Chapter II
Global Solutions and Uniform Boundedness of Nonlinear Evolution Equations

In this chapter, we present a set of abstract theorems to deal with the existence, regularity and uniform bounded-ness of global solutions for a class of nonlinear evolution equations. This theory can be regarded as a correspondence of acute angle principle of inner product operators to the nonlinear evolution equations . Applying the abstract results, we have dissussed the global solvibility, regularity and uniform boundedness for the initial boundary value problems of a class of nonlinear and fully nonlinear parabolic and hyperbolic equations (systems), in addition, we have also discussed some nonlinear evolution equations which are related with physics and mechanics.

2.1. Global Existence of General Nonlinear Evolution Equations of the First Order in Time

2.1.1. Preliminaries

Let X, Y be Banach spaces, H be a Hilbert space. $X, Y \hookrightarrow H$, and $G : R^+ \times X \to Y^*$ be a mapiing, $R^+ = [0, \infty)$.

We consider the abstract evolution equations

$$\begin{cases} \frac{du}{dt} + Gu = 0, & t > 0 \\ u(0) = \phi \end{cases} \tag{2.1.1}$$

Let $|\cdot|_i (1 \leq i \leq m, m \geq 1)$ be the seminorms on X, and

$$\|\cdot\|_X = \sum_{i=1}^{m} |\cdot|_i,$$

For $p = \{p_1, \cdots, p_m\} \geq 1$, we denote by

$$L^p((0,T),X) = \{u(t) \in X|\ a.e.\ 0 \leq t < T \leq \infty,\ and$$
$$\int_0^T \|u\|_X^p dt = \int_0^T \sum_{i=1}^m |u|_i^{p_i} dt < \infty\}$$

with the norm

$$\|u\| = \sum_{i=1}^m [\int_0^T |u|_i^{p_i} dt]^{\frac{1}{p_i}}.$$

where $p \geq 1$ means that $\forall i (1 \leq i \leq m), p_i \geq 1$.

Definition 2.1.1. Let $\phi \in H$, and $p \geq 1$ be some constant. $u \in L^\infty_{loc}((0,\infty),H) \cap L^p_{loc}((0,\infty),X)$ is called a global solution of (2.1.1), if for any $v \in Y, u$ satisfies

$$< u(t), v >_H + \int_0^t < Gu, v > dt = < \phi, v >_H \quad a.e.\ t \geq 0.$$

Definition 2.1.2. The solutions $u(t,\phi)$ of (2.1.1) are called to be uniformly bounded in some space $Z \subset H$, if for any bounded subset $\Sigma \subset Z$, there is a constant $c > 0$ such that

$$\|u(t,\phi)\|_Z \leq c, \quad \forall \phi \in \Sigma,\ and\ t \geq 0.$$

Definition 2.1.3. Let $u_n, u_0 \in L^p((0,T),X)$. We say that $u_n \rightharpoonup u_0$ uniformly in $L^p((0,T),X)$, if $\{u_n\}$ is bounded in $L^\infty((0,T),H)$, and

$$\left\{\begin{array}{l} u_n \rightharpoonup u_0\ in\ L^p((0,T),X),\ and \\ \lim_{n\to\infty} \int_0^T | < u_n - u_0, v >_H |^2 \cdot dt = 0, \quad \forall v \in H. \end{array}\right. \tag{2.1.2}$$

Definition 2.1.4. A mapping $G : R^+ \times X \to Y^*$ is p-weakly continuous if $\forall u_n \rightharpoonup u_0$ uniformly in $L^p((0,T),X), p = \{p_1, \cdots, p_m\}, 0 < T < \infty$, we have

$$\lim_{n\to\infty} \int_0^T < Gu_n, v > dt = \int_0^T < Gu_0, v > dt, \quad \forall v \in Y, T < \infty.$$

Definition 2.1.5. Let $L : X \to Y$ be a linear mapping (as $X = Y, L = id$) and

$$< u, Lv >_H = < u, v >_{H_1} .$$

A mapping $G : R^+ \times X \to Y^*$ is called to be p-coerceivly continuous, if

i) . G takes each bounded subset $\sum \subset L^p((0,T), X) \cap L^\infty([0,T), H_1)$ to a bounded subset $G(\sum)$ in $L^{p'}((0,T), Y^*), (\frac{1}{p'} + \frac{1}{p} = 1), \forall 0 < T < \infty$;

ii) . as $u_n \rightharpoonup u_0$ uniformly in $L^p((0,T), X)$, $\{u_n\}$ is bounded in $L^\infty((0,T), H_1)$ and

$$\lim_{n\to\infty} \int_0^t < Gu_n - Gu_0, Lu_n - Lu_0 > dt = 0$$

then we have

$$\lim_{n\to\infty} \int_0^t < Gu_n, v > dt = \int_0^t < Gu_0, v > dt, \quad \forall v \in Y.$$

In the following, we introduce two lemmas which are useful for the later discussion.

Lemma 2.1.6([LSU]). Let $u \in L^p_{loc}((-\infty, \infty), X)(P \geq 1)$. Denote by

$$u_h = \frac{1}{h} \int_t^{t+h} u(\tau) d\tau, \quad 0 < |h| < 1,$$

Then $\forall 0 < T < \infty, \{u_h\} \subset W^{1,p}((-T,T), X)$, and

$$\lim_{h\to 0} \int_{-T}^{T} \|u_h - u\|_X^p dt = 0$$

$$\lim_{h\to 0} \|u_h(t) - u(t)\|_X = 0, \ a.e. \ t \in (-\infty, \infty).$$

Lemma 2.1.7. Let H_1, H be Hilbert spaces, and $H_1 \hookrightarrow H$ be dense and compact. Then there exists a normal orthogonal base $\{e_i\} \subset H$, such that $\{e_i\}$ is also an orthogonal base of H_1.

Proof. Let $I : H_1 \to H$ be the embeding mapping. By the assumption, I is a compact operator. We define a mapping $A : H_1 \to H_1$ by

$$< Au, v >_{H_1} = < Iu, v >_H = < u, v >_H, \quad \forall v \in H_1.$$

Obviously, $A : H_1 \to H_1$ is a linear compact operator, which is symmetry and positive definite. Hence there is a sequence of eigenvalues of A

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

and a sequence of eigenvectors $\{\widetilde{e}_i\} \subset H_1$ with

$$\widetilde{e}_i = \lambda_i A\widetilde{e}_i, \quad i = 1, 2, \cdots.$$

and $\{\widetilde{e}_i\}$ is an orthogonal base of H_1. Thus

$$< \widetilde{e}_i, \widetilde{e}_j >_H = < A\widetilde{e}_i, \widetilde{e}_j >_{H_1} = < \lambda_i^{-1}\widetilde{e}_i, \widetilde{e}_j >_{H_1} = 0, \quad i \ne j$$

Hence $\{\widetilde{e}_i\}$ is an orthogonal sequence in H. Since $I : H_1 \to H$ is a dense embedding, $\{\widetilde{e}_i\}$ is an orthogonal base of H. Let $\{\widetilde{e}_i\}$ be normalized

$$e_i = \widetilde{e}_i / \|\widetilde{e}_i\|_H,$$

then $\{e_i\}$ is a normal orthogonal base of H. The proof is complete.

2.1.2 Upward weakly continuous operators

Let X be a linear space, X_1, X_2, H are the completion of X respectively with their norm. X_1, X_2 are separable and reflexive Banach spaces, H be a Hilbert space, $X_1, X_2 \hookrightarrow H$.

Let $G : R^+ \times X_1 \to X_2^*$ be a mapping. We make the following assumptions.

(A_1) . There is a $P = \{P_1, \cdots, P_m\} > 1$, such that

$$\int_0^t < Gu, u > dt \ge C_1 \int_0^t \|u\|_{X_1}^P dt - C_2 \int_0^t (\|u\|_H^2 + 1) dt$$

where $C_1, C_2 > 0$ are constants.

(A_2) . There is a number $\alpha (0 < \alpha < 1), \forall 0 < h < 1$ and $u \in C^1([0, \infty), X)$, we have

$$|\int_t^{t+h} < Gu, v > dt| \le ch^\alpha, \quad \forall v \in X, 0 \le t < T,$$

where $C > 0$ is only dependent on $T, \|v\|_{X_2}$ and $\int_0^T \|u\|_{X_1}^P dt$.

Theorem 2.1.8. Suppose that $G : R^+ \times X_1 \to X_2^*$ is p-weakly continuous which satisfies (A_1) and (A_2). Then the problem (2.1.1) has a global solution $u \in L^\infty_{loc}((0, \infty), H) \cap L^P_{loc}((0, \infty), X_1)$.

Proof. Because X_2, H are separable, and $X \subset X_2 \hookrightarrow H$ are dense, there exists a sequence $\{e_n\} \subset X$ which is a normal orthogonal base of H, and dense in X_2. Denote by

$$\begin{cases} X_n = \{\sum_{i=1}^n \alpha_i e_i | \quad \alpha_i \in R\} \\ \widetilde{X}_n = \{\sum_{i=1}^n \beta_i(t) e_i | \quad \beta_i \in C^1[0,\infty)\}. \end{cases} \tag{2.1.3}$$

We proceed the proof by using the Galerkin procedure. Consider the below ordinary differential equations

$$\begin{cases} \frac{dx_i}{dt} + < Gu_n, e_i >= 0, \quad 1 \le i \le n \\ x_i(0) =< \phi, e_i >_H \end{cases} \tag{2.1.4}$$

where $u_n = \sum_{i=1}^n x_i(t) e_i$. By the theory of ordinary differential equations, the equations (2.1.4) has local solutions

$$\{x_1(t), \cdots, x_n(t)\}, \quad 0 \le t < \tau$$

By (2.1.4) we have

$$< u_n(t), v >_H + \int_0^t < Gu_n, v > dt =< \phi, v >_H, \quad \forall v \in X_n \tag{2.1.5}$$

$$\int_0^t [< \frac{du_n}{dt}, v >_H + < Gu_n, v >] dt = 0, \quad \forall v \in \widetilde{X}_n \tag{2.1.6}$$

Replacing v by u_n in (2.1.6), one reads

$$\int_0^t < Gu_n, u_n > dt + \frac{1}{2} < u_n(t), u_n(t) >_H = \frac{1}{2} < \phi_n, \phi_n >_H$$

$$\phi_n = \sum_{i=1}^n < \phi, e_i >_H e_i$$

. From (A_1) we obtain

$$\|u_n\|_H^2 \le \|\phi\|_H^2 + 2C_2 \int_0^t \|u_n\|_H^2 dt + 2C_2 t \tag{2.1.7}$$

$$C_1 \int_0^t \|u_n\|_{X_1}^P dt \le \|\phi\|_H^2 + C_2 \int_0^t \|u_n\|_H^2 dt + C_2 t \tag{2.1.8}$$

By the Gronwell inequality, from (2.1.7) and (2.1.8) it follows

$$\|u_n\|_H^2 \le \|\phi\|_H^2 e^{2C_2 t} + 2C_2 e^{2C_2 t} \tag{2.1.9}$$

$$(2.1.10) \qquad C_1\int_0^t \|u_n\|_{X_1}^P dt \le \|\phi\|_H^2(1+e^{2C_2t})+Ce^{2C_2t}+Ct.$$

which implies that the solutions of (2.1.4) can be extended to $[0,\infty)$, and $\{u_n(t)\}$ is bounded in $L^\infty((0,T),H)\cap L^p((0,T),X_1), \forall 0<T<\infty$. It remains to verify that the sequence $\{u_n\}$ is uniformly weakly convergent to u_0 in $L^p((0,T),X_1)$. Let

$$u_n \rightharpoonup u_0 \ in \ L^p((0,T),X_1)$$

$$u_n \rightharpoonup^* u_0 \ in \ L^\infty((0,T),H)$$

Denote by

$$u_n=\sum_{i=1}^n x_i^n(t)e_i$$

$$u_0=\sum_{j=1}^\infty x_j^0(t)e_j$$

From (2.1.4) and (A_2), one derives

$$(2.1.11) \qquad |x_i^n(t+h)-x_i^n(t)|\le \left|\int_t^{t+h} <Gu_n,e_i>dt\right|$$

$$\le Ch^\alpha, \quad 0<h,\alpha<1, \quad t<\infty$$

According to (A_2) and (2.1.10), the number $C>0$ in (2.1.11) is independent of n. Consequently, for each fixed number $i, \{x_i^n(t)\}$ is uniformly bounded and equicontinuous in $[0,T]$. By the Arzera-Ascoli theorem, we have

$$(2.1.12) \qquad \lim_{n\to\infty}\sup_{0\le t\le T} <u_n-u_0,v>=0, \quad \forall v\in\cup_{n=1}^\infty X_n$$

Because $\cup_{n=1}^\infty X_n$ is dense in H, the equality (2.1.12) holds true for all $v\in H$, which implies, by the dominated convergence theorem, that (2.1.2) is fulfilled.

By the p-weak continuity of $G: R^+\times X_1\to X_2^*$, from (2.1.5) it follow that

$$<u_0(t),v>_H+\int_0^t <Gu_0,v>dt=<\phi,v>_H, \quad \forall v\in X_n$$

which implies, due to $\cup_{n=1}^\infty X_n$ being dense in X_2, that $u_0\in L^\infty_{loc}((0,\infty),H)\cap L^p_{loc}((0,\infty),X_1)$ is a solution of (2.1.1). The proof is complete.

