets. In fact, when we integrate (5.5.23) over Ω, we find

$$\frac{\partial}{\partial t}\int_\Omega u(x,t)dx=\int_\Omega \Delta K(u)dx=\int_{\partial\Omega}\frac{\partial}{\partial n}K(u)dx=0$$

$$\int_\Omega u(x,t)dx=\int_\Omega \phi(x)dx, \quad \forall t\geq 0$$

On the other hand, when the initial value ϕ in (5.5.25) satisfies

$$\int_\Omega \phi(x)dx=0$$

then the solution $u(x,t,\phi)$ of (5.5.23) has

$$\int_\Omega u(x,t,\phi)dx=0, \quad \forall t\geq 0$$

Thus, it makes sense for us to discuss the bifurcation problem for the equation (5.5.23) with the initial value and boundary condition (5.5.25)(5.5.26), or

(5.5.25)(5.5.27) in the spaces introduced by

$$H_1 = \{u \in H^4(\Omega) | \frac{\partial u}{\partial n}|_{\partial\Omega} = \frac{\partial \Delta u}{\partial n}|_{\partial\Omega} = 0, \int_\Omega u dx = 0\};\ for\ (5.5.26)$$
$$H_1 = \{u \in H^4(\Omega) | u(x + 2k\pi) = u(x), \int_\Omega u dx = 0\};\ for\ (5.5.27)$$
$$H = \{u \in L^2(\Omega) | \int_\Omega u dx = 0\}$$

When we say that the problem (5.5.23)(5.5.26)(or (5.5.23)(5.5.27)) bifurcates an invariant set $\sum_\lambda$ from $(u, \lambda) = (0, \lambda_0)$, it means that $\sum_\lambda \subset H_1$, and $\sum_\lambda \to 0$ in H as $\lambda \to \lambda_0$.

We denote by $\{\lambda_k\}$ and $\{e_k\}$ the eigenvalues and eigenfunctions in H_1 of the Laplace operator $-\Delta$ with the boundary condition (5.5.26), or (5.5.27)

$$(5.5.29) \qquad \begin{cases} -\Delta e_k = \lambda_k e_k \\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \lambda_k \to \infty, k \to \infty \end{cases}$$

It is known that the eigenfunction $\{e_k\}$ of (5.5.29) constitute the common orthogonal base of H_1 and H.

Then we have the following results.

Theorem 5.5.8. Under the condition (5.5.28), the following assertions hold true:

i). As $\lambda > \alpha\lambda_1$, the equation (5.5.23) bifurcates from $(u, \lambda) = (0, \alpha\lambda_1)$ an attractor of homotopy sphere S^{n-1} for the boundary condition (5.5.26)(as $n = 1$, which is the equilibrum attractor), and an attractor of homotopy sphere S^{2n-1} for (5.5.27).

ii). For any eigenvalue λ_k of (5.5.29) having multiplicity m, as $\lambda > \alpha\lambda_k$ the problem (5.5.23)(5.5.26)(or (5.5.23)(5.5.27)) will bifurcate from $(u, \lambda) = (0, \alpha\lambda_k)$ an invariant set of homotopy sphere S^{m-1}.

Proof. We define the mappings $L_\lambda = -A + B_\lambda$ and $G : H_1 \to H$ by

$$Au = \alpha\Delta^2 u$$
$$B_\lambda u = -\lambda\Delta u$$
$$Gu = \Delta(\sum_{p=2}^{2k+1} a_p u^p)$$

It is readily to check that the conditions (5.4.2)-(5.4.7) and (5.4.5)′ in the center manifold theorems are satisfied.

Hence, for an eigenvalue λ_k of (5.5.29) having multiplicity m the bifurcation equation of (5.5.23) can be written as

$$\frac{dx_i}{dt} = \lambda_k(\lambda - \alpha\lambda_k)x_i - \lambda_k \int_\Omega \sum_{p=2}^{2k+1} a_p(v + h(v))^p e_i^* dx \tag{5.5.30}$$

$1 \leq i \leq m$, where $v = \sum_{j=1}^m x_j e_j^*$, $\{e_j^* | 1 \leq j \leq m\}$ the eigenfunctions of (5.5.29) corresponding to λ_k, and

$$h(v) \in H_k^\perp = \{u \in H_1 | \quad < u, e_i^* >_{H_1} = 0, \quad \forall 1 \leq i \leq m\}$$

the center manifold function, which satisfies

$$h(v) = o(|x|), \quad |x| = \sqrt{\sum_{i=1}^m x_i^2}$$

By (5.5.28), from (5.5.30) we can obtain

$$\frac{d}{dt}|x|^2 = \lambda_k(\lambda - \alpha\lambda_k)|x|^2 - 2\lambda_k\beta \int_\Omega |v|^4 dx + o(|x|^4)$$

It means that when $\lambda = \alpha\lambda_k, x = 0$ is asymptotically stable for (5.5.30). Hence, by Theorem 5.3.4, the conclusion ii) is proved.

Because the first eigenvalue λ_1 of (5.5.29) has multiplicity $m = n$ for the boundary condition (5.5.26), and $m = 2n$ for the boundary condition (5.5.27), the conclusion i) follows. Indeed, the eigenfunctions corresponding to λ_1 are

$$\cos x_i (1 \leq i \leq n) \ \textit{for b.c.} \ (5.5.26)$$

$$\sin x_i, \cos x_i (1 \leq i \leq n) \ \textit{for b.c.} \ (5.5.27)$$

where $x = (x_1, \cdots, x_n) \in \Omega = (0, 2\pi)^n$. Thus, the proof is complete.

Now, we consider the Kuramoto-Sivashinsky equation in space dimension one, which is given by

$$\begin{cases} \frac{\partial u}{\partial t} + \mu\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}(\frac{\partial u}{\partial x})^2 = 0, \quad 0 < x < 2\pi \\ u(x + 2\pi, t) = u(x, t) \\ u(x, 0) = \phi(x) \end{cases} \tag{5.5.31}$$

where $\mu > 0$, the boundary condition is space periodic

Alternative, we consider the equation obtained by differentiation of (5.5.31) with respect to x:

$$\begin{cases} \frac{\partial v}{\partial t} = -\mu\frac{\partial^4 v}{\partial x^4} - \frac{\partial^2 v}{\partial x^2} - v\frac{\partial v}{\partial x}, \quad 0 < x < 2\pi \\ v(x + 2\pi, t) = v(x, t); \int_0^{2\pi} v(x, t)dx = 0, \forall t \geq 0 \\ v(x, 0) = \psi \end{cases} \tag{5.5.32}$$

For the problem (5.5.32) we introduce

$$H_1 = \{u \in H^4(0,2\pi)| \quad u(x+2\pi) = u(x), \int_0^{2\pi} udx = 0\}$$

$$H = \{u \in L^2(0,2\pi)| \int_0^{2\pi} udx = 0\}$$

Then we have the below results.

Theorem 5.5.9. When $\mu \geq 1, v = 0$ is a global asymptotically stable equilibrum point of (5.5.32) in H, and for each integer $k(k = 1, 2, \cdots)$ the problem (5.5.32) will bifurcate from $(v, \mu) = (0, k^{-2})$ a periodic orbit. Especially, from $(v, \mu) = (0, 1)$ it bifurcates the S^1-attractor.

Proof. We define the mapping $L_\mu = -A + \mu^{-1}B$ and $G : H_1 \to H$ by

$$Au = -\frac{d^4u}{dx^4}; \quad Bu = -\frac{d^2u}{dx^2}; \quad Gu = -u\frac{du}{dx}$$

The problem (5.5.32) can be written as the abstract form

$$\text{(5.5.33)} \qquad \begin{cases} \frac{du}{dt} = \mu L_\mu u + Gu, u \in H_1, \mu > 0 \\ u(0) = \phi, \phi \in H_1 \end{cases}$$

We know that the eigen-problem

$$\begin{cases} -\frac{d^2e_k}{dx^2} = \lambda_k e_k \\ e_k(x+2\pi) = e_k(x) \end{cases}$$

has the following eigenvalues and eigenfunctions

$$\lambda_k = k^2, \quad k = 1, 2, \cdots$$

$$e_k(x) = \sin kx; \quad \cos kx$$

Hence the operator L_μ has the eigenvalues and eigenvectors

$$\text{(5.5.34)} \qquad \begin{cases} \beta_k(\mu) = \mu^{-1}k^2 - k^4, \quad k = 1, 2, \cdots \\ \{\sin kx, \cos kx| \quad k = 1, 2, \cdots\} \end{cases}$$

Each eigenvalue of L_μ has the multiplicity $m = 2$, and the eigenvectors of L_μ constitute the common orthogonal base of H_1 and H. It is readily to check that for the equation (5.5.33) the conditions in the center manifold theorem (Theorem 5.4.1 (b)) are fulfilled. Hence, near every eigenvalue $\mu^{-1}k^2 - k^4 =$

0, the bifurcation equation of (5.5.33) is a 2-dimensional system of ordinary differential equation, which reads

$$\frac{dy_1}{dt} = (k^2 - \mu k^4)y_1 - \int_0^{2\pi} u\frac{du}{dx}\sin kx dx \tag{5.5.35}$$

$$\frac{dy_2}{dt} = (k^2 - \mu k^4)y_2 - \int_0^{2\pi} u\frac{du}{dx}\cos kx dx$$

where $u = y_1 \sin kx + y_2 \cos kx + h(y_1, y_2)$, and

$$h(y_1, y_2) \in \{v \in H_1 | \int_0^{2\pi} v \sin kx dx = \int_0^{2\pi} v \cos kx dx = 0\}$$

is the center manifold function. By Theorem 5.3.9, from (5.5.35) it follows that the equation (5.5.33) bifurcates from $(u, \mu) = (0, k^{-2})$ a S^1-invariant set. Meanwhile, the equation (5.5.32) has no nonzero equilibrum points. Indeed, if (5.5.32) has a nonzero equilibrum point $v \neq 0$ in H_1, then $u = \int v dx \neq 0$ in H_1 is an equilibrum point of (5.5.31), thus it satisfies

$$\begin{aligned} 0 &= \int_0^{2\pi} [\mu\frac{d^4u}{dx^4} + \frac{d^2u}{dx^2} + \frac{1}{2}(\frac{du}{dx})^2]dx \\ &= \frac{1}{2}\int_0^{2\pi} (\frac{du}{dx})^2 dx \end{aligned}$$

It is a contradiction to that $u \neq 0$ in H_1. Hence the bifurcated S^1-invariant sets of (5.5.32) at $\mu = k^{-2}(k = 1, 2, \cdots)$ are the periodic orbits.