The upward operators deal with the existence of weak solutions, but if $G: R^+\times X_1\to X_2^*$ is differentiable, then with some conditions, we can obtain a regularity result as follows.

Let there be the space sequence

$$X \subset H_1 \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow H$$

where X, X_1, H are as in Theorem 2.1.8, and X_3 is a Banach space, H_1 is a Hilbert space, $H_1 \hookrightarrow H$ is compact.

Theorem 2.1.9. Under the hypotheses of Theorem 2.1.7, if $G : R^+ \times X_1 \to X_2^*$ is differentiable, which satisfies

$$< DG(u)v, v > \geq -c\|v\|_H^2, \quad c \geq 0, \quad \forall u, v \in X. \tag{2.1.13}$$

$$\begin{cases} | < Gu, v > | \leq \frac{1}{2}\|v\|_H^2 + g(u) \\ g : X_3 \to R^+ \ \textit{is a bounded continuous functional} \end{cases} \tag{2.1.14}$$

then for any $\phi \in X_3$ the solution $u \in L^\infty_{loc}((0,\infty), H) \cap L^p_{loc}((0,\infty), X_1)$ of (2.1.1) is unique which has the regularity $u \in W^{1,\infty}_{loc}((0,\infty), H)$.

Proof. The sequence $\{u_n\}$ which satisfy (2.1.5)-(2.1.10) also fulfile, by (2.1.4), that

$$\int_0^t [< \frac{d^2u_n}{dt^2}, v >_H + < DG(u_n)\frac{du_n}{dt}, v >]dt = 0, \tag{2.1.15}$$

$\forall v \in \widetilde{X}_n$. Replacing v by $\frac{du_n}{dt}$ in (2.1.15) it follows

$$\begin{aligned} \frac{1}{2} < \frac{du_n}{dt}, \frac{du_n}{dt} >_H + \int_0^t < DG(u_n)\frac{du_n}{dt}, \frac{du_n}{dt} > dt \\ = \frac{1}{2} < \frac{du_n(0)}{dt}, \frac{du_n(0)}{dt} >_H . \end{aligned}$$

According to (2.1.13) we have

$$\frac{1}{2}\|\frac{du_n(t)}{dt}\|_H^2 \leq \frac{1}{2}\|\frac{du_n(0)}{dt}\|_H^2 + C\int_0^t \|\frac{du_n}{dt}\|_H^2 dt \tag{2.1.16}$$

On the other hand

$$< \frac{du_n(0)}{dt}, v >_H + < Gu_n(0), v > = 0, \quad \forall v \in X_n \tag{2.1.17}$$

Putting $v = \frac{du_n(0)}{dt}$ in (2.1.17), from (2.1.14) it follows

$$\begin{aligned} \|\frac{du_n(0)}{dt}\|_H^2 &\leq | < G\phi_n, \frac{du_n(0)}{dt} > | \\ &\leq \frac{1}{2}\|\frac{du_n(0)}{dt}\|_H^2 + g(\phi_n) \end{aligned}$$

Hence from (2.1.16) we get

$$\frac{1}{2}\|\frac{du_n(0)}{dt}\|_H^2 \leq g(\phi_n) + C\int_0^t \|\frac{du_n}{dt}\|_H^2 dt \tag{2.1.18}$$

By Lemma 2.1.7, we take the normal orthogonal base $\{e_n\}$ of H is also an orthogonal base of H_1.Because the boundedness of $\{\|\phi_n\|_{X_3}\}$ can not be directly obtained from the space X_3, we first take $\phi \in H_1$. Since $\phi_n \to \phi$ in H_1 and $H_1 \hookrightarrow X_3$, we have

$$\phi_n \to \phi \ in \ X_3 \tag{2.1.19}$$

By the Gronwell inequality, from (2.1.18)(2.1.19) and (2.1.9)(2.1.10) it follows that

$$\{u_n\} \subset W^{1,\infty}((0,T),H) \cap L^p((0,T),X_1)$$

is bounded $\forall 0 < T < \infty$. Let

$$u_n \rightharpoonup^* u_0 \ in \ W^{1,\infty}((0,T),H)$$

$$u_n \rightharpoonup u_0 \ in \ L^p((0,T),X_1)$$

Then, as the proof of Theorem 2.1.8, $u_0 \in W^{1,\infty}_{loc}((0,\infty),H) \cap L^p_{loc}((0,\infty),X_1)$ is a solution of (2.1.1). And, from (2.1.18) (2.1.19) and (2.1.10), we can get the estimate

$$\|\frac{du_0}{dt}\|_H^2 + \int_0^t \|u_0\|_{X_1}^p dt \tag{2.1.20}$$

$$\leq C[1 + g(\phi) + g(\phi)e^{Ct} + \|\phi\|_H^2 e^{Ct}]$$

Now we take $\phi \in X_3$. As $H_1 \hookrightarrow X_3$ is dense (as X is their common dense subspace), there is a sequence $\{\psi_n\} \subset H_1$ such that $\psi_n \to \phi$ in X_3. Owing to the estemate (2.1.20), the solutions $\{u(t,\psi_n)\}$ of (2.1.1) are bounded and weakly (*) convergent to $u(t,\phi)$ in $W^{1,\infty}((0,T),H) \cap L^p((0,T),X_2), \forall 0 < T < \infty$. In the same fashion as used in Theorem 2.1.8, one can derive that $u(t,\phi) \in W^{1,\infty}_{loc}((0,\infty),H) \cap L^p_{loc}((0,\infty),X_1)$ is a solution of (2.1.1).

Next we verify the uniqueness. It is easy to see that

$$u \in W^{1,\infty}_{loc}((0,\infty),H) \Rightarrow u \in c([0,\infty),H)$$

Assume that there are two solutions $u_1, u_2 \in c([0,\infty),H)$, then $u_0 = u_2 - u_1$ satisfies

$$\int_0^t [< \frac{du_0}{dt}, v >_H + < G(u_2) - G(u_1), v >] dt = 0$$

$$u_0(0) = 0$$

$\forall v \in X_2$. Thanks to the mean value theorem

$$< G(u_2) - G(u_1), v >= \int_0^1 < DG(su_2 + (1-s)u_1)u_0, v > ds$$

Hence we have

$$\int_0^t [< \frac{du_0}{dt}, v >_H + \int_0^1 < DG(su_2 + (1-s)u_1)u_0, v > ds]dt = 0 \tag{2.1.21}$$

Putting $v = u_0$ in (2.1.21), from (2.1.13) one can derive that

$$\frac{1}{2}\|u_0(t)\|_H^2 \leq c \int_0^t \|u_0\|_H^2 dt$$

which implies that, by the Gronwell inequality, that $u_0 = 0$ in $H \forall t \geq 0$. The proof is complete.

2.1.3 Downward weakly continuous operators

Let there be the embedding space sequence as to write

$$X \subset X_2 \hookrightarrow X_1 \hookrightarrow H \tag{2.1.22}$$

$$X_2 \hookrightarrow H_2 \hookrightarrow H \tag{2.1.23}$$

where X is a linear space, and X_2, X_1, H_2, H are the completion of X respectively with their norm. In addition, X_2, X_1 are separable and reflexive Banach spaces, H_2, H are Hilbert spaces.

Suppose that $H_2 \hookrightarrow H$ is compact, and there exists an one to one dense linear mapping $L_2 : X \to X_1$ which satisfies

$$< u, L_2 v >_H = < u, v >_{H_2}, \quad \forall u, v \in X. \tag{2.1.24}$$

and there is a sequence $\{e_n\} \subset X$ such that span$\{e_n\}$ is dense in X_1, and

$$L_2 e_n = \lambda_n e_n, \quad \lambda_n \neq 0 \tag{2.1.25}$$

moreover, $\{e_n\}$ is also a normal orthogonal base of H.

For the mapping $G : R^+ \times X_2 \to X_1^*$, we have the following global existence theorem.

Theorem 2.1.10. Let the conditions (2.1.22)-(2.1.25) be fulfilled. If there is a $p = \{p_1, \cdots, p_m\} > 1$ such that

$$\int_0^t < Gu, L_2 u > dt \geq c_1 \int_0^t \|u\|_{X_2}^p dt - c_2 \int_0^t [\|u\|_{H_2}^2 + 1]dt \tag{2.1.26}$$

$\forall u \in X$, and $G : R^+ \times X_2 \to X_1^*$ is p-weakly continuous, then for any $\phi \in H_2$ the equation (2.1.1) has a global solution

$$u \in L^\infty_{loc}((0,\infty), H_2) \cap L^p_{loc}((0,\infty), X_2).$$

Proof. Take $\{e_n\}$ as in (2.1.25), and define X_n and $\widetilde{X}_n$ as (2.1.3). It is clear that $L_2\widetilde{X}_n = \widetilde{X}_n$. As the proof o f Theorem 2.1.8, there exist $u_n \in c^1([0,\infty), X)$ which satisfy (2.1.5) and (2.1.6). Inserting $v = L_2 u_n$ in (2.1.6), and by (2.1.24) one derives that

$$\int_0^t < Gu_n, L_2 u_n > dt + \frac{1}{2} < u_n(t), u_n(t) >_{H_2} = \frac{1}{2} < \phi_n, \phi_n >_{H_2}$$

From (2.1.26) and the Gronwell inequality we can get

$$\|u_n\|^2_{H_2} + \int_0^t \|u\|^p_{X_2} dt \le c[1 + e^{ct} + \|\phi\|^2_{H_2} e^{ct}].$$

Hence $\{u_n\} \subset L^\infty((0,T), H_2) \cap L^p((0,T), X_2)$ is bounded for any $0 < T < \infty$. Let u_n weakly (*) converges to u_0 in $L^\infty((0,T), H_2) \cap L^p((0,T), X_2)$. Because $H_2 \hookrightarrow H$ is compact,

$$u_n(t) \to u_0(t) \ in\ H\ a.e.\ t \ge 0$$

By the dominated convergence theorem, it follows that $u_n \rightharpoonup u_0$ uniformly in $L^p((0,T), X_2), \forall 0 < T < \infty$. In the same fashion as used in Theorem 2.1.8, we can derive that $u_0 \in L^\infty_{loc}((0,\infty), H_2) \cap L^p_{loc}((0,\infty), X_2)$ is a solution of (2.1.1). The proof is complete.

Now, we shall discuss the further regularity of global solutions of (2.1.1). To this end, it is necessary to augment the space sequences (2.1.22) and (2.1.23) to the following sequences

$$X \subset H_3 \hookrightarrow X_3 \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow H \tag{2.1.27}$$

$$X_2 \hookrightarrow H_2 \hookrightarrow H_1 \hookrightarrow H. \tag{2.1.28}$$

where X, X_2, X_1, H_2 and H are as in (2.1.22) and (2.1.23), and X_3 is a Banach space, H_3, H_1 are Hilbert spaces.

Besides the linear mapping $L_2 : X \to X_1$ which satisfies (2.1.24) and (2.1.25), we assume that the sequence $\{e_n\}$ in (2.1.25) is also an orthogonal base of H_3, and there exists a linear mappings

$$L_1 : H_2 \to H$$

which satisfies

$$\begin{cases} < u, L_1 v >_H = < u, v >_{H_1}, & \forall u, v \in X \\ L_1 e_i = \mu_i e_i, & \mu_i \neq 0 \end{cases} \tag{2.1.29}$$

In addition, we assume that $G : R^+ \times X_2 \to X_1^*$ is differentiable, and $\forall u, v \in X$

$$< DG(u)v, L_1 v > \geq c_1 \|v\|_{H_2}^2 - c_2 \|v\|_{H_1}^2 \tag{2.1.30}$$

$$\begin{cases} | < Gu, L_1 v > | \leq \frac{1}{2}\|v\|_{H_1}^2 + g(u) \\ g : X_3 \to R^+ \ is\ a\ bounded\ continuous\ functional \end{cases} \tag{2.1.31}$$

Theorem 2.1.11. Under the hypotheses of Theorem 2.1.10, if the conditions (2.1.27)-(2.1.31) are satisfied, then for any $\phi \in X_3$, the solution of (2.1.1) is unique, and which has the further regularity

$$u_t \in L^\infty_{loc}((0, \infty), H_1) \cap L^2_{loc}((0, \infty), H_2).$$

Proof. We know that the sequence $\{u_n\}$ in the proof of Theorem 2.1.10 also satisfies (2.1.15). By (2.1.29), $L_1 \widetilde{X}_n = \widetilde{X}_n$, and inserting $v = L_1 \frac{du_n}{dt}$ in (2.1.15) it follows

$$\frac{1}{2}\|\frac{du_n}{dt}\|_{H_1}^2 + \int_0^t < DG(u_n)\frac{du_n}{dt}, L_1 \frac{du_n}{dt} > dt$$
$$= \frac{1}{2}\|\frac{du_n(0)}{dt}\|_{H_1}^2.$$

Due to (2.1.30) we get

$$\frac{1}{2}\|\frac{du_n(t)}{dt}\|_{H_1}^2 + c_1 \int_0^t \|\frac{du_n}{dt}\|_{H_2}^2 dt$$
$$\leq \frac{1}{2}\|\frac{du_n(0)}{dt}\|_{H_1}^2 + c_2 \int_0^t \|\frac{du_n}{dt}\|_{H_1}^2 dt$$

Because u_n satisfies (2.1.17), and by (2.1.31) one can get

$$\frac{1}{2}\|\frac{du_n(0)}{dt}\|_{H_1}^2 \leq g(\phi_n)$$

Hence we obtain

$$\|\frac{du_n}{dt}\|_{H_1}^2 + \int_0^t \|\frac{du_n}{dt}\|_{H_2}^2 dt \leq c[1 + g(\phi_n) + g(\phi_n)e^{ct}].$$

The remainder proof is similar to that of Theorem 2.1.9. The proof is complete.