Now, we return to prove that $v = 0$ is a global asymptotically stable equilibrum point of (5.5.32) for $\mu \geq 1$ by using Theorem 3.2.8. In fact, we see that $G : H_1 \to H$ is an orthogonal operator

$$< Gu, u >= -\int_0^{2\pi} u^2\frac{du}{dx}dx = -\frac{1}{3}\int_0^{2\pi} \frac{du^3}{dx}dx = 0$$

On the other hand, the eigenvalues (5.5.34) of L_μ are negative for all $\mu > 1$:

$$\beta_k(\mu) = k^2(\mu^{-1} - k^2) < 0, \quad \forall k = 1, 2, \cdots, \ and\ \mu > 1.$$

When $\mu = 1, \beta_1(1) = 0$ and its eigenvectors are $\{\sin x, \cos x\}$. We see that for any $\alpha_1 \sin x + \alpha_2 \cos x, |\alpha_1| + |\alpha_2| \neq 0$,

$$\begin{aligned} G(\alpha_1 \sin x + \alpha_2 \cos x) &= (\alpha_1 \sin x + \alpha_2 \cos x)(\alpha_1 \cos x - \alpha_2 \sin x) \\ &= \frac{1}{2}(\alpha_1^2 - \alpha_2^2) \sin 2x + \alpha_1\alpha_2 \cos 2x \end{aligned}$$

Namely, $\forall u \in E_0 = span\{\sin x, \cos x\}, u \neq 0$

$$Gu \in E_0^{\perp} = \{u \in H| < u, v >_H = 0, \forall v \in E_0\},\ and$$

$$Gu \neq 0$$

Thus, the desired conclusion follows from Theorem 3.2.8. The proof is complete.

5.5.4. Reaction-diffusion equations

In this subsection we study the bifurcation of invariant sets associated with reaction-diffusion equations. We consider a boundary value problem involving a vector function $u = (u_1, \cdots, u_m)$ which satisfies the equation

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} = A\Delta u + B_\lambda u + G(x,u), & x \in \Omega \subset R^n (1 \leq n \leq 3) \\ u|_{\partial\Omega} = 0\ (or\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0) & \\ u(x,0) = \phi & \end{array} \right. \tag{5.5.36}$$

where A is a positive diagnal matrix of diffusion coeficients

$$A = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m \end{pmatrix} \tag{5.5.37}$$

and B_λ is a $m \times m$ parameterized constant matrix

$$B_\lambda = \begin{pmatrix} b_{11}(\lambda) & \cdots & b_{1m}(\lambda) \\ \vdots & \ddots & \vdots \\ b_{m1}(\lambda) & \cdots & b_{mm}(\lambda) \end{pmatrix} \tag{5.5.38}$$

and $G = \{G_1, \cdots, G_m\}$ is continuous on $\bar{\Omega} \times R^m$, and

$$G(x,\xi) = o(|\xi|), \quad \xi \in R^m \tag{5.5.39}$$

Let $\rho_k \geq 0 (k = 1, 2, \cdots)$ be the k-th eigenvalue of the Laplacian

$$\left\{ \begin{array}{l} -\Delta u = \rho_k u \\ u|_{\partial\Omega} = 0\ (or\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0) \end{array} \right. \tag{5.5.40}$$

For the mathematical setting we introduce

$$H_1 = H^2(\Omega, R^m) \bigcap H_0^1(\Omega, R^m)$$

$$(or\ H_1 = \{u \in H^2(\Omega, R^m)| \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\})$$

$$H = L^2(\Omega, R^m)$$

Obviously, the operator $A + B_\lambda + G : H_1 \to H$ defined by (5.5.37)-(5.5.39) satisfies the conditions in (5.4.20). In view of Theorem 5.4.3-Theorem 5.4.6 we have the following results

Theorem 5.5.10. Assume that the eigenvalues $\beta_j(\lambda)(1 \leq j \leq m)$ of the matrix $-\rho_1 A + B_\lambda = (b_{ij}(\lambda) - \delta_{ij}\rho_1\mu_j)$ satisfy that (here ρ_1 is the first eigenvalue of (5.5.40))

$$Re\beta_l(\lambda) = \begin{cases} < 0, & \lambda < \lambda_0 \ (or\ \lambda > \lambda_0) \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0 \ (or\ \lambda < \lambda_0) \end{cases}$$

$$Re\beta_j(\lambda_0) < 0, \quad l + 1 \leq j \leq m$$

and as $\lambda = \lambda_0, u = 0$ is locally asymptotically stable for (5.5.36), then the problem (5.5.36) must bifurcate from $(u, \lambda) = (0, \lambda_0)$ an attractor with dimension $d \leq l - 1$.

Theorem 5.5.11. Assume that the eigenvalues $\beta_j(\lambda)(1 \leq \lambda \leq m)$ of the matrix $-\rho_k A + B_\lambda = (b_{ij}(\lambda) - \delta_{ij}\rho_k\mu_j)$ satisfy that

$$Re\beta_1(\lambda) = Re\beta_2(\lambda) = \begin{cases} < 0, & \lambda < \lambda_0 \ (or\ \lambda > \lambda_0) \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0 \ (or\ \lambda > \lambda_0) \end{cases}$$

$$Re\beta_j(\lambda_0) \neq 0, \quad \forall 3 \leq j \leq m$$

then the problem (5.5.36) will bifurcate from $(u, \lambda) = (0, \lambda_0)$ an S^1-invariant set.

In the following, we give some examples of reaction-diffusion equations satisfying the conditions in Theorem 5.5.10-5.5.11.

Example 5.5.12. These equations arise in the study of super-conductivity of liquids (see [Te]). We have $m = n$, and $u = (u_1, \cdots, u_n)$ is a solution of the equations

$$\begin{cases} \frac{\partial u}{\partial t} = A\Delta u + u - |u|^2 u \\ u|_{\partial\Omega} = 0 \end{cases} \tag{5.5.41}$$

where the parameters taken are μ_i in the diagnal matrix A. The eigenvalues β_j of the matrix

$$-\rho_1 A + I = \begin{pmatrix} 1 - \rho_1\mu_1 & & 0 \\ & \ddots & \\ 0 & & 1 - \rho_1\mu_n \end{pmatrix}$$

are as $\beta_j = 1 - \rho_1\mu_j, \rho_1$ the first eigenvalue of (5.5.40). If $\mu_1 = \cdots = \mu_l < \mu_{l+1}, \cdots, \mu_n (1 \leq l \leq n)$, then by Theorem 5.5.10, the system (5.5.41) will bifurcate from $(u, \mu_1) = (0, \rho_1^{-1})$ a S^{l-1}-attractor. In fact, the behaviors of solutions of (5.5.41) are alike to that of the Landau-Ginzburg equation discussed in Subsection 5.5.2. If the multiplicity ρ_k of (5.5.40) is r, and

$$\mu_{i_1} = \cdots = \mu_{i_l} = \rho_k^{-1} \quad (1 \leq l \leq m)$$

then it is not difficult to show that the system (5.5.4) will bifurcate from $(u, \mu) = (0, \rho_k^{-1})$ a S^{rl-1}-invariant set.

Example 5.5.13. These equations arise in ecology which discribe the ecological balance (for instance the predator-prey systems, the colony growth atc.), which read

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \lambda_1 u + uG_1(u,v) + \int_0^t G_2(u(s), v(s))ds \\ \frac{\partial v}{\partial t} = \Delta v + \lambda_2 v + vF_1(u,v) + \int_0^t F_2(u(s), v(s))ds \end{cases} \tag{5.5.42}$$

where

$$G_1(0,0) = 0, \quad G_2(z_1, z_2) = o(|z_1| + |z_2|)$$
$$F_1(0,0) = 0, \quad F_2(z_1, z_2) = o(|z_1| + |z_2|)$$

when the parameters $\lambda_1 = \lambda_2$, or the multiplicity of the eigenvalue ρ_k of (5.5.40) is two, the system (5.5.42) will bifurcate from $(u, \lambda) = (0, \rho_1)$(or from $(0, \rho_k)$) a S^1-invariant set. Furthermore if (5.5.42) has no nonzero stationary solution near $u = 0$, then the S^1-invariant set is a periodic orbit.

Example 5.5.14. The final example is the equations which serve as a model for the Belousov-Zhabotinsky reactions in chemical dynamics (see [Te]). Here $m = 3$ and $u = (u_1, u_2, u_3)$ satisfies

$$\frac{\partial u_1}{\partial t} = \mu_1 \Delta u_1 + \lambda u_1 + \lambda u_2 - \lambda u_1 u_2 - \lambda\beta u_1^2$$

$$\frac{\partial u_2}{\partial t} = \mu_2 \Delta u_2 - \frac{1}{\lambda} u_2 + \frac{\gamma}{\lambda} u_3 - \frac{1}{\lambda} u_1 u_2 \tag{5.5.43}$$

$$\frac{\partial u_3}{\partial t} = \mu_3 \Delta u_3 - \delta u_3 + \delta u_1$$

$$u_i|_{\partial\Omega} = 0, \quad i = 1, 2, 3.$$

where $\lambda, \beta, \gamma, \delta > 0$. For the domain Ω, we here take $\Omega = (0, 2\pi)^2 \subset R^2$. In this case, we know that the eigenvalues of (5.5.40) are

$$\rho_{nm} = m^2 + n^2, \quad n, m \geq 1 \textit{ are integers},$$

and when $m = n$, the multiplicity of ρ_{nm} is one, when $m \neq n$, the multiplicity is > 1(generally $= 2$). On the other hand, if the multiplicity of ρ_k equals 2, and $\lambda_0 > 0$ is a solution of the algebraic equation

$$det \begin{pmatrix} \lambda - \mu_1\rho_k & \lambda & 0 \\ 0 & -(\lambda^{-1} + \mu_2\rho_k) & \lambda^{-1}\gamma \\ \delta & 0 & -(\delta + \mu_3\rho_k) \end{pmatrix} = 0$$

namely $\lambda_0 > 0$ satisfies

$$(\lambda - \mu_1\rho_k)(\mu_2\rho_k\lambda_1)(\mu_3\rho_3 + \delta) + \gamma\delta\lambda = 0$$

then the system (5.5.43) will bifurcate from $(u, \lambda) = (0, \lambda_0)$ a S^1-invariant.