2.1.4 Coerceively continuous operators

We take the sequence (2.1.27) and (2.1.28) as the space framework to discuss (2.1.1).

Suppose that $X_2 \hookrightarrow H_1$ is compact, and there is an linear mapping $L : X_2 \to X_1$ which is one to one, bounded and dense. Moreover, L satisfies

$$< u, Lv >_H = < u, v >_{H_1}, \quad \forall u, v \in X_2 \tag{2.1.32}$$

and there exists a sequence $\{e_n\} \subset X$ such that span$\{e_n\}$ is dense in X_2, and

$$Le_n = \lambda_n e_n, \quad \lambda_n \neq 0 \tag{2.1.33}$$

furthermore, $\{e_n\}$ is a normal orthogonal base of H.

We are now in a position to state and prove the global existence and regularity theorems for the coerceivly continuous operators $G : R^+ \times X_2 \to X_1^*$.

Theorem 2.1.12. Let the conditions (2.1.32) and (2.1.33) be fulfilled. If there is a $p = \{p_1, \cdots, p_m\} > 1$ such that

$$\int_0^t < Gu, Lu > dt \geq c_1 \int_0^t \|u\|_{X_2}^p dt - c_2 \int_0^t [\|u\|_{H_1}^2 + 1] dt \tag{2.1.34}$$

and $G : R^+ \times X_2 \to X_1^*$ is p-coerceivly continuous, then for each $\phi \in H_1$, the equation (2.1.1) has a global solution

$$u \in L^\infty_{loc}((0,\infty), H_1) \cap L^p_{loc}((0,\infty), X_2).$$

Theorem 2.1.13. Under the hypotheses of Theorem 2.1.12, if G is differentiable, and $\forall u, v \in X$

$$< DG(u)v, Lv > \geq c_1 \|v\|_{H_2}^2 - c_2 \|v\|_{H_1}^2 \tag{2.1.30$'$}$$

$$\begin{cases} | < Gu, Lv > | \leq \frac{1}{2}\|v\|_{H_1}^2 + g(u) \\ g : X_3 \to R^+ \ is\ a\ bounded\ continuous\ functional \end{cases} \tag{2.1.31$'$}$$

in addition, the sequence $\{e_n\}$ in (2.1.33) is also an orthogonal base of H_3, then for each $\phi \in X_3$, the solution of (2.1.1) is unique, which has the further regularity

$$u \in W^{1,\infty}_{loc}((0,\infty), H_1) \cap W^{1,2}_{loc}((0,\infty), H_2).$$

The proof of Theorem 2.1.13 is similar to that of Theorem 2.1.11. In the following, we only need to prove Theorem 2.1.12.

Proof of Theorem 2.1.12. We take the normal orthogonal base $\{e_n\}$ of H as required by (2.1.33), and due to (2.1.32), $\{e_n\}$ is also an orthogonal base of H_1.

By using the standard Galerkin procedure as in the proof of Theorem 2.1.8, there is a sequence $\{u_n\}, u_n \in c^1([0,\infty), X_n)$ which satisfies (2.1.5) and (2.1.6). Putting $v = Lu_n$(By (2.1.33), $L\widetilde{X}_n = \widetilde{X}_n$) in (2.1.6), from (2.1.32) and (2.1.34) it follows that

$$\frac{1}{2}\|u_n\|^2_{H_1} + c_1\int_0^t \|u_n\|^p_{X_2}dt \leq \frac{1}{2}\|\phi_n\|^2_{H_1} + c_2\int_0^t [\|u_n\|^2_{H_1} + 1]dt$$

which implies that the sequence $\{u_n\} \subset L^\infty((0,T), H_1) \cap L^p((0,\infty), X_2)$ is bounded for any $0 < T < \infty$. Without loss of generality, let

$$\left\{ \begin{array}{l} u_n \rightharpoonup u_0 \ in \ L^p((0,T), X_2) \\ u_n \rightharpoonup^* u_0 \ in \ L^\infty((0,T), H_1), \quad \forall 0 < T < \infty \end{array} \right. \tag{2.1.35}$$

Now, we need to show that the sequence $\{u_n\}$ is uniformly weakly convergent to u_0 in $L^p((0,T), X_2)$.

According to (2.1.6)(2.1.32) and $LX_n = X_n$, we have

$$< u_n(t), v >_{H_1} + \int_0^t < Gu_n, Lv > dt =< \phi_n, v >_{H_1}, \quad \forall v \in X_n$$

Hence $\forall 0 < h < 1$, we have

$$< \frac{u_n(t+h) - u_n(t)}{h}, v >_{H_1} = -\frac{1}{h}\int_t^{t+h} < Gu, Lv > d\tau \tag{2.1.36}$$

Replacing v by $u_n(t+h) - u_n(t)$ in (2.1.36), one reads

$$\|\frac{u_n(t+h) - u_n(t)}{h^{\frac{1}{2}}}\|^2_{H_1} = -\frac{1}{h}\int_t^{t+h} < Gu_n(\tau), Lu_n(t+h) - Lu_n(t_1) > dt$$

$$= -\int_0^1 < Gu_n(t+\tau h), Lu_n(t+h) - Lu_n(t) > d\tau$$

Integrating the both sides of above equality for $t \in (0,T)$, we get that

$$\int_0^T \|\frac{u_n(t+h) - u_n(t)}{h^{\frac{1}{2}}}\|^2_{H_1}dt = \int_0^1 d\tau \int_0^T < Gu_n(t+\tau h), Lu_n(t) > dt \tag{2.1.37}$$

$$-\int_0^1 d\tau \int_0^T < Gu_n(t+\tau h), Lu_n(t+h) > dt$$

Because $\{u_n\} \subset L^p((0,T), X_2)$ is bounded, and G takes any bounded subset in $L^p((0,T), X_2)$ to a bounded subset in $L^{p'}((0,T), X_1^*) \forall 0 < T < \infty$, noticing that $L : X_2 \to X_1$ is bounded, from (2.1.37) we derive that

$$\int_0^T \| \frac{u_n(t+h) - u_n(t)}{h^{\frac{1}{2}}} \|_{H_1}^2 dt \tag{2.1.38}$$

$$\leq 2[\int_0^{T+h} \|Gu_n\|_{X_1^*}^{p'} dt]^{\frac{1}{p'}} [\int_0^{T+h} \|Lu_n\|_{X_1}^p dt]^{\frac{1}{p}}$$

$$\leq c \int_0^{T+h} \|u_n\|_{X_2}^p dt \leq \alpha,$$

where $\alpha > 0$ is a constant independent of n. Letting $u_n = \sum_{i=1}^n x_i^n(t) e_i$, from (2.1.38) we can derive that for ach fixed i, there is a constant $c_i > 0$ independent of n, such that

$$|x_i^n(t+h) - x_i^n(t)| \leq c_i h^{\frac{1}{2}}$$

which implies, by the Arzera-Ascoli theorem, that

$$\lim_{n\to\infty} \int_0^T | < u_n - u_0, v >_H |^2 dt = 0, \quad \forall v \in X_n. \tag{2.1.39}$$

Since $U_{n=1}^{\infty} X_n$ is dense in H, and $\{u_n\} \subset L^\infty(0,T), H)$ is bounded, (2.1.39) holds true for any $v \in H$. Therefore $u_n \rightharpoonup u_0$ uniformly in $L^p((0,T), X_2), \forall 0 < T < \infty$.

It remains to verify that

$$\lim_{n\to\infty} \int_0^t < Gu_n - Gu_0, Lu_n - Lu_0 > dt = 0 \tag{2.1.40}$$

By (2.1.6) we have

$$\int_0^t < Gu_n - Gu_0, Lu_n - Lu_0 > dt = \int_0^t < Gu_0, Lu_0 - Lu_n > dt \tag{2.1.41}$$

$$-\int_0^t < Gu_n, Lu_0 > dt - \frac{1}{2} < u_n, u_n >_{H_1} + \frac{1}{2} < \phi_n, \phi_n >_{H_1}$$

Owing to (2.1.35) and $\phi_n \to \phi$ in H_1, we get

$$\lim_{n\to\infty} \int_0^t < Gu_0, Lu_0 - Lu_n > dt = 0 \tag{2.1.42}$$

(2.1.43) $$\lim_{n\to\infty} < \phi_n, \phi_n >_{H_1} = < \phi, \phi >_{H_1}$$

On the other hand, by (2.1.38), the function $f_n(t) = \|u_n(t)\|_{H_1}$ satisfy that

$$\int_0^T |f_n(t+h) - f_n(t)| dt \le \int_0^T \|u_n(t+h) - u_n(t)\|_{H_1} dt$$

$$\le \alpha h^{\frac{1}{2}}, . \quad \forall 0 < h < 1$$

and for any subset $\Sigma \subset [0,T]$,

$$\int_\Sigma |f_n(t)| dt \le |\Sigma|^{\frac{1}{2}} [\int_0^T \|u_n\|_{H_1}^2 dt]^{\frac{1}{2}} \le c|\Sigma|^{\frac{1}{2}}$$

Hence, by Theorem 2.21 in [Ad], it follows that $\{f_n(t)\}$ is a precompact set in $L^1((0,T))$, and by the compactness of $X_2 \hookrightarrow H_1$ and (2.1.35), for almost all $t \in [0,T]$,

(2.1.44) $$\lim_{n\to\infty} < u_n(t), u_n(t) >_{H_1} = < u_0(t), u_0(t) >_{H_1}$$

In addition, by (2.1.6) we have

$$\int_0^t < Gu_n, Lv > dt = \int_0^t < u_n, \frac{dv}{dt} >_{H_1} dt$$

$$- < u_n, v >_{H_1} + < \phi_n, v(0) >_{H_1}, \quad \forall v \in \widetilde{X}_n$$

Passing to the limit $n \to \infty$, it follows

$$\lim_{n\to\infty} \int_0^t < Gu_n, v > dt = \int_0^t < u_0, \frac{dv}{dt} >_{H_1} dt$$

$$- < u_0, v >_{H_1} + < \phi, v(0) >_{H_1}$$

By the boundedness of $\{u_n\}$ in $L^p((0,t), X_2)$, $\{Gu_n\}$ is bounded in $L^{p'}((0,t), X_1^*)$, therefore there exists a $w \in L^{p'}((0,t), X_1^*)$ such that $Gu_n \rightharpoonup w$, namely

$$\lim_{n\to\infty} \int_0^t < Gu_n, Lv > dt = \int_0^t < w, Lv > dt$$

$$= \int_0^t < u_0, \frac{dv}{dt} >_H dt - < u_0, v >_{H_1} + < \phi, v(0) >_{H_1}$$

$\forall v \in \widetilde{X}_n$. Since $U_{n=1}^\infty X_n$ is dense in X_2, $U_{n=1}^\infty \widetilde{X}_n$ is dense in $W^{1,p}((0,t), X_2) \forall 0 < t < \infty$. Hence

(2.1.45) $$\int_0^t < w, Lv > dt = \int_0^t < u_0, \frac{dv}{dt} >_{H_1} dt$$

$$- < u_0, v >_{H_1} + < \phi, v(0) >_{H_1}$$

$\forall v \in W^{1,p}((0,t), X_2)$ and $0 < t < \infty$.