5.6. Navier-Stokes Equations $(n = 2)$

We shall conclude this chapter by consider the $2D$ Navier-Stokes equations with the perioidic boundary condition, the free boundary condition, and the Dirichlet boundary condition. The equations are given by

(5.6.1)
$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \mu\Delta u - \nabla\rho + f(x, \lambda), x \in \Omega \subset R^2 \\ divu = 0 \end{cases}$$

which are the same as in Subsection 3.3.1. Here $\lambda \in R^1$ is a parameter, and $\bar{\Omega} \subset R^2$ is a compact manifold with boundary.

The equations (5.6.1) is supplemented with the initial value condition

$$u(x, 0) = u_0(x)$$

Three cases of the boundary conditions will be considered.

The periodic boundary condition: in which case $\Omega = (0, 2\pi)^2$, and

(5.6.3)
$$u(x + 2K\pi, t) = u(x, t), \quad \forall t \geq 0$$

$K = (k_1, k_2), k_i$ the integers.

The free boundary condition:

(5.6.4)
$$u_n|_{\partial\Omega} = 0, \quad \frac{\partial u_\tau}{\partial n}|_{\partial\Omega} = 0$$

where $u_n = u \cdot n, u_\tau = u \cdot \tau, n$ and τ respectively the unit normal and tangent vectors on the boundary $\partial\Omega$.

The Dirichlet boundary condition

(5.6.5)
$$u|_{\partial\Omega} = 0$$

For the periodic boundary condition, we also impose the condition that the unknown function u has the zero average

$$\int_{\Omega} u(x,t)dt = 0, \quad \forall t \geq 0 \tag{5.6.6}$$

We shall latter see that the condition (5.6.6) means that u does not contain the harmonic fields.

5.6.1. The Hodge decomposition

We say that a vector field $u \in H^k(\Omega, R^2)(k \geq 0)$ is a Hamiltonian if there is a function $\psi \in H^{k+1}(\Omega)$ such that

$$u = (u_1, u_2) = curl\psi, \quad u_1 = \frac{\partial \psi}{\partial x_2}, \quad u_2 = -\frac{\partial \psi}{\partial x_1}$$

In this case, ψ is called the Hamiltonian function, or in some time it is also called the stream function. We set

$$\begin{aligned}
&D^k(\Omega, R^2) = \{u \in H^k(\Omega, R^2)| \quad divu = 0, u_n|_{\partial\Omega} = 0\} \\
&\mathcal{H}^k(\Omega, R^2) = \{u \in D^k(\Omega, R^2)| \quad u \ is \ a \ Hamiltonian\} \\
&G^k(\Omega, R^2) = \{\nabla p| \quad p \in H^{k+1}(\Omega), \frac{\partial p}{\partial \tau}|_{\partial\Omega} = 0\}
\end{aligned}$$

For a general domain $\Omega \subset R^2$, it is known that there is a decomposition

$$H^k(\Omega, R^2) = D^k(\Omega, R^2) \bigoplus G^k(\Omega, R^2)$$

But, if $\bar{\Omega} \subset R^2$ is a compact manifold with boundary, then we have the Hodge decomposition theorem as follows, which is very useful for our discussion on the $2D$ Navier-Stokes equations. The following version of the Hodge decomposition theorem on a compact manifold with boundary is due to G. Schwartz [Sw].

Theorem 5.6.1(Hodge decomposition). Let $\bar{\Omega} \subset R^2$ be a compact manifold with boundary. Then, for any $u \in H^k(\Omega, R^2)(k \geq 1), u$ can be uniquely written as

$$\begin{aligned}
&u = curl\psi + \nabla\phi + v \\
&\frac{\partial \psi}{\partial \tau}|_{\partial\Omega} = 0, \quad \frac{\partial \phi}{\partial \tau}|_{\partial\Omega} = 0, \quad \psi, \phi \in H^{k+1}(\Omega) \\
&v \in E^k(\Omega, R^2) = \{v \in D^k(\Omega, R^2)| \quad curlv = 0\}
\end{aligned}$$

Moreover, $E^k(\Omega, R^2)$ is finite dimensional and

$$dimE^k(\Omega, R^2) = \beta_1(\Omega), \ the \ first \ Betti \ number \ of \ \Omega.$$

The vectors in $E^k(\Omega, R^2)$ are called the harmonic fields.

Proof. For convenience, we prove this theorem in the k-th differentiable space $C^k(\Omega, R^2)(k \geq 1)$. Let $F^k(\Omega)$ be the space of all C^k differential one form on Ω. By the Hodge decomposition theorem (see [Sw]), any $\omega \in F^k(\Omega)$ can be uniquely decomposed into

$$\omega = d\psi + \delta\beta + v^*$$

$$\frac{\partial \psi}{\partial \tau}|_{\partial\Omega} = 0, \quad \psi \in C^{k+1}(\Omega)$$

$$\delta\beta \cdot n|_{\partial\Omega} = \frac{\partial \phi}{\partial \tau}|_{\partial\Omega} = 0, \quad \phi \in C^{k+1}(\Omega) \tag{5.6.7}$$

$$v^* \in E_h(\Omega)$$

where

$$\beta = \phi dx_1 \bigwedge dx_2$$
$$\delta\beta = -\frac{\partial \phi}{\partial x_2} dx_1 + \frac{\partial \phi}{\partial x_1} dx_2$$
$$v^* = v_1^* dx_1 + v_2^* dx_2$$

and $E_H(\Omega)$ is the space of C^k harmonic 1-forms:

$$E_H(\Omega) = \{v^* \in F^k(\Omega) | \delta v^* = 0, dv^* = 0, \ and$$

$$v_1^* \cos(\tau, x_1) + v_2^* \cos(\tau, x_2)|_{\partial\Omega} = 0\}$$

Under the symplectic homeomorphism $J : F^k(\Omega) \to C^k(\Omega, R^2)$ given by the area form $\omega_0 = dx_1 \bigwedge dx_2$ of Ω, we infer from (5.6.7) that

$$u = J\omega = curl\psi - \nabla\phi + v$$

$$\frac{\partial \psi}{\partial \tau}|_{\partial\Omega} = 0, \quad \frac{\partial \phi}{\partial \tau}|_{\partial\Omega} = 0 \tag{5.6.8}$$

$$divv = 0, \quad curlv = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0$$

where $v = Jv^*$. In fact, the symplectic homeomorphism J can be expressed as that for any $\omega = v_1 dx_1 + v_2 dx_2$,

$$J\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} \in C^k(\Omega, R^2)$$

Hence, for any $u \in C^k(\Omega, R^2)$ we deduce from (5.6.8) that

$$\begin{aligned} & u = curl\psi + \nabla\phi + v \\ & divv = 0, \quad curlv = 0 \\ & \frac{\partial\psi}{\partial\tau}|_{\partial\Omega} = 0, \quad \frac{\partial\phi}{\partial\tau}|_{\partial\Omega} = 0, \quad v \cdot n|_{\partial\Omega} = 0 \end{aligned}$$

This is the first conclusion of this theorem.

By Theorem 2.2.2 in [Sw], it is easy to see that $E^k(\Omega, R^2) = JE_H(\Omega)$, and

$$dimE^k(\Omega, R^2) = dimE_H(\Omega) = \beta_1(\Omega)$$

This proof is complete.

By the Hodge decomposition theorem, we see that a vector field on a two dimensional manifold is a direct sum of a Hamiltionian, a gradient field and a harmonic field. On the general manifolds of dimention two, the harmonic fields are not the Hamiltonian, however, on $\Omega \subset R^2$ the harmonic fields are the Hamiltonian, which are characterized by the following theorem

Theorem 5.6.2. Let $\bar{\Omega} \subset R^2$ be a compact manifold with boundary and $\beta_1(\Omega) \neq 0$. Then for any $v \in E^k(\Omega, R^2)$, there is a function $\psi \in H^{k+1}(\Omega)$ such that

$$\begin{cases} v = curl\psi \\ \Delta\psi = 0 \\ \psi|_{\partial\Omega} \neq 0, \frac{\partial\psi}{\partial\tau}|_{\partial\Omega} = 0 \end{cases}$$

Proof. We know that the first Betti number β_1 represent that there are β_1 holes in the interior of Ω. Hence, the boundary $\partial\Omega$ has $\beta_1 + 1$ components, each of which is homeomorphic to S^1:

$$\partial\Omega = \bigcup_{k=0}^{\beta_1} \Gamma_k$$

It is well known that each of the below boundary value problems has an unique solution

$$(5.6.9) \qquad \begin{cases} \Delta\psi_k = 0 \\ \psi_k|_{\Gamma_k} = 1, \ for\ some\ one\ k\ (0 \leq k \leq \beta_1) \\ \psi_k|_{\Gamma_j} = 0, \forall j \neq k \end{cases}$$

Thus, we obtain $\beta_1 + 1$ functions $\psi_0, \cdots, \psi_{\beta_1}$. Obviously, there are only β_1 functions in $\{\psi_k | 0 \leq k \leq \beta_1\}$ which are linear independent up to a constant.

For instance

$$\psi_0 - 1 = -\sum_{k=1}^{\beta_1} \psi_k$$

Hence, the vector field $v_j = curl\psi_j (1 \leq j \leq \beta_1)$ are linear independent, and $v_j \in E^k(\Omega, R^2)$ are the β_1 harmonic fields. Thus, this theorem follows from theorem 5.6.1. The proof is complete.

It is easy to see that the harmonic fields enjoy the properties

$$\Delta v = 0, \ and \ \int_\Omega v dx \neq 0, \quad \forall v \in E^k(\Omega, R^2) \tag{5.6.10}$$

For the periodic boundary condition (5.6.3), the equations (5.6.1) can be regarded as defined on the torus T^2. Thus the space of the harmonic fields on T^2 is as follows

$$E^k(T^2, R^2) = \{u = (a, b) | \quad a, b \in R^1\} = R^2$$

Hence, the vector fields $u \in \mathcal{H}^k(\Omega, R^2)$ with the periodic boundary condition if and only if u satisfy the condition (5.6.6).

5.6.2. Mathematical setting

According to the above subsection, it is sufficient for us to discuss the problems ((5.6.1)-(5.6.6)) only in the spaces $\mathcal{H}^k(\Omega, R^2) \bigoplus G^k(\Omega, R^2) (k \geq 0)$.

For any $f \in \mathcal{H}^k(\Omega, R^2) \bigoplus G^k(\Omega, R^2)$, we have

$$f = curl\psi + \nabla\phi, \quad \psi, \phi \in H^{k+1}(\Omega)$$

Then the equation (5.6.1) can be written as

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \mu\Delta u - \nabla p_1 + curl\psi \\ divu = 0 \\ p_1 = p - \phi \end{cases}$$

Hence, the gradient term $\nabla\phi$ in the given function f does not influence the behaviors of solutions of (5.6.1). Thus, we only need to consider the given functions f in the spaces $\mathcal{H}^k(\Omega, R^2)$.

For the given parameterized terms $f(x, \lambda)$ in (5.6.1), we introduce the following space

$$C(R^1, \mathcal{H}^k) = \{f(\cdot, \lambda) \in \mathcal{H}^k(\Omega, R^2) | \lambda \in R^1, f(x, 0) = 0,$$

$$and \ f(x, \lambda) \ is \ continuous \ on \ \lambda\}$$

endowed with the topology that $B_r(f) \subset C(R^1, \mathcal{H}^k)$ is an open disk with center f and radius r defined by

$$B_r(f) = \{g \in C(R^1, \mathcal{H}^k) | \quad \sup_{\lambda} \|g - f\|_{H^k} < r\}$$

Obviously, $C(R^1, \mathcal{H}^k)$ is a linear topological space, but not a Banach space.

For the vector field with the boundary condition, we set

$$\begin{aligned}
\mathcal{H}^k_p(\Omega, R^2) &= \{u \in \mathcal{H}^k(\Omega, R^2) | \quad u \ is \ \Omega - periodic\} \\
\mathcal{H}^k_F(\Omega, R^2) &= \{u \in \mathcal{H}^k(\Omega, R^2) | \quad u_n = \frac{\partial u_\tau}{\partial n} = 0 \ on \ \partial\Omega\} \\
\mathcal{H}^k_D(\Omega, R^2) &= \{u \in \mathcal{H}^k(\Omega, R^2) | \quad u|_{\partial\Omega} = 0\}
\end{aligned}$$

The spaces of the associated Hamiltonian function are given by

$$\begin{aligned}
H^k_p(\Omega) &= \{\psi \in H^k(\Omega) | \psi \ is \ \Omega - periodic\} \\
H^k_F(\Omega) &= \{\psi \in H^k(\Omega) | \frac{\partial \psi}{\partial \tau} = 0, \frac{\partial^2 \psi}{\partial n^2} = 0 \ on \ \partial\Omega\} \\
H^k_D(\Omega) &= \{\psi \in H^k(\Omega) | D\psi = 0 \ on \ \partial\Omega\}
\end{aligned}$$

Usually, we uniformly denote the above spaces by $\mathcal{H}^k_B(\Omega, R^2)$ and $H^k_B(\Omega)$.

It is readily to check that

$$\Delta u \neq 0, \quad \forall u \in \mathcal{H}^k_B(\Omega, R^2) \ and \ u \neq 0 \quad (k \geq 2)$$

Otherwise we can get that

$$\int_\Omega \Delta u \cdot u dx = -\int_\Omega |\nabla u|^2 dx = 0$$

which is a contradiction with that $u \neq 0$. Hence, by (5.6.10) the spaces $\mathcal{H}^k_B(\Omega, R^2)$ have no the harmonic fields.

When $\Omega = (0, 2\pi)^2$, there are three cases of the boundary conditions in which cases the vector fields can be expressed by the Fourier series.

The periodic case, $u \in \mathcal{H}^k_p(\Omega, R^2)$ has the Fourier expansion:

$$u = (u_1, u_2) \in \mathcal{H}^k_p(\Omega, R^2)$$

$$u_1 = \sum_{n,m=-\infty}^{\infty} a_{nm} e^{i(nx+my)}$$

$$u_2 = \sum_{n,m=-\infty}^{\infty} b_{nm} e^{i(nx+my)} \tag{5.6.11}$$

$$a_{nm} = \bar{a}_{-n-m}, \quad b_{nm} = \bar{b}_{-n-m}$$

$$na_{nm} = -mb_{nm}, \quad a_{00} = b_{00} = 0$$

The associated Homiltomian function is given by

$$\psi = -i \sum_{m,n=-\infty n\neq 0, m\neq 0}^{\infty} \frac{1}{m} a_{nm} e^{i(nx+my)} - i \sum_{m=-\infty m\neq 0}^{\infty} \frac{1}{m} a_{0m} e^{imy} \tag{5.6.12}$$

$$-i \sum_{n=-\infty n\neq 0}^{\infty} \frac{1}{n} b_{n0} e^{inx}$$

The free boundary condition, $u \in \mathcal{H}_F^k(\Omega, R^2)$ satisfies

$$\begin{aligned} &u_1(0,y) = u_1(2\pi, y) = 0, \quad 0 \le y \le 2\pi \\ &u_2(x,0) = u_2(x, 2\pi) = 0, \quad 0 \le x \le 2\pi \\ &\frac{\partial u_1(x,0)}{\partial y} = \frac{\partial u_1(x,2\pi)}{\partial y} = 0, \quad 0 \le x \le 2\pi \\ &\frac{\partial u_2(0,y)}{\partial x} = \frac{\partial u_2(2\pi,y)}{\partial x} = 0, \quad 0 \le y \le 2\pi \end{aligned}$$

Therefore, $u \in \mathcal{H}_F^k(\Omega, R^2)$ has the Fourier expansion

$$u = (u_1, u_2) \in \mathcal{H}_F^k(\Omega, R^2)$$

$$u_1 = \sum_{n,m=1}^{\infty} a_{nm} \sin nx \cos my$$

$$u_2 = \sum_{n,m=1}^{\infty} b_{nm} \cos nx \sin my \tag{5.6.13}$$

$$na_{nm} + mb_{nm} = 0$$

The associated Hamilton function is given by

$$\psi = \sum_{n,m=1}^{\infty} \frac{1}{m} a_{nm} \sin nx \sin my \tag{5.6.14}$$

The semi-periodic boundary condition, in this case it can be regarded as the free boundary condition defined on an annulus. This condition is given by

$$u_1(x + 2k\pi, y) = u_1(x, y)$$

$$\frac{\partial u_1}{\partial y}(x,0) = \frac{\partial u_1}{\partial y}(x,2\pi) = 0 \tag{5.6.15}$$

$$u_2(x,0) = u_2(x,2\pi) = 0$$

Under the condition, $u = (u_1, u_2)$ has the Fourier expansion

$$u_1 = \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} a_{nm}\cos mye^{inx}$$

$$u_2 = \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} b_{nm}\sin mye^{inx} \tag{5.6.16}$$

$$a_{nm} = \bar{a}_{-nm}, \quad b_{nm} = \bar{b}_{-nm}$$

$$ina_{nm} + mb_{nm} = 0, \quad a_{00} = 0$$

The associated Hamiltonian function reads

$$\psi = \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{m}a_{nm}\sin mye^{inx} \tag{5.6.17}$$

5.6.3. Invariance of the eigen-spaces

Let $\{\rho_k\} \subset R^1$ and $\{e_k\} \subset \mathcal{H}_B^\infty(\Omega, R^2)$ be the eigenvalues and eigenfunctions of the following system

$$\begin{cases} -\Delta e_k = \rho_k e_k \quad (\rho_k > 0) \\ dive_k = 0 \end{cases} \tag{5.6.18}$$

A special property of $2D$ Navier-Stokes equations is the invariance of the eigen-spaces of (5.6.18), which is given by the following theorem.

We investigate the equations

$$\begin{cases} \frac{\partial u}{\partial t} = \mu\Delta u - (u\cdot\nabla)u - \nabla\rho + e_k \\ divu = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{5.6.19}$$

where e_k is the eigenfunction of (5.6.11) corresponding to ρ_k.

Theorem 5.6.3. For the problem (5.6.19) with the periodic boundary condition (5.6.3), or with the Dirichlet boundary condition (5.6.5), the following assertions hold:
1). $v_k = \mu^{-1}\rho_k^{-1}e_k$ is a stationary solution of (5.6.19), and the eigen-space

$E_k \subset \mathcal{H}_B^\infty(\Omega, R^2)$ corresponding to ρ_k is invariant, namely if the initial value $u_0 \in E_k$, then the solution of (5.6.19) satisfies that $u(x,t,u_0) \in E_k, \forall t \geq 0$.

2). The stationary solution v_k is stable in E_k, i.e_1 the solution $u(x,t,u_0) \to v_k$ provided $u_0 \in E_k$.

Proof. Let $\mathcal{H}_1^k(\Omega, R^2)$ be the space of all the Hamiltonian which are not the harmonic fields. Then, by Theorem 5.6.1 and Theorem 5.6.2 we have

$$H^k(\Omega, R^2) = \mathcal{H}_1^k(\Omega, R^2) \bigoplus E^k(\Omega, R^2) \bigoplus G^k(\Omega, R^2)$$

Denote by

$$\begin{aligned} P_1 &: H^k(\Omega, R^2) \to \mathcal{H}_1^k(\Omega, R^2) \\ P_2 &: H^k(\Omega, R^2) \to E^k(\Omega, R^2) \\ P_3 &: H^k(\Omega, R^2) \to G^k(\Omega, R^2) \end{aligned}$$

the projection operators. Then the stationary equation of (5.6.19) can be decomposed into

$$P_1[\mu\Delta u - (u \cdot \nabla)u] + e_k = 0$$

$$P_2[\mu\Delta u - (u \cdot \nabla)u] = 0 \tag{5.6.20}$$

$$P_3[(u \cdot \nabla)u] - \nabla p = 0$$

In order to prove the conclusion 1), it is necessary to show that.

$$P_1[(u \cdot \nabla)u] = 0, \quad \forall u \in E_k \subset \mathcal{H}_B^\infty(\Omega, R^2) \tag{5.6.21}$$

$$P_2[(u \cdot \nabla)u] = 0, \quad \forall u \in E_k \tag{5.6.22}$$

Because $u \in E_k$ is a Hamiltonian,

$$u = curlh = \left(\frac{\partial h}{\partial x_2}, -\frac{\partial h}{\partial x_1}\right), \quad h \in H^\infty(\Omega)$$

Noting that u satisfies (5.6.18), we have

$$-\Delta h = \rho_k h \tag{5.6.23}$$

We see that

$$\frac{\partial}{\partial x_2}(u \cdot \nabla)u_1 - \frac{\partial}{\partial x_1}(u \cdot \nabla)u_2 \tag{5.6.24}$$

$$= \frac{\partial}{\partial x_2}\left(\frac{\partial h}{\partial x_2}\frac{\partial^2 h}{\partial x_1 \partial x_2} - \frac{\partial h}{\partial x_1}\frac{\partial^2 h}{\partial x_2^2}\right) - \frac{\partial}{\partial x_1}\left(-\frac{\partial h}{\partial x_2}\frac{\partial^2 h}{\partial x_1^2} + \frac{\partial h}{\partial x_1}\frac{\partial^2 h}{\partial x_1 \partial x_2}\right)$$
$$= \frac{\partial h}{\partial x_2}\frac{\partial}{\partial x_1}\Delta h - \frac{\partial h}{\partial x_1}\frac{\partial}{\partial x_2}\Delta h$$
$$= 0, \quad (by\ (5.6.23))$$

From (5.6.24) we obtain the equality (5.6.21).