Respectively putting

$$v = u_h = \frac{1}{h}\int_t^{t+h} u_0 d\tau, \ and$$

$$v = u_{-h} = \frac{1}{h}\int_{t-h}^{t} \widetilde{u}_0 d\tau, \quad \widetilde{u}_0 = \begin{cases} u_0(t), & t \geq 0 \\ 0, & t \leq 0 \end{cases}$$

in (2.1.45) we get

$$\int_0^t < w, Lu_h > dt = \int_0^t < u_0, \frac{u_0(\tau+h) - u_0(\tau)}{h} >_{H_1} d\tau$$

$$- < u_0, u_h >_{H_1} + < \phi, u_h(0) >_{H_1}$$

and (naticing $u_{-h}(t) = 0, \forall t \leq 0$)

$$\int_0^{t+h} < w, Lu_{-h} > d\tau = \int_t^{t+h} < u_0, \frac{u_0(\tau) - u_0(\tau-h)}{h} >_{H_1} d\tau$$

$$+\frac{1}{h}\int_0^h < u_0, u_0 >_{H_1} d\tau - < u_0, u_{-h} >_{H_1}$$

On the other hand

$$\begin{aligned} & \int_0^t < u_0, \frac{u_0(\tau+h) - u_0(\tau)}{h} >_{H_1} d\tau + \int_h^{t+h} < u_0, \frac{u_0(\tau) - u_0(\tau-h)}{h} >_{H_1} d\tau \\ = & \int_0^t [< u_0(\tau), \frac{u_0(\tau+h) - u_0(\tau)}{h} >_{H_1} + < u_0(\tau+h), \frac{u_0(\tau+h) - u_0(\tau)}{h} >_{H_1}] d\tau \\ = & \frac{1}{h}\int_0^t [< u_0(\tau+h), u_0(\tau+h) >_{H_1} - < u_0(\tau), u_0(\tau) >_{H_1}] d\tau \\ = & \frac{1}{h}\int_t^{t+h} < u_0, u_0 >_{H_1} d\tau - \frac{1}{h}\int_0^h < u_0, u_0 >_{H_1} d\tau \end{aligned}$$

Consequently we obtain that

$$\int_0^t < w, Lu_h > dt + \int_0^{t+h} < w, Lu_{-h} > dt$$

$$= - < u_0, u_h >_{H_1} - < u_0, u_{-h} >_{H_1} + < \phi, u_h(0) >_{H_1} + \frac{1}{h}\int_t^{t+h} < u_0, u_0 >_{H_1} d\tau$$

Letting $h \to 0^+$, by Lemma 2.1.6, it follows

$$\int_0^t < w, Lu_0 > dt = \frac{1}{2} < \phi, \phi >_{H_1} - \frac{1}{2} < u_0, u_0 >_{H_1} \quad a.e.\ t \geq 0$$

Hence we have

$$\lim_{n\to\infty} \int_0^t < Gu_n, Lu_0 > dt = \int_0^t < w, Lu_0 > dt$$

$$= \frac{1}{2} < \phi, \phi >_{H_1} - \frac{1}{2} < u_0, u_0 >_{H_1}, \ a.e.\ t \geq 0$$

Combining with (2.1.41)-(2.1.44), one can derive the equality (2.1.40). The proof is complete.

Remark 2.1.14. When the spaces in (2.1.27) and (2.1.28) are of $X_2 = X_1, H_1 = H$, and the mapping $L = id : X_2 \to X_1$, then the mapping $G : X_2 = X_1 \to X_1^*$ is the usual horizonal operator, which can be used to treat the quasilinear problems.

2.1.5 Monotone operators

Here, we still take (2.1.27) and (2.1.28) as the space framework to discuss (2.1.1), and assume that there is a linear mappings $L : X_2 \to X_1$ which is the same as in previous subsection.

Similarily, for the monotone operators, we also have the global existence and regularity results as follows.

Theorem 2.1.15. Let the condition (2.1.32) and (2.1.33) be fulfilled. If $G : L^p((0,T), X_2) \to L^{p'}((0,T), X_1^*)$ is continuous $\forall 0 < T < \infty$, and satisfies

$$< Gu - Gv, Lu - Lv > \geq 0, \quad \forall u, v \in X_2 \tag{2.1.46}$$

$$\int_0^t < Gu, Lu > dt \geq \int_0^t c_1 \|u\|_{X_2}^p dt - c_2 \int_0^t [\|u\|_{H_1}^2 + 1] dt \tag{2.1.47}$$

then for any $\phi \in H_1$, the equation (2.1.1) has a global solution $u \in L^\infty_{loc}((0,\infty), H_1) \cap L^p_{loc}((0,\infty), X_2)$.

Theorem 2.1.16. Under the hypotheses of Theorem 2.1.13, if the sequence $\{e_n\}$ in (2.1.33) is also an orthogonal base of H_3, and G is differentiable which satisfies (2.1.30)' and (2.1.31)', then the conclusions of Theorem 2.1.13 hold true.

Remark 2.1.17. In the regularity theorems (Theorem 2.1.11, Theorem 2.1.13 and Theorem 2.1.16), the constant c_1 may permit to $c_1 \geq 0$. In this case, the conclusions of regularity should be revised to read

$$\begin{cases} u \in W^{1,\infty}_{loc}((0,\infty), H_1) \cap W^{1,2}_{loc}((0,\infty), H_2), \ as \ c_1 > 0 \\ u \in W^{1,\infty}_{loc}((0,\infty), H_1), \ as \ c_1 = 0 \end{cases}$$

Likewise, we only need to prove Theorem 2.1.15, and the proof of Theorem 2.1.16 is parallel to that of Theorem 2.1.11.

Proof of Theorem 2.1.15. We know that there is a bounded sequence $\{u_n\} \subset L^\infty((0,T), H_1) \cap L^p((0,T), X_2) \forall 0 < T < \infty$¡ which satisfies (2.1.5)(2.1.6) and (2.1.35).

According to the monotone condition (2.1.46), from (2.1.6) and (2.1.32), $\forall v \in \widetilde{X}_k, n \geq k$, it follows

$$\begin{aligned} 0 \ \leq \ & \int_0^t < Gv - Gu_n, Lv - Lu_n > dt \\ = \ & \int_0^t < Gv, Lv - Lu_n > dt + \int_0^t < \frac{du_n}{dt}, Lv - Lu_n >_H dt \\ = \ & \int_0^t [< Gv, Lv - Lu_n > - < u_n, \frac{d}{dt} Lv >_H] dt \\ & + \frac{1}{2} < \phi_n, \phi_n >_{H_1} - \frac{1}{2} < u_n, u_n >_{H_1} + < u_n, v >_{H_1} - < \phi_n, v(0) >_{H_1} \end{aligned}$$

Letting $n \to \infty$, by (2.1.35) we get

$$\begin{aligned} & \int_0^t [< Gv, Lv - Lu_0 > - < u_0, \frac{dv}{dt} >_{H_1}] dt \\ & + \frac{1}{2} < \phi, \phi >_{H_1} + < u_0, v >_{H_1} - < \phi, v(0) >_{H_1} \\ & - \frac{1}{2} \underline{\lim}_{n \to \infty} < u_n, u_n >_{H_1} \geq 0, \quad \forall v \in \widetilde{X}_n. \end{aligned} \tag{2.1.48}$$

Because $U^\infty_{n=1} \widetilde{X}_n$ is dense in $W^{1,p}((0,T), H_1) \cap L^p((0,T), X_2) \forall 0 < T < \infty$, the equality (2.1.48) holds for all $v \in W^{1,p}((0,T), H_1) \cap L^p((0,T), X_2)$.

Replacing v in (2.1.48) respectively by

$$u_h + \lambda w = \frac{1}{h} \int_t^{t+h} u_0 d\tau + \lambda w \quad (w \in X_2), \ and$$

$$u_{-h} + \lambda w = \frac{1}{h} \int_{t-h}^t \widetilde{u}_0 d\tau + \lambda w \quad (w \in X_2)$$

$$\widetilde{u}_0 = \begin{cases} u_0(t), & as \ t \geq 0 \\ 0, & as \ t < 0 \end{cases}$$

we obtain

$$\int_0^t < G(u_h + \lambda w), L(u_h - u_0 + \lambda w) > d\tau -$$
$$-\int_0^t < u_0, \frac{u_0(\tau + h) - u_0(\tau)}{h} >_{H_1} d\tau + < u_0(t), u_h(t) + \lambda w >_{H_1}$$
$$+\frac{1}{2} < \phi, \phi >_{H_1} - < \phi, u_h(0) + \lambda w >_{H_1} - \frac{1}{2}\underline{\lim}_{n\to\infty} < u_n, u_n >_{H_1} \geq 0$$

and (noticing $u_{-h}(t) = 0, \forall t \leq 0$)

$$\int_0^{t+h} < G(u_{-h} + \lambda w), L(u_{-h} - u_0 + \lambda w) > d\tau -$$
$$-\int_h^{t+h} < u_0, \frac{u_0(\tau) - u_0(\tau - h)}{h} >_{H_1} d\tau - \frac{1}{h}\int_0^h < u_0, u_0 >_{H_1} d\tau$$
$$+ < u_0(t+h), u_{-h}(t+h) + \lambda w >_{H_1} + \frac{1}{2} < \phi, \phi >_{H_1} - < \phi, \lambda w >_{H_1}$$
$$-\frac{1}{2}\underline{\lim}_{n\to\infty} < u_n, u_n >_{H_1} \geq 0$$

In addition,

$$\begin{aligned} & \int_0^t < u_0(\tau), \frac{u_0(\tau + h) - u_0(\tau)}{h} >_{H_1} d\tau + \int_h^{t+h} < u_0(\tau), \frac{u_0(\tau) - u_0(\tau - h)}{h} >_{H_1} d\tau \\ = \; & \int_0^t [< u_0(\tau), \frac{u_0(\tau + h) - u_0(\tau)}{h} >_{H_1} + < u_0(\tau + h), \frac{u_0(\tau + h) - u_0(\tau)}{h} >_{H_1}] d\tau \\ = \; & \frac{1}{h}\int_0^t [< u_0(\tau + h), u_0(\tau + h) >_{H_1} - < u_0(\tau), u_0(\tau) >_{H_1}] d\tau \\ = \; & \frac{1}{h}\int_t^{t+h} < u_0, u_0 >_{H_1} d\tau - \frac{1}{h}\int_0^h < u_0, u_0 >_{H_1} d\tau \end{aligned}$$

Hence we have

$$\int_0^t < G(u_h + \lambda w), L(u_h - u_0 + \lambda w) > dt +$$
$$+\int_0^{t+h} < G(u_{-h} + \lambda w), L(u_{-h} - u_0 + \lambda w > dt + < \phi, \phi >_{H_1}$$
$$-\frac{1}{2}\underline{\lim}_{n\to\infty} < u_n(t), u_n(t) >_{H_1} - \frac{1}{2}\underline{\lim}_{n\to\infty} < u_n(t+h), u_n(t+h) >_{H_1}$$
$$+ < u_0(t), u_0(t) + \lambda w >_{H_1} + < u_0(t+h), u_{-h}(t+h) + \lambda w >_{H_1}$$
$$- < \phi, u_h(0) + \lambda w >_{H_1} - < \phi, \lambda w >_{H_1} - \frac{1}{h}\int_t^{t+h} < u_0, u_0 >_{H_1} d\tau \geq 0$$

Thanks to Lemma 2.1.6 and the continuity of $G : L^p((0,T), X_2) \to L^{p'}((0,T), X_1^*) \forall 0 < T < \infty$, letting $h \to 0^+$, it follows

$$\int_0^t < G(u_0 + \lambda w), \lambda L w > dt + \lambda < u_0, w >_{H_1} - \lambda < \phi, w >_{H_1} \tag{2.1.49}$$
$$\geq \frac{1}{2}[\underline{\lim}_{n\to\infty} \|u_n\|^2_{H_1} - \|u_0\|^2_{H_1}]$$

$\forall \lambda \in R, w \in X_2$ and $t \geq 0$, a.e.. Because the functional $\|u\|^2_{H_1}$ is weakly inferior semi-continuous, we have

$$\underline{\lim}_{n\to\infty} \|u_n\|^2_{H_1} \geq \|u_0\|^2_{H_1}$$

Hence, from (2.1.49) we get

$$\int_0^t < G(u_0 + \lambda w), Lw > dt + < u_0, w >_{H_1} - < \phi, w >_{H_1} \geq 0$$

$\forall \lambda > 0$ and $w \in X_2$. Letting $\lambda \to 0^+$, and by (2.1.32), one gets

$$\int_0^t < Gu_0, Lv > dt + < u_0, Lv >_H - < \phi, Lv >_H \geq 0,$$

$\forall v \in X_2$. Since $L : X_2 \to X_1$ is dense, we have

$$< u_0, v >_H + \int_0^t < Gu_0, v > dt = < \phi, v >_H, \forall v \in X_1$$

The proof is complete.

Remark 2.1.18. If the operator G is relaxed by

$$G = A + B : R^+ \times X_2 \to X_1^*$$

and the condition (2.1.46) is revised as to write

$$< Au - Av, Lu - Lv > \geq 0, \quad \forall u, v \in X,$$

$$\begin{cases} \lim_{n\to\infty} \int_0^t < Bu_n, Lu_n > dt = \int_0^t < Bu_0, Lu_0 > dt \\ \textit{provided } u_n, u_0 \textit{ satisfing } (2.1.35) \end{cases}$$

the Theorem 2.1.15 and Theorem 2.1.16 still hold true.

2.1.6 Remarks and example

In the previous global existence and regularity theorems, Theorem 2.1.8 and Theorem 2.1.9 can be applied treat the initial boundary value problems of degenerate parabolic equations and the equations with nonnegative characteristic form; and Theorem 2.1.10-Theorem 2.1.16 can be used to deal with the initial boundary value problem of a class of fully nonlinear parabolic equations.

In the following, we shall give an example to show how to apply the basic theorems to partial differential equations. To this end, it is necessary to introduce two lemmas, which play the role of the compact embedding theorems.

Lemma 2.1.19. If $\Omega \subset R^n$ is bounded, and $\{e_n\} \subset L^2(\Omega)$ is an orthogonal base, then for any $\epsilon > 0$ there exists $N_\epsilon > 0$ such that $\forall u \in W^{1,p}(\Omega)(p > 2n/n+2, n \geq 2)$

$$[\int_\Omega |u|^2 dx]^{\frac{1}{2}} \leq \epsilon [\int_\Omega |\nabla u|^p + |u|^p dx]^{\frac{1}{p}} + [\sum_{k=1}^{N_\epsilon} (\int_\Omega u \cdot e_k dx)^2]^{\frac{1}{2}}$$

This lemma is Lemma 2.4 of Ch.II in [LSU].

Lemma 2.1.20. Let $\{u_n\} \subset L^p((0,T), W^{m,p}(\Omega))$ be bounded $(m \geq 1)$, and $u_n \rightharpoonup u_0$ in $L^p((0,T), W^{m,p}(\Omega))(p \geq 2)$. If

$$\lim_{n\to\infty} \int_0^T [\int_\Omega (u_n - u_0) v dx]^2 dt = 0, \quad \forall v \in c_0^\infty(\Omega) \tag{2.1.50}$$

then $\forall |\alpha| \leq m-1$, we have

$$D^\alpha u_n \to D^\alpha u_0 \ in \ L^2((0,T) \times \Omega). \tag{2.1.51}$$

Proof. Because $c_0^\infty(\Omega)$ is separable with the L^2-norm and dense in $L^2(\Omega)$, by the Gram-Schmidt standard orthogonal procedure, there exists $\{e_n\} \subset c_0^\infty(\Omega)$ such that $\{e_n\}$ is an orthogonal base of $L^2(\Omega)$. By Lemma 2.1.19, we have

$$\int_0^T \int_\Omega |D^\alpha u_n - D^\alpha u_0|^2 dx dt \leq \epsilon \int_0^T [\int_\Omega \sum_{i=1}^n |D_i D^\alpha u_n - D_i D^\alpha u_0|^p + \tag{2.1.52}$$

$$+ |D^\alpha u_n - D^\alpha u_0|^p dx]^{\frac{2}{p}} dt + \sum_{k=1}^{N_\epsilon} \int_0^T [\int_\Omega (D^\alpha u_n - D^\alpha u_0) e_k dx]^2 dt$$

$\forall |\alpha| \leq m-1$. Due to $e_k \in c_0^\infty(\Omega)$,

$$\int_\Omega (D^\alpha u_n - D^\alpha u_0) e_k dx = (-1)^{|\alpha|} \int_\Omega (u_n - u_0) D^\alpha e_k dx$$

Thanks to the boundedness of $\{u_n\}$ in $L^p((0,T),W^{m,p}(\Omega))$ and $2/p \leq 1$, from (2.1.50) and (2.1.52) one can derive (2.1.51). The proof is complete.

Example 2.1.21. We consider the below quasilinear degenerate parabolic equations of second order

$$\text{(2.1.53)} \qquad \begin{cases} \frac{\partial u}{\partial t} - D_i(a_{ij}(x,u)D_j u + b_i(x)u) + c(x,u) = f(x,t), \\ u|_{\sum_2 \cup \sum_3} = 0, \quad \forall t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

where a_{ij}, b_i, c and $\sum_2, \sum_3 \subset \partial\Omega$ are the same as in Theorem 1.6.3, which satisfy the condition $(L_1)-(L_4)$ in Subsection 1.6.2.

Take X, X_1, X_2 as in Theorem 1.6.3, $H = L^2(\Omega)$, and $G : R^+ \times X_1 \to X_2^*$ defined by

$$< Gu, v >= \int_\Omega [a_{ij}(x,u)D_i u D_j v + b_i(x)u D_i v +$$

$$+ c(x,u)v - f(x,t)v]dx - \int_{\sum_1} \bar{b} \cdot \bar{N} u \cdot v ds$$

As the proof of Theorem 1.6.3, the condition (A_1) in Theorem 2.1.8 is readily checked, and when $u_n \rightharpoonup u_0$ uniformly in $L^p((0,T),X_1)$, by Lemma 2.1.20, one can derive that u_n converges to u_0 on $\Omega \times [0,T]$ in measure, which implies that $G : R^+ \times X_1 \to X_2^*$ is p-weakly continuous. It remains to check the condition (A_2). We see that

$$\begin{aligned} & |\int_t^{t+h} \int_\Omega a_{ij}(x,u)D_i u D_j v dx dt| \leq c|\int_t^{t+h} \int_\Omega a_{ij}(x,0)D_i u D_j v dx dt| \\ \leq \ & c[\int_t^{t+h} \int_\Omega a_{ij}(x,0)D_i u D_j u dx dt]^{\frac{1}{2}} [\int_t^{t+h} \int_\Omega a_{ij}(x,0)D_i v D_j v dx dt]^{\frac{1}{2}} \\ \leq \ & c[\int_0^T \int_\Omega a_{ij}(x,0)D_i u D_j u dx dt]^{\frac{1}{2}} \cdot [\int_\Omega |\nabla v|^2 dx]^{\frac{1}{2}} \cdot h^{\frac{1}{2}} \\ \leq \ & c \cdot h^{\frac{1}{h}}, \forall 0 < h < 1 \ and \ v \in c^1(\overline{\Omega}) \end{aligned}$$

In the same fashion, we can derive

$$|\int_t^{t+h} < Gu, v > dt| \leq ch^\alpha, \alpha = min\{\frac{1}{2}, \frac{1}{k'}\}.$$

and the condition (A_2) is checked.

Hence, by Theorem 2.1.8, for any $\phi \in L^2(\Omega), f \in L^{k'}(\Omega)$, the problem (2.1.53) has a global weak solution

$$u \in L^\infty_{loc}((0,\infty), L^2(\Omega)) \cap L^k_{loc}((0,\infty), L^k(\Omega)) \cap L^2_{loc}((0,\infty), H^1_{loc}(\Omega))$$

In addition, if $a_{ij}(x,z) = a_{ij}(x)$, and the coeficients of (2.1.53) are differentiable, $c'_z(x,z) \geq -c(c \geq 0)$, then by Theorem 2.1.19, $\forall\phi \in W^{2,2}(\Omega) \cap X_1 \cap L^{2k}(\Omega)$, the solution of (2.1.53) is unique and belongs to $W^{1,\infty}_{loc}((0,\infty), L^2(\Omega))$.

2.2. Uniform Boundedness of Nonlinear Evolution Equations with Variational Structure of the First Order in Time

2.2.1. Weakly continuous horizontal operators

Let the space framework be given by

$$X \subset H_2 \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow H_1 \hookrightarrow H \tag{2.2.1}$$

where H, H_1, H_2 are Hilbert spaces, X_1, X_2 are Banach spaces, and X_1 is reflexive. Let $X_1 \hookrightarrow H$ be compact.

We continue to consider the nonlinear evolution equations as to read

$$\begin{cases} \frac{du}{dt} + Gu = 0 \\ u(0) = \phi \end{cases} \tag{2.2.2}$$

where

$$\begin{aligned} &G = DF + B : R^+ \times X_1 \to X_1^* \\ &F : X_1 \to R \; is \; a \; c^1 \; functional \\ &B : R^+ \times X_1 \to X_1^* \; is \; a \; mapping \end{aligned} \tag{1}$$

Suppose that

$$\|u\|_{X_1} \to \infty \Leftrightarrow F(u) \to \infty \tag{2.2.3}$$

$$| < Bu, v > | \leq \frac{1}{2}\|v\|_H^2 + CF(u) + c \tag{2.2.4}$$

We are now in a position to state and prove the uniform boundedness and regularity of global solutions of (2.2.2).

Theorem 2.2.1. Let $G : R^+ \times X_1 \to X_1^*$ be weakly continuous, and $\phi \in X_1$.

i) . If $G = DF$ satisfies (2.2.3), then (2.2.2) has a global solution

$$u \in W^{1,2}((0,\infty), H) \cap L^{\infty}((0,\infty), X_1)$$

and the solutions of (2.2.2) are uniformly bounded in X_1

ii) . If $G = DF + B$ satisfies (2.2.3) and (2.2.4), then (2.2.2) has a global solution

$$u \in W^{1,2}_{loc}((0,\infty), H) \cap L^{\infty}_{loc}((0,\infty), X_1).$$

Theorem 2.2.2. Let $G = DF : X_1 \to X_1^*$ be weakly continuous and differentiable, which satisfies (2.2.3) and

$$< DG(u)v, v > \geq c_1\|v\|^2_{H_1} - c_2\|v\|^2_H, \quad \forall u, v \in X,$$

$$\begin{cases} | < Gu, v > | \leq \frac{1}{2}\|v\|^2_H + g(u) \\ g : X_2 \to R \ is\ a\ bounded\ continuous\ functional \end{cases}$$

then for any $\phi \in X_2$, the solution of (2.2.2) is unique, which has the further regularity

$$u \in W^{1,\infty}((0,\infty), H) \cap W^{1,2}((0,\infty), H_1),\ as\ c_1 > 0$$

$$u \in W^{1,\infty}((0,\infty), H),\ as\ c_1 = 0$$

moreover, for any bounded subset $\sum \subset X_2$, the solutions of (2.2.2) are uniformly bounded for $(u_t(t,\phi), u(t,\phi))$ in $H \times X_1$ with $\phi \in \sum$, i.e. there exists a constant $c > 0$ such that for all $\phi \in \sum$, the solutions $u(t,\phi)$ satisfy the boundedness

$$\|u_t\|_H + \|u\|_{X_1} \leq c, \quad \forall t \geq 0.$$

We only prove Theorem 2.2.1, and the proof of Theorem 2.2.2 is parallel to that of Theorem 2.1.9.

Proof of Theorem 2.2.1. By Lemma 2.1.7, we take the normal orthogonal base $\{e_n\}$ of H, which is also an orthogonal base of H_2.

By using the Galerkin procedure, there is a sequence $\{u_n\}, u_n \in c^1([0,\infty), X_n)$, which satisfies

$$< u_n, v >_H + \int_0^t < Gu_n, v > dt =< \phi_n, v >_H, \quad \forall v \in X_n \tag{2.2.25}$$

$$\int_0^t [< \frac{du_n}{dt}, v >_H + < Gu_n, v >]dt = 0, \quad \forall v \in \widetilde{X}_n \tag{2.2.6}$$

We first verify the claim i). Inserting $v = \frac{du_n}{dt}$ in (2.2.6), we have

$$\begin{aligned} 0 &= \int_0^t [< \frac{du_n}{dt}, \frac{du_n}{dt} >_H + < DFu_n, \frac{du_n}{dt} >] dt \\ &= \int_0^t < \frac{du_n}{dt}, \frac{du_n}{dt} >_H dt + \int_0^t \frac{d}{dt} F(u_n) dt \\ &= \int_0^t < \frac{du_n}{dt}, \frac{du_n}{dt} >_H dt + F(u_n(t)) - F(\phi_n) \end{aligned}$$

Hence we get

$$\int_0^t \| \frac{du_n}{dt} \|_H^2 dt + F(u_n) = F(\phi_n) \tag{2.2.7}$$

Because the boundedness of $\|\phi_n\|_{X_1}$ can not be directly obtained from X_1, we first take $\phi \in H_2$. Due to $\{e_n\}$ being also an orthogonal base of $H_2, \phi_n \to \phi$ in H_2, and by $H_2 \hookrightarrow X_1$, we obtain

$$\phi_n \to \phi \ in \ X_1 \tag{2.2.8}$$

Thus, from (2.2.7)(2.2.8) and (2.2.3) one derives that $\{u_n\} \subset W^{1,2}((0,\infty), H) \cap L^\infty((0,\infty), X_1)$ is bounded. Let

$$u_n \rightharpoonup u_0 \ in \ W^{1,2}((0,\infty), H)$$

$$u_n \rightharpoonup^* u_0 \ in \ L^\infty((0,\infty), X_1)$$

By the weak continuity of G and the dominated convergence theorem, from (2.2.5) it follows

$$< u_0, v >_H + \int_0^t < Gu_0, v > dt = < \phi, v >_H, \quad \forall v \in X_n$$

Because $U_{n=1}^\infty X_n$ is dense in X_1, we get

$$< \frac{du_0}{dt}, v >_H + < Gu_0, v > = 0, \ a.e. \ t > 0, v \in X_1$$

On the other hand, owing to (2.2.3)(2.2.7) and (2.2.8), for any $\phi \in H_2, \|\phi\|_{X_1} \leq R$ there is a constant $c > 0$ only dependent on R, such that the solutions $u(t, \phi)$ of (2.2.2) satisfy

$$\int_0^\infty \|u_t\|_H^2 dt + \|u\|_{X_1} \leq c, \quad \forall t \geq 0 \tag{2.2.9}$$

Now, we take $\phi \in X_1, \|\phi\|_{X_1} \leq R$, then there is a sequence $\{\psi_n\} \subset H_2$ with $\|\psi_n\|_{X_1} \leq R$, and

$$\lim_{n\to\infty} \psi_n = \phi \ in \ X_1$$

By the estimate (2.2.9), the solutions $\{u(t,\psi_n)\}$ of (2.2.2) are bounded in $W^{1,2}((0,\infty),H) \cap L^\infty((0,\infty),X_1)$. It is easy to see that $u(t,\psi_n)$ weakly (*) converges to $u(t,\phi)$ in $W^{1,2}((0,\infty),H) \cap L^\infty((0,\infty),X_1)$, and $u(t,\phi)$ is a solution of (2.2.2) which satisfies (2.2.9).