The equality (5.6.22) is equivalent to that

$$\int_\Omega (u\cdot\nabla)u\cdot v dx = 0, \quad \forall u \in E_k, \quad v \in E^k(\Omega, R^2) \tag{5.6.25}$$

For the periodic boundary condition, v=constant in $E^k(\Omega, R^2)$, the equality (5.6.25) obviously holds. For the Dirichlet boundary condition, by Theorem 5.6.2, there is a $\psi \in H^{k+1}(\Omega)$ such that

$$v = curl\psi = \left(\frac{\partial \psi}{\partial x_2} \quad -\frac{\partial \psi}{\partial x_1}\right)$$

Hence we have (by $u|_{\partial\Omega} = 0$)

$$\begin{aligned}\int_\Omega (u\cdot\nabla)u\cdot v dx &= -\int_\Omega \psi\left[\frac{\partial}{\partial x_2}((u\cdot\nabla)u_1) - \frac{\partial}{\partial x_1}((u\cdot\nabla)u_2)\right]dx \\ &= 0, \quad (by\ (5.6.24))\end{aligned}$$

Thus, we derive the equality (5.6.22)

In view of (5.6.21)(5.6.22), the equations (5.6.20) restricted on E_k is equivalent to the following equation

$$\mu\Delta u + e_k = 0, \quad \forall u \in E_k$$

Thus, the conclusion 1) is proven.

Because the eigen-space E_k is invariant, the equation (5.6.19) restricted on E_k can be written as

$$\begin{cases} \frac{\partial u}{\partial t} = \mu\Delta u + e_k, & u \in E_k \\ u(x,0) = u_0, & u_0 \in E_k \end{cases} \tag{5.6.26}$$

Let $E_k = span\{e_{k_1}, \cdots, e_{k_m}\}, u = \sum_{i=1}^m x_i(t)e_{k_i}, u_0 = \sum_{i=1}^m \alpha_i e_{k_i}$, and $e_k = e_{k_1}$. Then the equation (5.6.26) is equivalent to the ordinary differential equations

$$\begin{cases} \frac{dx_1}{dt} = -\mu\rho_k x_1 + 1, \\ \frac{dx_j}{dt} = -\mu\rho_k x_j; & 2 \le j \le m \\ x_i(0) = \alpha_i \end{cases} \tag{5.6.27}$$

The solutions of (5.6.27) read

$$x_1(t) = \alpha_1 e^{-\mu\rho_k t} + \mu^{-1}\rho_k^{-1}(1 - e^{-\mu\rho_k t})$$
$$x_j(t) = \alpha_j e^{-\mu\rho_k t}, \quad 2 \leq j \leq m$$

It is clear that

$$u(x,t) = \sum_{i=1}^{m} x_i(t) e_{k_i} \to \mu^{-1}\rho_k^{-1} e_{k_1} = \mu^{-1}\rho_k e_k, \qquad as\ t \to \infty$$

The proof is complete.

Remark 5.6.4. When the Betti number $\beta_1(\Omega) > 0$, in general the theorem of invariant eigen-spaces does not hold for the free boundary condition. The reasion is that the equality (5.6.22) is not true. Indeed, we find that

$$\int_\Omega (u\cdot\nabla)u\cdot v dx = \int_\Omega (u\cdot\nabla)u\cdot curl\psi dx$$
$$= \int_{\partial\Omega} (u\cdot\nabla)u\cdot\tau\psi ds \qquad (by\ (5.6.24))$$

$\forall u \in E_k, v \in E^k(\Omega, R^2)$. By Theorem 5.6.2,

$$\psi|_{\Gamma_i} = C_i, \quad 0 \leq i \leq \beta_1, \quad (\partial\Omega = \bigcup_{j=1}^{\beta_1}\Gamma_j)$$

where $C_i (0 \leq i \leq \beta_1)$ are constant, and there is a $C_j \neq 0$. Thus

$$\begin{aligned}\int_{\partial\Omega}(u\cdot\nabla)u\cdot\tau\psi ds &= \sum_{i=0}^{\beta_1} C_i \int_{\Gamma_i}(u\cdot\nabla)u\cdot\tau ds \\ &= \sum_{i=0}^{\beta_1} C_i \int_{\Gamma_i}|u|\frac{\partial}{\partial\tau}u\cdot\tau ds \qquad (by\ u_n|_{\partial\Omega} = 0) \\ &= \sum_{i=0}^{\beta_1} C_i \int_{\Gamma_i}[|u|\frac{\partial u_\tau}{\partial\tau} - |u|k(x)u_n]ds \\ &= \sum_{i=0}^{\beta_1} C_i \int_{\Gamma_i}|u|\frac{\partial u_\tau}{d\tau}ds\end{aligned}$$

where $k(x)$ is the carvature of $\partial\Omega$ at x. Usually $\frac{\partial u_\tau}{\partial\tau}|_{\partial\Omega} \neq 0$, it means that the eigen-space E_k is not invariant for the free boundary condition in general.

But, if the Betti number $\beta_1(\Omega) = 0$, i.e. Ω is homeomorphic to an open disk, or the equations (5.6.19) are defined on anulus, i.e. supplemented with the semi-periodic boundary condition (5.6.15), then the theorem of invariant eigen-spaces still holds true.

Theorem 5.6.5. For the free boundary condition, if $\beta_1(\Omega) = 0$, or the condition (5.6.15) is imposed, then the conclusions of Theorem 5.6.3 hold true.

Proof. If $\beta_1(\Omega) = 0$, then $E^k(\Omega, R^2) = \{0\}$. Hence the projections (5.6.21) and (5.6.22) are valid. When the boundary condition is (5.6.15) the harmonic space reads

$$E^k(\Omega, R^2) = \{u = (a, 0) | a \in R^1\} = R^1$$

In this case, it is obviously that the equalities (5.6.21)(5.6.22) hold. Thus, this theorem is derived.

5.6.4. Global stability

This subsection is devoted to the study of global stability for the following problems

$$\begin{cases} \frac{\partial u}{\partial t} = \mu\Delta u - (u \cdot \nabla)u - \nabla p + \lambda e_1, \quad x \in \Omega = (0, 2\pi)^2 \\ divu = 0 \\ u(x, 0) = u_0 \end{cases} \tag{5.6.28}$$

where e_1 is an eigenfunction of (5.6.18) corresponding to the first eigenvalue ρ_1, and the associated boundary condition is one of the conditions ((5.6.3),(5.6.4) and (5.6.15)).

In view of ((5.6.11)-(5.6.17)) it is clear that the eigen-spaces E_1 of (5.6.18) corresponding to ρ_1 are respectively as follows

$$\begin{aligned} &E_1 = span\{(\sin x_2, 0), (\cos x_2, 0), (0, \sin x_1), (0, \cos x_1)\}; \ for \ (5.6.3) \\ &E_1 = span\{(\sin x_1 \cos x_2, -\cos x_1 \sin x_2)\}, \ for \ (5.6.4) \\ &E_1 = span\{(\cos x_2, 0)\}, \ for \ (5.6.15) \end{aligned}$$

and the eigenvalue $\rho_1 = 1$ for (5.6.3) and (5.6.15), $\rho_1 = 2$ for (5.6.4).

We know from Theorem 5.6.3 and Theorem 5.6.5 that $v_\lambda = \lambda\mu^{-1}\rho_1^{-1}e_1 \in E_1$ is a stationary solution of (5.6.28). In the following, we shall prove that for any $\lambda \in R^1$, the stationary solution v_λ is globally asymptotically stable for (5.6.28), i.e. for any initial value $u_0 \in \mathcal{H}_B^k(\Omega, R^2)(k \geq 3)$, the solution of (5.6.28) satisfies

$$\lim_{t\to\infty} \|u(x, t, u_0) - v_\lambda\|_{H^1} = 0.$$

Theorem 5.6.6. For any $\lambda \in R^1$ the stationary solution $v_\lambda = \lambda\mu^{-1}\rho_1^{-1}e_1$ is global asymptotically stable in the H^1-norm for (5.6.28) provided the initial value $u_0 \in \mathcal{H}_B^3(\Omega, R^2)$, and locally asymptotically stable in the H^k-norm for any $k \geq 1$.

Proof. Let the eigenfucntions $\{e_k\} \subset \mathcal{H}_B^\infty(\Omega, R^2)$ of (5.6.18) be as follows

$$e_k = curl h_k = \{\frac{\partial h_k}{\partial x_2}, -\frac{\partial h}{\partial x_1}\}$$

Obviously, the functions h_k satisfy

$$-\Delta h_k = \rho_k h_k \tag{5.6.29}$$

For the boundary conditions ((5.6.3),(5.6.4) and (5.6.15)) from ((5.6.11)-(5.6.17)) we can see that $\{e_k\}$ and $\{h_k\}$ respectively constitutes the orthogonal base of $\mathcal{H}_B^k(\Omega, R^2)$ and $H_B^{k+1}(\Omega), \forall k \geq 0$. Moreover, the Hamiltonian function $\psi \in H_B^{k+1}(\Omega)$ satisfy the boundary condition

$$\begin{cases} \psi\ is\ \Omega-periodic,\ for\ (5.6.3) \\ \psi|_{\partial\Omega} = 0, \Delta\psi|_{\partial\Omega} = 0,\ for\ (5.6.4) \\ \psi\ is\ x_1-periodic,\ and\ \psi = \Delta\psi = 0\ on\ y = 0, 2\pi,\ for\ (5.6.15) \end{cases} \tag{5.6.30}$$

Let

$$u = curl\psi = \{\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\}$$

From (5.6.28) and (5.6.29) it follows that the Hamiltonian function ψ satisfies

$$\begin{cases} \frac{\partial\Delta\psi}{\partial t} = \mu\Delta^2\psi + [\psi, \Delta\psi] - \lambda\rho_1 h_1 \\ \psi(x, 0) = \psi_0 \end{cases} \tag{5.6.31}$$

where

$$[\psi, \phi] = \frac{\partial\psi}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\phi}{\partial x_1}$$

In order to prove this theorem, we only need to investigate the problems (5.6.31) with the boundary conditions (5.6.30). Let

$$\phi = \psi - \lambda\mu^{-1}\rho_1^{-1}h_1$$

where $\lambda\mu^{-1}\rho_1^{-1}h_1$ is the stationary solution of (5.6.31). Then we get

$$\begin{aligned} \frac{\partial\Delta\phi}{\partial t} &= \mu\Delta^2\phi + \lambda\mu^{-1}\rho_1^{-1}([h_1, \Delta\phi] + [\phi, \Delta h_1]) + [\phi, \Delta\phi] \\ &= \mu\Delta^2\phi + \lambda\mu^{-1}\rho_1^{-1}[h_1, \Delta\phi + \rho_1\phi] + [\phi, \Delta\phi] \end{aligned} \tag{5.6.32}$$

$$\phi(x,0)=\phi_0=\psi_0-\lambda\mu^{-1}\rho_1 h_1 \tag{5.6.33}$$

Multiplying the equation (5.6.32) by $\Delta\phi+\rho_1\phi$ and integrating it, we can derive that

$$\frac{d}{dt}\int_\Omega[|\Delta\phi|^2+\rho_1\Delta\phi\cdot\phi]dx=\mu\int_\Omega[\Delta^2\phi\cdot\Delta\phi+\rho_1|\Delta\phi|^2]dx \tag{5.6.34}$$

In the above equality, we have employed the boundary condition (5.6.30) and the following fact

$$\int_\Omega[f,g]fdx=-\int[g,f]fdx=0,\quad \forall f,g\in H^k_B(\Omega),\quad k\geq 1.$$

Because $\{h_j\}$ is a common orthogonal base of all the spaces $H^k_B(\Omega)(k\geq 0)$, as $\phi\in H^k_B(\Omega),\phi=\sum_{j=1}^\infty\phi_j h_j$, and

$$\|\phi\|_{H^k}=\int_\Omega(-1)^k\Delta^k\phi\cdot\phi dx=\sum_{j=1}^\infty\phi_j^2\rho_j^k,\quad (k\geq 0)$$

here we have made the normalization

$$\int_\Omega|h_j|^2dx=1,\quad \forall j=1,2,\cdots.$$

Noting that $\rho_j>\rho_1\geq 1,\forall j\geq m+1,m$ the multiplicity of ρ_1, from (5.6.34) it follows that

$$\frac{d}{dt}\sum_{j=m+1}^\infty\rho_j(\rho_j-\rho_1)\phi_j^2=-\mu\sum_{j=m+1}^\infty\rho_j^2(\rho_j-\rho_1)\phi_j^2$$

$$\leq -\rho_{m+1}\mu\sum_{j=m+1}^\infty\rho_j(\rho_j-\rho_1)\phi_j^2 \tag{5.6.35}$$

By the Gromwell inequality, we infer from (5.6.35) that the solution of (5.6.32)(5.6.33) satisfies

$$\sum_{j=m+1}^\infty\rho_j(\rho_j-\rho_1)\phi_j^2\leq Ce^{-\mu\rho_{m+1}t}$$

where $C=\|(\Delta^2+\rho_1\Delta)\phi_0\|_{L^2}$ is a constant. Thus, for the solutions of (5.6.32)(5.6.33) we obtain that

$$\|\phi-P\phi\|_{H^2}\leq Ce^{-\frac{1}{2}\mu\rho_{m+1}t},\quad C>0 \textit{ a constant}. \tag{5.6.36}$$

where

$$P : H_B^k(\Omega) \to \widetilde{E}_1 \textit{ the projection.}$$

$\widetilde{E}_1$ the eigen-space of (5.6.29) corresponding to ρ_1. We denote by

$$\widetilde{\phi} = \phi - P\phi$$

Then, from (5.6.32) it follows that $P\phi$ satisfies

$$\rho_1 \frac{d}{dt} P\phi = -\mu_1 \rho_1^2 P\phi - P[P\phi, \Delta\widetilde{\phi} + \rho_1 \widetilde{\phi}] -$$

$$-P[\widetilde{\phi}, \Delta\widetilde{\phi}] - \lambda\mu^{-1}\rho_1^{-1} P[h_1, \Delta\widetilde{\phi} + \rho_1\widetilde{\phi}]$$

Thus, we get that

$$P\phi = P\phi_0 e^{-\mu\rho_1 t} - \rho_1^{-1} e^{-\mu\rho_1 t} \int_0^t e^{\mu\rho_1 \tau} [P[P\phi, \Delta\widetilde{\phi} + \rho_1\widetilde{\phi}] +$$

$$+P[\widetilde{\phi}, \Delta\widetilde{\phi}] + \lambda\mu^{-1}\rho_1^{-1} P[h_1, \Delta\widetilde{\phi} + \rho_1\widetilde{\phi}]] d\tau \tag{5.6.37}$$

From (5.6.36) we can derive the estimates

$$\begin{aligned}
|P[\widetilde{\phi}, \Delta\widetilde{\phi}]| &= \left| \int_\Omega \left[\frac{\partial\widetilde{\phi}}{\partial x_1} \frac{\partial \Delta\widetilde{\phi}}{\partial x_2} - \frac{\partial\widetilde{\phi}}{\partial x_2} \frac{\partial\widetilde{\phi}}{\partial x_1} \right] h_i dx \right| \\
&= \left| \int_\Omega \left[\frac{\partial\widetilde{\phi}}{\partial x_1} \Delta\widetilde{\phi} \frac{\partial h_i}{\partial x_2} - \frac{\partial\widetilde{\phi}}{\partial x_2} \Delta\widetilde{\phi} \frac{\partial h_i}{\partial x_1} \right] dx \right| \\
&\leq C \int_\Omega |\Delta\widetilde{\phi}| \cdot |\nabla\widetilde{\phi}| dx \\
&\leq C \|\widetilde{\phi}\|_{H^2}^2 \\
&\leq C e^{-\mu\rho_{m+1} t}
\end{aligned}$$

where $h_i \in \widetilde{E}_1, (1 \leq i \leq m), C > 0$ the constant dependent on the moduli of the first eigen-functions h_i.

$$|P[h_1, \Delta\widetilde{\phi} + \rho_1\widetilde{\phi}]| \leq C e^{-\frac{1}{2}\mu\rho_{m+1} t}$$

$$|P[P\phi, \Delta\widetilde{\phi} + \rho_1\widetilde{\phi}]| \leq C e^{-\frac{1}{2}\mu\rho_{m+1} t} \|P\phi\|$$

By the estimates above, from (5.6.37) we obtain (noting that $\rho_{m+1} \geq 2\rho_1$)

$$\|P\phi\| \leq C e^{-\mu\rho_1 t} + C e^{-\mu\rho_1 t} \int_0^t \|P\phi\| d\tau \tag{5.6.38}$$

Let

$$\|P\phi\| = e^{-\mu\rho_1 t} z(t) \quad (z(t) \geq 0) \tag{5.6.39}$$

From (5.6.38) we get

$$z \leq C + C\int_0^t e^{-\mu\rho_1\tau} z(\tau)d\tau \tag{5.6.40}$$

By the Gromwell inequality, from (5.6.40) we obtain

$$z(t) \leq Ce^{-\frac{1}{\mu\rho_1}(e^{-\mu\rho_1 t}-1)} \leq const.$$

Thus, we infer from (5.6.39) that $P\phi \to 0$ in $\widetilde{E}_1$. Therefore we deduce from (5.6.36) that

$$\lim_{t\to\infty} \|\psi - \lambda\mu^{-1}\rho_1^{-1}h_1\|_{H^2} = \lim_{t\to\infty} \|\phi\|_{H^2} = 0$$

The first conclusion of this theorem is proven.

It is know that the solutions of (5.6.31)(5.6.30) are C^∞ provided the initial value $\psi_0 \in C^\infty$, and the eigenvalue of

$$\Delta^2 h_k = \rho_k^2 h_k$$

have the asymptotical property

$$\rho_k^2 \sim ck^2$$

On the other hand, the eigen-problem

$$\begin{cases} -\mu\Delta^2\phi - \lambda\mu^{-1}\rho_1^{-1}[h_1, \Delta\phi + \rho_1\phi] = \beta(\lambda)\phi \\ \phi \in H_B^k(\Omega), k \geq 4 \end{cases} \tag{5.6.41}$$

has no the eigenvalues possessing the nonnegative real parts for all $\lambda \in R^1$, otherwise, there must exist a $\lambda_0 \in R^1$ such that $\beta(\lambda_0) = 0$(because each eigenvalue of (5.6.41) is isolated, it is continuously dependent on λ(see [Ka]), and $\beta(0) = -\rho_k^2, k = 1, 2, \cdots$), then the eigenfunction $\psi_0 \neq 0$ of (5.6.4) satisfies

$$\mu\Delta^2\psi_0 + \lambda_0\rho_1^{-1}\mu^{-1}[h_1, \Delta\psi_0 + \rho_1\psi_0] = 0 \tag{5.6.42}$$

In the same fashion as above one can deduce from (5.6.42) that $\psi_0 \in \widetilde{E}_1$, and it means that $\psi_0 = 0$, a contradiction.

Hence, for the equation (5.6.32) the conditions in Theorem 3.2.6 are satisfied. Thus, the second conclusion is proven. This proof is complete.

From Theorem 5.6.6, we can deduce the local stability of (5.6.1) in a neighborhhod of $\lambda e_1, \forall \lambda \in R^1$.

Theorem 5.6.7. For any $\lambda \in R^1$, there is a neighborhhod $O \subset \mathcal{H}_B^k(\Omega, R^2)(k \geq 1)$ of λe_1 such that for any $f \in O$ the equations (5.6.1) are locally stable, i.e. the stationary solution of (5.6.1) for $f \in O$ is locally asymptotically stable.