Next we shall verify the chaim ii). Putting $v = \frac{du_n}{dt}$ in (2.2.6) we get

$$\int_0^t \|\frac{du_n}{dt}\|_H^2 dt + F(u_n) = F(\phi_n) - \int_0^t < Bu_n, \frac{du_n}{dt} > dt$$

According to (2.2.4) one gets

$$\frac{1}{2}\int_0^t \|\frac{du_n}{dt}\|_H^2 dt + F(u_n) \leq F(\phi_n) + c\int_0^t [F(u_n)+1]dt \tag{2.1.10}$$

Thanks to the Gronwell inequality, from (2.2.3 and (2.2.10) we can derive that $\{u_n\} \subset W^{1,2}((0,T),H) \cap L^\infty((0,T),X_1)$ is bounded $\forall 0 < T < \infty$, and the remainder proof is parallel to that of the claim i). The proof is complete.

2.2.2 Weakly continuous downward operators

Let there be the embedding space sequences

$$X \subset H_3 \hookrightarrow X_4 \hookrightarrow X_3 \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow H \tag{2.2.11}$$

$$X_3 \hookrightarrow H_2 \hookrightarrow H_1 \hookrightarrow H \tag{2.2.12}$$

where X is a linear space, $X_i (1 \leq i \leq 4)$ and $H, H_j(1 \leq j \leq 3)$ are the completion of X respectively with their norm, $X_i(1 \leq i \leq 4)$ are Banach spaces, H and $H_j(1 \leq j \leq 3)$ are Hilbert spaces. Let X_3 be reflexive.

Suppose that there are two linear mappings

$$L_1 : X_2 \to X_1$$

$$L_2 : X \to X_1$$

which are one to one and dense, and satisfy

$$< u, L_1 v >_H = < u, v >_{H_1} \tag{2.2.13}$$

$$< u, L_2 v >_H = < u, v >_{H_2} \tag{2.2.14}$$

Let there exist a sequence $\{e_n\} \subset X$ which is respectively a normal orthogonal base of H and an orthogonal base of H_3, and

$$\begin{cases} L_1 e_n = \lambda_n e_n, & \lambda_n \neq 0 \\ L_2 e_n = \mu_n e_n, & \mu_n \neq 0 \end{cases} \tag{2.2.15}$$

It is clear that span $\{e_n\}$ is dense in X_1.

Let us consider the equations (2.2.2) with the mapping

$$G : X_3 \to X_1^*$$

Assume that there exists a c^1 functional $F : X_2 \to R$, such that for all $u, v \in X$

$$< Gu, L_1 v > = < DFu, v >_{X_2}, \tag{2.2.16}$$

$$\|u\|_{X_2} \to \infty \Leftrightarrow F(u) \to \infty \tag{2.2.17}$$

and there is a $p = \{p_1, \cdots, p_m\} > 1$ such that

$$\int_0^t < Gu, L_2 u > dt \geq c_1 \int_0^t \|u\|_{X_3}^p dt - c_2 \int_0^t [\|u\|_{H_2}^2 + 1] dt \tag{2.2.18}$$

Theorem 2.2.3. Let $G : X_3 \to X_1^*$ be weakly continuous and satisfy the conditions (2.2.16)-(2.2.18). Then for any $\phi \in X_2$, (2.2.2) has a global solution

$$u \in L^\infty((0,\infty), X_2) \cap L^p_{loc}((0,\infty), X_3)$$

$$u_t \in L^2((0,\infty), H_1),$$

and the solutions of (2.2.2) are uniformly bounded in X_2.

Theorem 2.2.4. Under the hypotheses of Theorem 2.2.3, if $G : X_3 \to X_1^*$ is differentiable, and satisfies

$$< DG(u)v, L_1 v > \geq c_1 \|v\|_{H_2}^2 - c_2 \|v\|_{H_1}^2, \quad \forall u, v \in X.$$

$$\begin{cases} |< Gu, L_1 v >| \leq \frac{1}{2}\|v\|_{H_1}^2 + g(u), \\ g : X_4 \to R^+ \ \textit{is a bounded continuous functional} \end{cases}$$

then for any $\phi \in X_4$, the solution of (2.2.2) is unique, which has the further regularity

$$u \in W^{1,\infty}((0,\infty), H_1) \cap W^{1,2}((0,\infty), H_2), \ as\ c_1 > 0$$

$$u \in W^{1,\infty}((0,\infty), H_1), \ as\ c_1 = 0$$

moreover, for any bounded subset $\sum \subset X_4$, the solutions $u(t,\phi)$ of (2.2.2) are uniformly bounded for $(u_t(t,\phi), u(t,\phi))$ in $H_1 \times X_2$ with $\phi \in \sum$.

We only prove Theorem 2.2.3, and the proof of Theorem 2.2.4 is parallel to that of Theorem 2.1.11.

Proof of Theorem 2.2.3. We take the sequence $\{e_n\} \subset X$ as in (2.2.15) which is respectively a normal orthogonal base of H and an orthogonal base of H_3. It is clear that $\text{Li}(\widetilde{X}_n) = \widetilde{X}_n (i = 1, 2)$.

It is known that there is a sequence $\{u_n(t)\}, u_n \in c^1([0,\infty), X_n)$ which satisfies (2.2.5) and (2.2.6). Putting $v = \frac{d}{dt} L_1 u_n$ in (2.2.6), and by (2.2.13) and (2.2.16) we get

$$\int_0^t [< \frac{du_n}{dt}, \frac{du_n}{dt} >_{H_1} + < DFu_n, \frac{du_n}{dt} >_{X_2}]dt$$

$$= \int_0^t \|\frac{du_n}{dt}\|_{H_1}^2 dt + F(u_n(t)) - F(\phi_n) = 0$$

namely

$$\int_0^t \|\frac{du_n}{dt}\|_{H_1}^2 dt + F(u_n) = F(\phi_n), \forall t \geq 0 \tag{2.2.19}$$

Again, inserting $v = L_2 u_n$ in (2.2.6), and by (2.2.14)(2.2.18) it follows that

$$\frac{1}{2}\|u_n(t)\|_{H_2}^2 + c_1 \int_0^t \|u_n\|_{X_3}^p dt \tag{2.2.20}$$

$$\leq \frac{1}{2}\|\phi_n\|_{H_2}^2 + c_2 \int_0^t [\|u_n\|_{H_2}^2 + 1]dt$$

By using the Gronwell inequality, from (2.2.19)(2.2.20) and the condition (2.2.17), one can derive that the sequence $\{u_n\} \subset W^{1,2}((0,\infty), H_1) \cap L^\infty((0,\infty), X_2) \cap L^p_{loc}((0,\infty), X_3)$ is bounded provided $\phi \in H_3$. The remainder proof is parallel to that of Theorem 2.2.1. The proof is complete.

2.2.3 Operators with monotone structure

In this subsection, we take the sequences (2.1.27) and (2.1.28) as our space framework to discuss (2.2.2), where $X_2 \hookrightarrow H_1$ is compact.

Let there be an one to one and dense linear bounded mapping $L : X_2 \to X_1$, which satisfies (2.1.32) and (2.1.33). Moreover, $\{e_n\}$ is respectively a normal orthogonal base of H and an orthogonal base of H_3.

We consider the equations (2.2.2) with the operators

$$G = A + B : R^+ \times X_2 \to X_1^*$$

Suppose that there exists a c^1 functional $F : X_2 \to R$ such that

$$< Au, Lv >=< DFu, v >_{X_2}, \quad \forall u, v \in X_2 \tag{2.2.21}$$

$$\|u\|_{X_2} \to \infty \Leftrightarrow F(u) \to \infty \tag{2.2.22}$$

$$| < Bu, Lv > | \leq \frac{1}{2}\|v\|^2_{H_1} + cF(u) + c \tag{2.2.23}$$

We say that $G : R^+ \times X_2 \to X_1^*$ is coerceivly continuous, if for any $u_n \rightharpoonup^* u_0$ in $L^\infty((0, \infty), X_2)$, and

$$\lim_{n\to\infty} \int_0^t < Gu_n - Gu_0, Lu_n - Lu_0 > dt = 0, \quad \forall 0 < t < \infty$$

then for any $v \in X_1$

$$\lim_{n\to\infty} \int_0^t < Gu_n, v > dt = \int_0^t < Gu_0, v > dt, \quad \forall 0 < t < \infty.$$

Theorem 2.2.5. Let $G : R^+ \times X_2 \to X_1^*$ be coerceivly continuous, and $\phi \in X_2$.
i). If $G = A$ satisfies (2.2.21) and (2.2.22), then (2.2.2) has a global solution

$$u \in W^{1,2}((0, \infty), H_1) \cap L^\infty((0, \infty), X_2)$$

and the solutions of (2.2.2) are uniformly bounded in X_2.
ii). If $G = A + B$ satisfies (2.2.21)-(2.2.23), then (2.2.2) has a global solution

$$u \in W^{1,2}_{loc}((0, \infty), H_1) \cap L^\infty_{loc}((0, \infty), X_2).$$

Theorem 2.2.6. Let $G : R^+ \times X_2 \to X_1^*$ be bounded continuous $\phi \in X_2$, and

$$< Au - Av, Lu - Lv >\geq 0, \quad \forall u, v \in X_2 \tag{2.2.24}$$

i). If $G = A$ satisfies (2.2.21)(2.2.22) and (2.2.24), then (2.2.2) has a global solution

$$u \in W^{1,2}((0, \infty), H_1) \cap L^\infty((0, \infty), X_2)$$

and the solutions of (2.2.2) are uniformly bounded in X_2
ii). If $G = A+B$ satisfies (2.2.21)-(2.2.24), and for any $u_n \rightharpoonup^* u_0$ in $L^\infty((0,t), X_2), \forall 0 < t < \infty$,

$$\lim_{n\to\infty} \int_0^t < Bu_n, Lu_n > dt = \int_0^t < Bu_0, Lu_0 > dt$$

$$\lim_{n\to\infty} \int_0^t < Bu_n, v > dt = \int_0^t < Bu_0, v > dt$$

then (2.2.2) has a global solution

$$u \in W^{1,2}_{loc}((0,\infty), H_1) \cap L^\infty_{loc}((0,\infty), X_2).$$

Theorem 2.2.7. Under the hypotheses of Theorem 2.2.5(or Theorem 2.2.6), if the conditions $(2.1.30)'$ and $(2.1.31)'$. in Theorem 2.1.13 are fulilled, then for any $\phi \in X_3$ the solution of (2.2.2) is unique, which has the further regularity

$$u \in W^{1,\infty}_{loc}((0,\infty), H_1) \cap L^{1,2}_{loc}((0,\infty), H_2) \ as\ c_1 > 0\ in\ (2.1.30)'$$

$$u \in W^{1,\infty}_{loc}((0,\infty), H_1),\ as\ c_1 = 0.$$

Moreover, if $G = A$, then for any bounded subset $\sum \subset X_3$, the solutions of (2.2.2) are uniformly bounded for $(u_t(t,\phi), u(t,\phi))$ in $H_1 \times X_2$ with $\phi \in \sum$

The proofs of Theorem 2.2.5-2.2.7 are similar to these of Theorem 2.1.12-2.1.13 and Theorem 2.1.15.

In fact, the proofs of Theorem 2.2.5 and Theorem 2.2.6 are more simple than these of Theorem 2.1.12 and Theorem 2.1.15, because the approximative sequence $\{u_n\}$ in Theorem 2.2.5 and Theorem 2.2.6 is bounded in $W^{1,2}((0,\infty), H_1) \cap L^\infty((0,\infty), X_2)$, thus we can directly obtain the limits

$$\lim_{n\to\infty} \int_0^t < Gu_n, Lu_0 > dt = -\lim_{n\to\infty} \int_0^t < \frac{du_n}{dt}, u_0 >_{H_1} dt$$

$$= -\int_0^t < \frac{du_0}{dt}, u_0 >_{H_1} dt = -\frac{1}{2} < u_0, u_0 >_{H_1} -\frac{1}{2} < \phi, \phi >_{H_1}$$

and

$$\lim_{n\to\infty} \int_0^t < \frac{du_n}{dt}, v >_{H_1} dt = \int_0^t < \frac{du_0}{dt}, v >_{H_1} dt.$$

Here, we omit the details of the proofs.

2.3. Abstract Theory for Nonlinear Evolution Equations of Second Order in Time

2.3.1. Abstract nonlinear hyperbolic equations

Let the space sequences be given by

$$X \subset H_2 \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow H \tag{2.3.1}$$

$$X_2 \hookrightarrow H_1 \hookrightarrow H$$

Let $L : X \to X_1$ be an one to one and dense linear mapping, which satisfies

$$< u, Lv >_H =< u, v >_{H_1}, \quad \forall u, v \in X, \tag{2.3.2}$$

$$Le_i = \lambda_i e_i, \quad \lambda_i \neq 0 \tag{2.3.3}$$

where $\{e_i\} \subset X$ is respectively a normal orthogonal base and on orthogonal base of H_2.

We consider the equations as follows

$$\begin{cases} \frac{d^2u}{dt^2} + k\frac{du}{dt} + Gu = 0, & k \geq 0 \\ u(0) = \phi, \quad u_t(0) = \psi \end{cases} \tag{2.3.4}$$

where $G : R^+ \times X_2 \to X_1^*$.