Proof. For any $f \in O, f = \lambda e_1 + g(x)$, and

$$\|g\|_{H^k} < \epsilon, \ \ for\ some\ \epsilon > 0 \tag{5.6.43}$$

Let $g = curl\beta, \beta \in H^{k+1}(\Omega)$, and $u = curl\psi$. Then we have

$$\begin{cases} \frac{\partial \Delta\psi}{\partial t} = \mu\Delta^2\psi + [\psi, \Delta\psi] - \lambda\rho_1 h_1 + \Delta\beta \\ \psi \in H_B^{k+3}(\Omega), \psi(x,0) = \psi_0 \end{cases} \tag{5.6.44}$$

We consider the stationary equation of (5.6.44)

$$\mu\Delta^2\psi + [\psi, \Delta\psi] = \lambda\rho_1 h_1 - \Delta\beta \tag{5.6.45}$$

We define the mapping $A + G : H_B^{k+3}(\Omega) \to H^{k+1}(\Omega)(k \geq 1)$ by

$$A\psi = \mu\Delta^2\psi, \quad G\psi = [\psi, \Delta\psi]$$

By Theorem 5.6.6 we can derive that $\forall \lambda \in R^1$ the solution $v_\lambda = \lambda\mu^{-1}\rho_1^{-1}h_1$ of $A\psi + G\psi = \lambda\rho_1 h_1$ is unique, and the derivative operator

$$A + DG(v_\lambda) : H_B^{k+3}(\Omega) \to H^{k-1}(\Omega) \tag{5.6.46}$$

which is a linear completely continuous field, is a homeomorphism because it has no zero eigenvalues. Indeed, (5.6.41) and (5.6.42) are the eigenvalue equations of (5.6.46).

By the inverse function theorem, for any $\epsilon > 0$ sufficiently small there is a $\delta > 0$ such that the equation (5.6.45) has unique solution $\widetilde{\psi} = \lambda_1\rho_1^{-1}\mu^{-1}h_1 + \widetilde{\phi}$ with

$$\|\widetilde{\phi}\|_{H^{k+3}} < \delta \quad (\delta \to 0\ as\ \epsilon \to 0) \tag{5.6.47}$$

provided the condition (5.6.43) satisfied.

Let $\phi = \psi - \widetilde{\psi}$. Then the stability problem of (5.6.44) on the stationary solution $\widetilde{\psi}$ is equivalent to that of the below equation on the stationary solution $\phi = 0$

$$-\frac{\partial \Delta\phi}{\partial t} = -\mu\Delta^2\phi + \lambda\rho_1^{-1}\mu^{-1}[\Delta\phi + \rho_1\phi, h_1] +$$
$$+[\Delta\phi, \widetilde{\phi}] + [\Delta\widetilde{\phi}, \phi] + [\Delta\phi, \phi]$$

Because the eigenvalues of (5.6.41) have the negative real parts for all $\lambda \in R^1$. Hence, $\forall\lambda \in R^1$ there exists $\delta > 0$ such that if (5.6.47) is fulfilled the eigenvalue problem

$$-\mu\Delta^2\phi - \lambda\rho_1^{-1}\mu^{-1}[h_1, \Delta\phi + \rho_1\phi] + [\Delta\widetilde{\phi}, \phi]$$
$$-[\widetilde{\phi}, \Delta\phi] = \beta\phi, \quad \phi \in H_B^k(\Omega), k \geq 4$$

has no the eigenvalues having the nonnegative real parts. Therefore, by Theorem 3.2.2 we can derive this theorem. This proof is complete.

5.6.5. Taylor vortex type of the periodic structure

The invariance of eigen-spaces is related with the phenomena of the Taylor vortices. The Taylor vortices appear in the case of fluid flows contained between two rotating cylinders, which is studied originally in Taylor's 1923[Ty]. In fact, such periodic structure appears in many problems of mathematics and physics, for instance see [FP] and [BLP].

Let the domain $\Omega = (0, 2\pi)^2$. In this subsection, we shall restrict our attention to the boundary conditions ((5.6.3),(5.6.4) and (5.6.15)). The Taylor fields on Ω with the boundary conditions considered are referred to the Hamiltonian defined by

$$v = curl\psi, \psi \; satisfy \; the \; boundary \; conditions \; (5.6.30),$$

$$\psi \; take \; the \; functions \; below \tag{5.6.48}$$

$$\{\cos nx_1 \sin mx_2, \cos nx_1 \cos mx_2, \sin nx_1 \sin mx_2\}$$

The Taylor vortices are referred to the periodic structure of phase diagram of the Taylor field (5.6.48) as shown in Fig 5.15 below

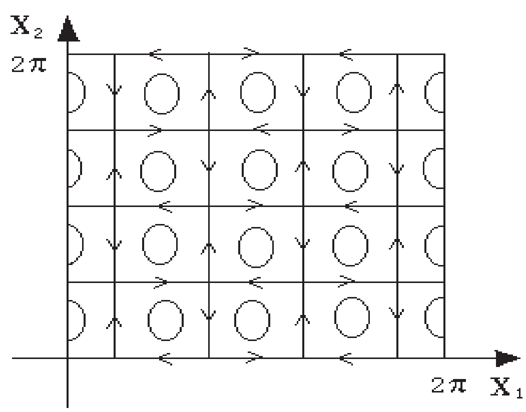


Fig. 5.15. The phase diagram of $v = (\cos 2x_1 \cos 2x_2, \sin 2x_1 \sin 2x_2)$

In the following, we hsall illustrate the applications of the previous theorems in Subsections 5.6.3-5.6.4 to the Taylor vortex type of periodic structure by some examples.

The problem is given by

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \mu \Delta u - \nabla P + f(x) \\ divu = 0 \\ u(x,0) = u_0 \end{cases} \tag{5.6.49}$$

Example 5.6.8. Let the periodic boundary condition be imposed, the initial value $u_0 = 0$, and the function f in (5.6.49) be taken by

$$f = (\sin x_1 \cos x_2, -\cos x_1 \sin x_2) \tag{5.6.50}$$

By Theorem 5.6.3, the solution of (5.6.49) reads as

$$u = \frac{1}{2\mu}(1 - e^{-2\mu t})\{\sin x_1 \cos x_2, -\cos x_1 \sin x_2\} \tag{5.6.51}$$

When the periodic eternal force (5.6.50) is exerted, the fluid motion is governed by the Taylor field (5.6.51) whose topological structure is illustrated by Fig. 5.16(a)

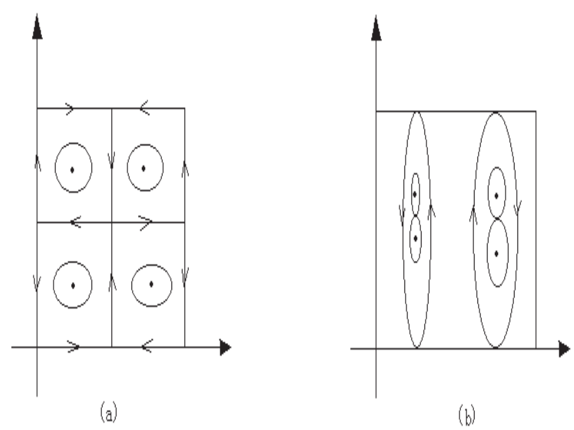


Fig. 5.16.

In [MW,1], the structure evolution of the Taylor vortices is studied. A basic result (Theorem 4.3 in [MW,1] says that there is an open and dense set

$O \subset \mathcal{H}_P^k(\Omega, R^2)$ such that for any $g \in O$, under the small perturbation $f + \epsilon g$, the topological structure of the perturbated solution u_ϵ of (5.6.49) in a time inteval $0 < t < t_0$ is topologically equivalent to that as shown in Fig.5.16(b).

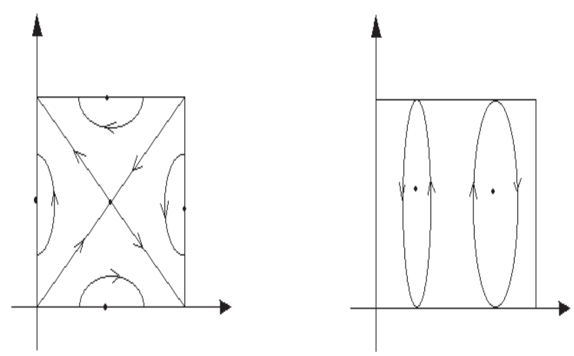
Example 5.6.9. We still consider the periodic boundary condition. The function f in (5.6.49) is given by

$$\begin{cases} f = e_1 + \epsilon g, |\epsilon| \geq 0 \; sufficiently \; small \\ e_1 = (\sin x_2, \cos x_1) \end{cases} \tag{5.6.52}$$

The function e_1 is the first eigenfunction of (5.6.18) in $\mathcal{H}_P^k(\Omega, R^2)$. By Theorem 5.6.7, the stationary solution $v_\epsilon(x)$ of (5.6.49) is locally asymptotically stable.

In [MW,1], a theorem (Theorem 6.2) says that there is an open and dense set $O \subset \mathcal{H}_P^k(\Omega, R^2)$, such that if $g \in O$, the stationary solution $v_\epsilon(x)(\epsilon \neq 0)$ of (5.6.49) with (5.6.52) is structure stable whose topological structure is topologically equivalent to that as shown in Fig. 5.17(b). Thus, we infer that there is a neighborhood $U \subset \mathcal{H}_P^k(\Omega, R^2)(k \geq 2)$ of $v_\epsilon(x)$, for any the initial value $u_0 \in U$, there exists a $T_0 \geq 0$ such that the solution $u(x, t, u_0)$ of (5.6.49) is topologically equivalent to $v_\epsilon(x)(\epsilon \neq 0)$ for any $t > T_0$.

The topological structures of v_ϵ at $\epsilon = 0$ and $\epsilon \neq 0$ are illustrated by Fig.5.17(a) and (b)



(a) structure of v_ϵ with $\epsilon = 0$ (b) structure of v_ϵ with $\epsilon \neq 0$

Fig. 5.17.

Example 5.6.10. We consider the free boundary condition (5.6.4). The function in (5.6.49) is given by

$$\begin{cases} f = e_1 + \epsilon g, |\epsilon| \geq 0 \; sufficiently \; small \\ e_1 = (\sin x_1 \cos x_2, -\cos x_1 \sin x_2) \end{cases} \tag{5.6.53}$$

The function e_1 is the first eigenfunction of (5.6.18) in $\mathcal{H}^k_F(\Omega, R^2)(k \geq 2)$, therefore the stationary solution $v_\epsilon(x)$ of (5.6.49) is locally asymptotically stable.