Definition 2.3.1. Let $(\phi, \psi) \in X_2 \times H_1 . u \in W^{1,\infty}_{loc}((0, \infty), H_1) \cap L^\infty_{loc}((0, \infty), X_2)$ is called a global solution of (2.3.4), if for any $v \in X_1$,

$$< u_t, v >_H + k < u, v >_H + \int_0^t < Gu, v > dt =< \psi, v >_H + k < \phi, v >_H$$

Definition 2.3.2. Let Y_1, Y_2 be Banach spaces. The solution $u(t, \phi, \psi)$ of (2.3.4) is called to be uniformly bounded in $Y_1 \times Y_2$, if for any bounded subset $B_1 \times B_2 \subset Y_1 \times Y_2$, there is a bounded open set $\Omega_1 \times \Omega_2 \subset Y_1 \times Y_2$ such that $\{u, u_t\} \in \Omega_1 \times \Omega_2 \quad \forall t \geq 0$ provided $(\phi, \psi) \in B_1 \times B_2$.

We suppose that

$$G = A + B : R^+ \times X_2 \to X_1^*$$

and there is a c^1 functional $F : X_2 \to R$ such that

$$< Au, Lv >=< DFu, v >_{X_2}, \quad \forall u, v \in X \tag{2.3.5}$$

$$\|u\|_{X_2} \to \infty \Leftrightarrow F(u) \to \infty \tag{2.3.6}$$

$$| < Bu, Lv > | \leq c_1 F(u) + c_2 \|v\|^2_{H_1} + g_1(t) \tag{2.3.7}$$

$$| < Gu, v > | \leq \frac{1}{2}\|v\|^2_H + cF(u) + g_2(t) \tag{2.3.8}$$

where $g_1, g_2 \in L^1_{loc}(0,\infty)$.

Theorem 2.3.3. Let $G : R^+ \times X_2 \to X_1^*$ be weakly continuous and $(\phi, \psi) \in X_2 \times H_1$. Then the below conclusions hold.

i) . If $G = A$ satisfies (2.3.5) and (2.3.6), then the equation (2.3.4) has a global solution $u \in W^{1,\infty}((0,\infty), H_1) \cap L^\infty((0,\infty), X_2)$, and the solutions of (2.3.4) are uniformly bounded in $X_2 \times H_1$.

ii) . If $G = A + B$ satisfies (2.3.5)-(2.3.7), then (2.3.4) has a global solution $u \in W^{1,\infty}_{loc}((0,\infty), H_1) \cap L^\infty_{loc}((0,\infty), X_2)$

iii) . In addition, if the condition (2.3.8) is also fulfilled then the solution $u \in W^{2,2}_{loc}((0,\infty), H)$.

Proof. Let $\{e_n\} \subset X$ be the common orthogonal base of H and H_2, which satisfies (2.3.3). Hence $L\widetilde{X}_n = \widetilde{X}_n$.

By means of the Galerkin procedure, there exist $u_n \in c^2([0,\infty), X_n)$ which satisfy

$$\begin{cases} < \frac{du_n}{dt}, v >_H + k < u_n, v >_H + \int_0^t < Gu_n, v > dt \\ = < \psi_n, v >_H + k < \phi_n, v >_H, & \forall v \in X_n \\ u_n(0) = \phi_n, \quad \frac{du_n(0)}{dt} = \psi_n \end{cases} \tag{2.3.9}$$

$$\int_0^t [< \frac{d^2u_n}{dt^2}, v >_H + k < \frac{du_n}{dt}, v >_H + < Gu_n, v >] dt = 0 \tag{2.3.10}$$

$\forall v \in \widetilde{X}_n$ and $0 < t < \infty$.

We first consider the case of $G = A$. Inserting $v = \frac{d}{dt} Lu_n$ in (2.3.10), from (2.3.2) and (2.3.5) we get

$$0 = \int_0^t [< DFu_n, \frac{du_n}{dt} >_{X_2} + \frac{1}{2}\frac{d}{dt} < \frac{du_n}{dt}, \frac{du_n}{dt} >_{H_1}$$

$$+ k < \frac{du_n}{dt}, \frac{du_n}{dt} >_{H_1}] dt$$

$$= F(u_n) - F(\phi_n) + \frac{1}{2}\|\frac{du_n}{dt}\|^2_{H_1} - \frac{1}{2}\|\psi_n\|^2_{H_1} + k\int_0^t \|\frac{du_n}{dt}\|^2_{H_1} dt$$

namely

$$F(u_n) + \frac{1}{2}\|\frac{du_n}{dt}\|^2_{H_1} + k\int_0^t \|\frac{du_n}{dt}\|^2_{H_1} dt = F(\phi_n) + \frac{1}{2}\|\psi_n\|^2_{H_1} \tag{2.3.11}$$

We first take $\phi \in H_2$. Because $\{e_n\}$ is a common orthogonal base of H and H_2, and by (2.3.2) and (2.3.3), $\{e_n\}$ is also an orthogonal base of H_1. Hence $\phi_n \to \phi$ in $H_2, \psi_n \to \psi$ in H_1. And by $H_2 \hookrightarrow X_2$, we have

$$\begin{cases} \phi_n \to \phi \ in\ X_2 \\ \psi_n \to \psi \ in\ H_1 \end{cases} \tag{2.3.12}$$

Due to (2.3.6), from (2.3.11) and (2.3.12) it follows that $\{u_n\} \subset W^{1,\infty}((0,\infty), H_1) \cap L^\infty((0,\infty), X_2)$ is bounded. Let

$$u_n \rightharpoonup^* u_0 \ in\ W^{1,\infty}((0,\infty), H_1) \cap L^\infty((0,\infty), X_2)$$

By the weak continuity of $G : X_2 \to X_1^*$, from (2.3.9) we can derive that u_0 is a solution of (2.3.4). In addition, from (2.3.11) and (2.3.12), one can deduce that for any $R > 0$ there is a constant $c > 0$ such that if

$$\|\phi\|_{X_2} + \|\psi\|_{H_1} \leq R \tag{2.3.13}$$

then the solution $u(t, \phi, \psi)$ of (2.2.4) satisfies

$$\|u(t, \phi, \psi)\|_{X_2} + \|u_t(t, \phi, \psi)\|_{H_1} \leq c, \quad \forall t \geq 0 \tag{2.3.14}$$

Now we take $(\phi, \psi) \in X_2 \times H_1$ which satisfies (2.3.13). Because $H_2 \to X_2$ is dense, we take $\phi_n \in H_2$ such that

$$\|\phi_n\|_{X_2} + \|\psi\|_H \leq R$$

$$\lim_{n\to\infty} \phi_n = \phi \ in\ X_2$$

By (2.3.14), the solutions $\{u(t, \phi_n, \psi)\}$ of (2.3.4) is bounded in $W^{1,\infty}((0,\infty), H_1) \cap L^\infty((0,\infty), X_2)$. Let

$$u(t, \phi_n, \psi) \rightharpoonup^* u \ in\ W^{1,\infty}((0,\infty), H_1) \cap L^\infty((0,\infty), X_2)$$

Then it is easy to see that $u(t)$ is a solution of (2.3.4) which satisfies the uniform boundedness (2.3.14). Thus, the claim i) is verified.

Next, we consider the case of $G = A + B$. Putting $v = \frac{d}{dt}Lu_n$ in (2.3.10) we get

$$F(u_n) + \frac{1}{2}\|\frac{du_n}{dt}\|^2_{H_1} + k\int_0^t \|\frac{du_n}{dt}\|^2_{H_1} dt \tag{2.3.15}$$

$$= -\int_0^t < Bu_n, \frac{d}{dt}Lu_n > dt + F(\phi_n) + \frac{1}{2}\|\psi_n\|_{H_1}^2$$

Due to (2.3.7), from (2.3.15) it follows

$$(2.3.16) \qquad \begin{cases} F(u_n) + \frac{1}{2}\|\frac{du_n}{dt}\|_{H_1}^2 \le c\int_0^t [F(u_n) + \frac{1}{2}\|\frac{du_n}{dt}\|_{H_1}^2]dt + k(t) \\ k(t) = \int_0^t g_1(t)dt + \frac{1}{2}\|\psi\|_{H_1}^2 + \sup_n F(\phi_n) \end{cases}$$

Thanks to the Gronwell inequality, from (2.3.16) one can deduce

$$(2.3.17) \qquad F(u_n) + \frac{1}{2}\|\frac{du_n}{dt}\|_{H_1}^2 \le k(0)e^{ct} + \int_0^t g_1(s)e^{c(t-s)}ds$$

The inequality (2.3.17) implies that for any $0 < T < \infty, \{u_n\}$ is a bounded sequence in $W^{1,\infty}((0,T),H_1) \cap L^\infty((0,T),X_2)$. In the same fashion as the above proof, one can obtain the claim ii).

Finally, if the condition (2.8) is also fulfilled, then inserting $v = \frac{d^2u_n}{dt^2}$ in (2.3.10) we have

$$\int_0^t < \frac{d^2u_n}{dt^2}, \frac{d^2u_n}{dt^2} >_H dt + \frac{1}{2}k < \frac{du_n}{dt}, \frac{du_n}{dt} >_H$$

$$\le \frac{k}{2}\|\psi_n\|_H^2 + \frac{1}{2}\int_0^t \|\frac{du_n}{dt}\|_H^2 dt + c\int_0^t [F(u_n) + g_2(t)]dt$$

which means, by (2.3.17), that

$$\int_0^t \|\frac{d^2u_n}{dt^2}\|_H^2 dt \le k\|\psi\|_H^2 + c\int_0^t [g_2 + k(0)e^{c\tau} + \int_0^\tau g_1(s)e^{c(\tau-s)}ds]d\tau$$

Thus, for any $0 < T < \infty, \{u_n\} \subset W^{2,2}((0,T),H)$ is bounded. It infers that the solution of (2.3.4) belongs to $W^{2,2}_{loc}((0,\infty),H)$. The proof is complete.

Remark 2.3.4. In Theorem 2.3.3, the condition (2.3.7) can be relaxed as that

$$(2.3.18) \qquad |\int_0^t < Bu, \frac{d}{dt}Lu > dt| \le \int_0^t [cF(u) + c\|\frac{du}{dt}\|_{H_1}^2 + g_2]dt$$

$$+\alpha F(u) + \beta.$$

$\forall u \in c^1([0,\infty),X), u(0) = \phi$, where $0 < \alpha < 1, 0 < \beta$, and $g_2 \in L^1_{loc}(0,\infty)$.

Remark 2.3.5. As $H_1 = H, X_2 = X_1$ and $L = id : X \to X_1$ is a including mapping, then for the horizonal mapping $G : R^+ \times X_1 \to X_1^*$, Theorem 2.3.3 still holds, which is garranteed by Lemma 2.1.7.

2.3.2. Equations with a strong damping term

In this subsection, the space framework is taken as to read

$$X \subset H_3 \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow H \tag{2.3.19}$$

$$X_2 \hookrightarrow H_2 \hookrightarrow H_1 \hookrightarrow H$$

where $H_2 \hookrightarrow H_1$ be compact.

Let there be an one to one and dense linear bounded mapping $L : X_2 \to X_1$, which satisfies

$$< u, Lv >_H = < u, v >_{H_1}, \quad \forall u, v \in X, \tag{2.3.20}$$

$$Le_n = \lambda_n e_n, \quad \lambda_n \neq 0, \tag{2.3.21}$$

where $\{e_n\}$ is respectively a normal orthogonal base of H and an orthogonal base of H_3.

We shall deal with the equations given by

$$\begin{cases} \frac{d^2u}{dt^2} + k\frac{d}{dt}\mathcal{L}u + Gu = 0, \quad k > 0 \\ u(0) = \phi, \quad u_t(0) = \psi \end{cases} \tag{2.3.22}$$

where $\mathcal{L} : X_2 \to X_1$ is a linear bounded operator, and

$$< \mathcal{L}u, Lv >_H = < u, v >_{H_2}, \quad \forall u, v \in X_2. \tag{2.3.23}$$

Let

$$G = A + B : R^+ \times X_2 \to X_1^*$$

where A satisfies (2.3.5) and (2.3.6), and B satisfies

$$| < Bu, Lv > | \leq cF(u) + \frac{k}{2}\|v\|^2_{H_2} + g(t) \tag{2.3.24}$$

where $g \in L^1_{loc}(0, \infty)$.

Theorem 2.3.6. Let $G : R^+ \times X_2 \to X_1^*$ be coerceivly continuous, and $(\phi, \psi) \in X_2 \times H_1$.

i) . If $G = A$ satisfies (2.3.5) and (2.3.6), then (2.3.22) has a global solution

$$u \in W^{1,\infty}((0, \infty), H_1) \cap W^{1,2}((0, \infty), H_2) \cap L^\infty((0, \infty), X_2)$$

and the solutions of (2.3.22) are uniformly bounded in $X_2 \times H_1$.

ii) . If $G = A + B$ satisfies (2.3.5)(2.3.6) and (2.3.24), then (2.3.22) has a global solution

$$u \in W^{1,\infty}_{loc}((0,\infty), H_1) \cap W^{1,2}_{loc}((0,\infty), H_2) \cap L^{\infty}_{loc}((0,\infty), X_2).$$

iii) . Furthermore, if (2.3.8) also satisfied, then the solution $u \in W^{2,2}_{loc}((0,\infty), H)$.

Similarily, for the monotone operators, we also have the global existence and uniform boundedness results.