The generic Theorem of structure stability on stationary solution for the Navier-Stokes equations with the free boundary condition is also valid (see [MW,2]). Hence there is an open and dense set $O \subset \mathcal{H}^k_F(\Omega, R^2)$ as well a neighborhhod $U \subset \mathcal{H}^k_F(\Omega, R^2)(k \geq 2)$ of e_1, such that for any $g \in O$ and the initial value $u_0 \in U$, there exists a $T_0 \geq 0$, at each moment $t > T_0$ the solution $u(x, t, u_0)$ of (5.6.49)(5.6.53) with $|\epsilon| > 0$ sufficiently small is topologically equivalent to the vector field which has the topological structure as shown in Fig.5.16.(b).

5.6.6. Asymptotically time-periodic solutions

We consider the Navier-Stokes equations where the given functions f are time dependent

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \mu\Delta u - \nabla p + f(x,t), x \in \Omega \subset R^2 \\ divu = 0 \\ u(x, 0) = u_0 \end{cases} \tag{5.6.54}$$

which are supplemented with one of the boundary conditions ((5.6.3)-(5.6.5)).

When the eternal forces f are given to be the time-periodic functions, in general the solutions of (5.6.54) are not time-periodic. However, we now concern this problem whether or when the solutions are the asymptotically time-periodic. To this end, we first introduce

Definition 5.6.11. Let the given function $f(x, t)$ in (5.6.54) be time-periodic. We say that a solution $u(x, t, u_0)$ of (5.6.54) is asymptotically time-periodic if there exists a time periodic function $v(x, t) \in \mathcal{H}^2_B(\Omega, R^2)$ such that

$$\lim_{t \to \infty} \|u(x, t, u_0) - v(x, t)\|_{H^2} = 0$$

Ofcourse, in general, for the arbitrary time-periodic functions and the initial values u_0 the solutions $u(x, t, u_0)$ of (5.6.54) may be not asymptotically time-periodic. But under some appropriate conditions they will be. By applying the invariance theorem of eigen-spaces, we can give some examples on the existence of asymptotically time-periodic solutions. The conditions that make the invariance of eigen-spaces hold are imposed in this subsection.

Theorem 5.6.12. Let $f \in L^2((0, 2\pi), E_k)$ be a time-periodic function, $E_k \subset \mathcal{H}^2_B(\Omega, R^2)$ an eigen-space of (5.6.18). Then, for any initial value $u_0 \in E_k$, the solution of (5.6.54) is asymptotically time-periodic.

Proof. By Theorems (5.6.3 and 5.6.5), the eigen-spaces of (5.6.18) are invariant for the system (5.6.54). By the projections (5.6.21)-(5.6.22), the system (5.6.54) restricted on E_k is equivalent to

$$\left\{ \begin{array}{l} \frac{dx_i}{dt} = -\mu\rho_k x_i + f_i(t), \quad 1 \leq i \leq m = dimE_k \\ x_i(0) = \alpha_i \end{array} \right. \tag{5.6.55}$$

where m is the multiplicity of the eigenvalue ρ_k,

$$f(x,t) = \sum_{i=1}^{m} f_i(t) e_{k_i}$$
$$u_0(x) = \sum_{i=1}^{m} \alpha_i e_{k_i}$$
$$u(x,t) = \sum_{i=1}^{m} x_i(t) e_{k_i}$$

$\{e_{k_1}, \cdots, e_{k_m}\}$ is the orthogonal base of E_k.

From (5.6.55) we can obtain

$$x_i(t) = \alpha_i e^{-\mu\rho_k t} + e^{-\mu\rho_k t} \int_0^t e^{\mu\rho_k \tau} f_i(\tau) d\tau$$

Because $f(x,t)$ is t-periodic, we have the below Fourier expansion:

$$f_i(t) = C_i + \sum_{m=1}^{\infty} [a_{im} \cos mt + b_{im} \sin mt]$$

$$\sum_{m=1}^{\infty} [a_{im}^2 + b_{im}^2] < \infty$$

We see that

$$\begin{aligned} & e^{-\mu\rho_k t} \int_0^t e^{\mu\rho_k \tau} \sin m\tau d\tau \\ = & \frac{1}{m}(1 + \frac{\mu^2\rho_k^2}{m^2})^{-1} [e^{-\mu\rho_k t} - \cos mt + \frac{\mu\rho_k}{m} \sin mt] \\ & e^{-\mu\rho_k t} \int_0^t e^{\mu\rho_k \tau} \cos m\tau d\tau \\ = & \frac{1}{m}(1 + \frac{\mu^2\rho_k^2}{m^2})^{-1} [-\frac{\mu\rho_k}{m} e^{-\mu\rho_k t} + \sin mt + \frac{\mu\rho_k}{m} \cos mt] \end{aligned}$$

From the equalities above it is easy to see that the solution of (5.6.55) is asymptotically time-periodic. The proof is complete.

5.6.7. Open problems on the stability and bifurcation

We wonder whether the following two theorems are true or not.

Theorem 5.6.13. For any $f \in C(R^1, \mathcal{H}^k)(k \geq 1)$, there two parameter values $-\infty \leq \lambda_1 < \infty$ and $0 < \lambda_2 \leq \infty$, such that the following assertions hold.

1). For any $\lambda_1 < \lambda < \lambda_2$, the stationary solution $v_\lambda \in \mathcal{H}_B^{k+2}(\Omega, R^2)$ of (5.6.1) is locally asymptotically stable in H^1-norm.

2). If $|\lambda_i| < \infty (i = 1, \text{or} = 2)$, then the system (5.6.1) must bifurcate from $(v_{\lambda_i}, \lambda_i)$ an invariant set.

Theorem 5.6.14. There exists an open and dense set $\mathcal{F} \subset C(R^1, \mathcal{H}^k)(k \geq 1)$, for any $f \in \mathcal{F}$, there are two parameter values $-\infty < \lambda_1 < 0$ and $0 < \lambda_2 \leq \infty$, such that the following assertions hold.

1). For any $\lambda_1 < \lambda < \lambda_2$, the stationary solution $v_\lambda \in \mathcal{H}_B^{k+2}(\Omega, R^2)$ of (5.6.1) is locally asymptotically stable in H^1-norm.

2). For the free and the Dirichlet boundary conditions, if $|\lambda_i| < \infty (i = 1$, or $= 2)$, then the system (5.6.1) must bifurcate from $(v_{\lambda_i}, \lambda_i)$ either a stationary solution or a periodic orbit.

3). For the periodic and the semi-periodic boundary conditions, if $|\lambda_i| < \infty$, then the system (5.6.1) bifurcates from $(v_{\lambda_i}, \lambda_i)$ a S^1-invariant set.

Bibliographical Notes and Remarks

This book is a summary of the author's works(partly with Q. Yu) on the subjects of nonlinear operator theory, bifurcation theory and partial differential equations. Most of the results were completed recently by the author, part results, which are collected in Section 1.5-1.6 and Section 4.3-4.4, are referred to T. Ma and Q. Yu[MY,1-4], T. Ma[Ma,1-2], Q. Yu and T. Ma[YM,1-2], and the contents in Section 4.5 are referred to T. Ma[Ma, 3-4], Q. Yu, Q. Yu and T. Ma[YM, 3-5]. Especially, the author would like to mention that the results on the local uniqueness of bifurcated branch of positive solutions for the nonlinear second order elliptic equations in Subsection 4.5.1-4.5.2([Ma,3-4]) were completed under the supervision by Prof. P. L. Lions during the author visited CEREMADA, University Paris 9, in 1990-1991.

We believe that some results in this book may be covered by the other works which we don't know, and thus they are not mentioned here. We would like to give some references closely related with the material presented.

Chapter I. The results introduced in Section 1.1 are classical, they can be found in many books and Texts. The remarkable references are partly the books of R. Temam [Te], S. Chow and J. K. Hale[CH], D. Gilbarg and N. S. Trudinger[GT], T. Aubin[Au] etc.

There are plentiful studies on the theories of abstract operatos on Banach spaces(e.g. the fixed point theory, the monotone operator theory, the variational methods, the operator semigroups etc.) and their applications to the partial differential equations. Some good references related with the material given in Section 1.2 are referred to E. Zeidler[Ze], M. A. Krasnoselskii[Kr], F. E. Browder[Bd,1,2], H. Brezis[Bz], M. Struwe[St], A. Pazy[Pa].

Some works related with the material in Section 1.5 are referred to O. A. Ladyzenskaja and N. N. Uralceva[LU], M. Giaquinta[Ma,1,2].

On the Keldys-Fichera boundary value problem for the linear equations with nonnegative characteristic form of second order, we refer to G. Fichera[Fi], M. V. Keldys[Ke], O. A. Cleinik and E. V. Radkevich[OR], J. Kohn and L. Nirenberg[KN].

Chapter II. There are lots of works on the global existence and regularity of the initial boundary value problems of quasi-linear and semilinear parabolic equations and systems, we briefly refer to O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Uralceva[LSU], J. L. Lions[Li], A. Haraux[Ha], A. Friedman[Fr]. On the initial boundary value problems of nonlinear wave equations we refer to K. Jorgens[Jr], D. H. Sattinger[Sa,1,2], R. Temam[Te].

Chapter III. On the eigenvalue problem of linearized operators we refer to the excellent monographs of T. Kato[Ka], N. Dunford and J. T. Schwartz[DS].

Chapter IV. A detail introduction on the Liapnov-Schmidt method is given by S. Chow and J. K. Hale[CH], W. Cheng[Ch]. The global bifurcation theorem(Theorem 4.1.2) can be found in P. Rabinowitz[Ra]. Theorem 4.1.5 is due to M. A. Krasnoselskii[Kr].

Chapter V. The Kaldor's model on the business cycle can be found in G. Gabisch and H. Lorenz[GL]. On the Hopf bifurcation and the center manifold theorem of nonlinear operator defined on Hilbert spaces we refer to G. Iooss and D. D. Joseph[IJ]. In addition, we would like to mention the works of M. Golubitsky and D. G. Schaeffer[GS], D. H. Sattinger[Sa, 3-4], V. I. Iudovich[Iu,1-2].

We need to point that the results in Section 5.3-5.4 follow the idea of T. Ma and C. Zhong[MZ], although there some errors in the note.

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