Theorem 2.3.7. Let $G : R^+ \times X_2 \to X_1^*$ be bounded continuous, $(\phi, \psi) \in X_2 \times H_1$, and

$$< Au - Av, Lu - Lv > \geq 0, \quad \forall u, v \in X_2. \tag{2.3.25}$$

i) . If $G = A$ satisfies (2.3.5)(2.3.6) and (2.3.25), then the conclusions in i) of Theorem 2.3.6 holds.

ii) . If $G = A + B$ satisfies (2.3.5)(2.3.6)(2.3.24) and (2.3.25), and for any $u_n \rightharpoonup^* u_0$ in $L^\infty((0,T), X_2) \forall 0 < T < \infty$,

$$\lim_{n\to\infty} \int_0^T < Bu_n, v > dt = \int_0^T < Bu_0, v > dt, \quad \forall v \in L^\infty((0,T), X_1)$$

$$\lim_{n\to\infty} \int_0^T < Bu_n, Lu_n > dt = \int_0^T < Bu_0, Lu_0 > dt,$$

then the conclusions in (ii) of Theorem 2.3.6 holds.

iii) Furthermore if (2.3.8) is also satisfied, then the solution of (2.3.22) belongs to $W^{2,2}_{loc}((0,\infty), H)$.

We here only prove Theorem 2.3.6.

Proof of Theorem 2.3.6. Here, we only verify the claim i) and the proof of the other claims are similar to these of the previous theorems.

In the same fashian as the proof of Theorem 2.3.3, by (2.3.5)(2.3.6)(2.3.20) and (2.3.23), we can obtain a sequence $\{u_n\}, u_n \in c^2([0,\infty), X_n)$ which satisfies

$$\begin{cases} < \frac{du_n}{dt}, v >_H + k < \mathcal{L}u_n, v >_H + \int_0^t < Gu_n, v > dt \\ = < \psi_n, v >_H + k < \mathcal{L}\phi_n, v >_H, \quad \forall v \in X_n \\ u_n(0) = \phi_n, \quad \frac{du_n(0)}{dt} = \psi_n \end{cases} \tag{2.3.26}$$

$$\int_0^t [< \frac{d^2 u_n}{dt^2}, v >_H + k < \frac{d}{dt}\mathcal{L}u_n, v >_H + < Gu_n, v >] dt = 0 \tag{2.3.27}$$

$\forall v \in \widetilde{X}_n$, and $0 < t < \infty$, moreover

$$(2.3.28) \qquad \begin{cases} u_n \rightharpoonup^* u_0 \ in \ W^{1,\infty}((0,\infty), H_1) \cap L^\infty((0,\infty), X_2) \\ u_n \rightharpoonup u_0 \ in \ W^{1,2}((0,\infty), H_2) \end{cases}$$

If we show that

$$(2.3.29) \qquad \lim_{n\to\infty}[\int_0^t < Gu_n - Gu_0, Lu_n - Lu_0 > dt + \frac{k}{2}\|u_n - u_0\|_{H_2}] = 0$$

then by the coerceiv continuity of G, from (2.3.26) and (2.3.28) one obtain claim i).

According to (2.3.23), we have

$$\begin{aligned} & \int_0^t < \frac{d}{dt}\mathcal{L}u_n - \frac{d}{dt}\mathcal{L}u_0, Lu_n - Lu_0 >_H dt \\ = & \frac{1}{2}\int_0^t \frac{d}{dt} < u_n - u_0, u_n - u_0 >_{H_2} dt \\ = & \frac{1}{2}\|u_n(t) - u_0(t)\|_{H_2} - \frac{1}{2}\|\phi_n - \phi\|_{H_2} \end{aligned}$$

Hence, from (2.3.27)(2.3.23) and (2.3.20) it follows

$$\begin{aligned} & \int_0^t < Gu_n - Gu_0, Lu_n - Lu_0 > dt + \frac{k}{2}\|u_n - u_0\|_{H_2} \\ = & \int_0^t [< Gu_n - Gu_0, Lu_n - Lu_0 > + k < \frac{d}{dt}\mathcal{L}u_n - \frac{d}{dt}\mathcal{L}u_0, Lu_n - Lu_0 >_H]dt \\ & + \frac{k}{2}\|\phi_n - \phi\|_{H_2} \\ = & \int_0^t [< Gu_0, Lu_0 - Lu_n > - < Gu_n, Lu_0 > - k < \frac{d}{dt}\mathcal{L}u_n, Lu_0 >_H \\ & + k < \frac{d}{dt}\mathcal{L}u_0, Lu_0 - Lu_n >_H - < \frac{d^2u_n}{dt^2}, Lu_n >_H]dt \\ & + \frac{k}{2}\|\phi_n - \phi\|_{H_2} \\ = & \int_0^t [< Gu_0, Lu_0 - Lu_n > - < Gu_n, Lu_0 > - k < \frac{du_n}{dt}, u_0 >_{H_2} \\ & + k < \frac{du_0}{dt}, u_0 - u_n >_{H_2} + < \frac{du_n}{dt}, \frac{du_n}{dt} >_{H_1}]dt \\ & - < \frac{du_n}{dt}, u_n >_{H_1} + < \psi_n, \phi_n > + \frac{k}{2}\|\phi_n - \phi\|_{H_2} \end{aligned}$$

Because $X_2 \hookrightarrow H_2$, and $\phi_n \to \phi$ in X_2, we have

$$\lim_{n\to\infty} \|\phi_n - \phi\|_{H_2} = 0.$$

By (2.3.28), we have

$$\lim_{n\to\infty}\int_0^t < Gu_0, Lu_n - Lu_0 > dt = 0$$

$$\lim_{n\to\infty}\int_0^t < \frac{du_0}{dt}, u_n - u_0 >_{H_2} dt = 0$$

Hence we get

$$\lim_{n\to\infty}\int_0^t < Gu_n - Gu_0, Lu_n - Lu_0 > dt + \frac{k}{2}\lim_{n\to\infty}\|u_n - u_0\|_{H_2} \tag{2.3.30}$$

$$= \lim_{n\to\infty}\int_0^t [- < Gu_n, Lu_0 > -k < \frac{du_n}{dt}, u_0 >_{H_2} +$$

$$+ < \frac{du_n}{dt}, \frac{du_n}{dt} >_{H_1}]dt - \lim_{n\to\infty} < \frac{du_n}{dt}, u_n >_{H_1} + < \psi, \phi >_{H_1}$$

From (2.3.27) and (2.3.28) one can deduce

$$\lim_{n\to\infty}\int_0^t < Gu_n, Lv > dt = -\int_0^t [k < \frac{du_0}{dt}, v >_{H_2} \tag{2.3.31}$$

$$- < \frac{du_0}{dt}, \frac{dv}{dt} >_{H_1}]dt - < \frac{du_0}{dt}, v >_{H_1} + < \psi, v(0) >_{H_1},$$

$\forall v \in U_{n=1}^{\infty}\widetilde{X}_n$. Because $U_{n=1}^{\infty}\widetilde{X}_n$ is dense in $W^{1,2}((0,T), H_2)\cap L^p((0,T), X_2) \forall p < \infty$ and $T < \infty$, the equality (2.3.31) holds true for any $v \in W^{1,2}_{loc}((0,\infty), H_2)\cap L^p_{loc}((0,\infty), X_2)$. Hence we have

$$\lim_{n\to\infty}\int_0^t < Gu_n, Lu_0 > dt = -\int_0^t [k < \frac{du_0}{dt}, u_0 >_{H_2} \tag{2.3.32}$$

$$- < \frac{du_0}{dt}, \frac{du_0}{dt} >_{H_1}]dt - < \frac{du_0}{dt}, u_0 >_{H_1} + < \psi, \phi >_{H_1}$$

By the compactness of $H_2 \hookrightarrow H_1$, from (2.3.28) it follows

$$\lim_{n\to\infty}\int_0^t < \frac{du_n}{dt}, \frac{du_n}{dt} >_{H_1} dt = \int_0^t < \frac{du_0}{dt}, \frac{du_0}{dt} >_{H_1} dt$$

and by (2.3.28), $u_n \to u_0$ in H_1 a.e. $t \geq 0$, hence we get

$$\lim_{n\to\infty} < \frac{du_n}{dt}, u_n >_{H_1} = < \frac{du_0}{dt}, u_0 >_{H_1}, \ a.e.\ t \geq 0$$

Obviously

$$\lim_{n\to\infty}\int_0^t < \frac{du_n}{dt}, u_0 >_{H_2} dt = \int_0^t < \frac{du_0}{dt}, u_0 >_{H_2} dt$$

Consequently, from(2.3.30) and (2.3.32) one can obtain the equality (2.3.29). The proof is complete.

2.4. Initial Boundary Value Problems of Nonlinear and Fully Nonlinear Parabolic Equations

2.4.1. A lemma

In order to convenience, in the following we introduce a lemma, which is owed to [LSU] and [Mj].

Lemma 2.4.1 Let $\Omega \subset R^n$ be a bounded domain, $0 < T < \infty$. Then for any $0 \le k < m$ there exists a $p_k > p$ such that $L^p((0,T), W^{m,p}(\Omega)) \cap L^\infty((0,T), L^2(\Omega)$ can be imbedded into $L^{p_k}((0,T), W^{k,p_k}(\Omega))$, specially

$$L^p((0,T), W^{1,p}(\Omega)) \cap L^\infty((0,T), L^2(\Omega)) \hookrightarrow L^{\frac{n+2}{n}p}((0,T)\times\Omega)$$

where $p \ge 2$.

Proof. First, when $p < n$, from Theorem 1 of §1.4.7 in [Mj], we have

$$[\int_\Omega |D^k u|^q dx]^{\frac{1}{q}} \le c[\int_\Omega |D^{k+1}u|^p dx]^{\frac{1}{q}}[\int_\Omega |u|^2 dx]^{\frac{1}{2}-\frac{p}{2q}}$$

where $q = (2k+n+2)p/(2k+n), 0 \le k < m$. From the inequality we can obtain that

(2.4.1)
$$[\int_0^T\int_\Omega |D^k u|^q dxdt]^{\frac{1}{q}} \le c[\int_0^T\int_\Omega |D^{k+1}u|^p dxdt]^{\frac{1}{q}} \times$$

$$\times \max_{0\le t\le T}[\int_\Omega u^2 dx]^{\frac{1}{2}(1-\frac{p}{q})}$$

$$\le c[\int_0^T\int_\Omega \sum_{|\alpha|\le m}|D^\alpha u|^p dxdt]^{\frac{1}{p}} + \max_{0\le t\le T}[\int_\Omega |u|^2 dx]^{\frac{1}{2}}$$

When $n \le p$, by Theorem 2.2 of Ch.II in [LSU], we have

$$[\int_\Omega |D^k u|^q dx]^{\frac{1}{q}} \le c[\int_\Omega |D^{k+1}u|^p dx]^{\frac{\alpha}{p}} [\int_\Omega |D^k u|^2 dx]^{\frac{1}{2}(1-\alpha)}$$

for $u \in W^{m,p}(\Omega), 0 \le k < m, \alpha = (\frac{1}{2} - \frac{1}{q})(\frac{1}{n} - \frac{1}{p} + \frac{1}{2})^{-1}$.

On the other hand, by the interpolation inequality

$$[\int_\Omega |D^k u|^2 dx]^{\frac{1}{2}(1-\alpha)} \le [\int_\Omega |D^{k+1}u|^2 dx]^{\frac{\beta}{2}(1-\alpha)} [\int_\Omega u^2 dx]^{\frac{1}{2}(1-\alpha)(1-\beta)}$$

$$\le c[\int_\Omega |D^{k+1}u|^p dx]^{\frac{1}{p}\beta(1-\alpha)} [\int_\Omega u^2 dx]^{\frac{1}{2}(1-\alpha)(1-\beta)}$$

where $\beta = \frac{k}{k+1}$. Hence, we may obtain

(2.4.2) $$[\int_\Omega |D^k u|^q dx]^{\frac{1}{q}} \le c[\int_\Omega |D^{k+1}u|^p dx]^{\frac{1}{q}} [\int_\Omega u^2 dx]^{\frac{1}{2}(1-\frac{p}{q})}$$

where

$$q \cdot \left(\frac{k}{k+1} + \frac{1}{2(k+1)} \left(\frac{1}{n} + \frac{1}{2} - \frac{1}{p} \right)^{-1} \right) = p + \frac{1}{k+1} \left(\frac{1}{n} + \frac{1}{2} - \frac{1}{p} \right)^{-1}$$

It is easy to check that $\frac{k}{k+1} + \frac{1}{2(k+1)} \left(\frac{1}{n} + \frac{1}{2} - \frac{1}{p} \right)^{-1} \le 1$ as $n \le p$. Hence from (2.4.2) we can obtain (2.4.1).

Specially, as $k = 0, m = 1$, we have $q = \frac{n+2}{n} p$. The lemma is proved.

2.4.2. Global existence of fully nonlinear first initial boundary value problems

Hereafter, the domain $\Omega \subset R^n$ is always assumed to be bounded and smooth, unless it is specially pointed out.

First, we shall discuss the global existence of strong solutions of the problem given by

(2.4.3) $$\begin{cases} \frac{\partial u}{\partial t} - F(x, u, \nabla u, \triangle u) = g(x, u, \nabla u, D^2 u), \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \\ u(x, 0) = \phi(x) \end{cases}$$

Suppose that

(2.4.4) $$F(x, z, \xi, y)y \ge c_1 |y|^p - c_2, \quad p \ge 2$$

$$(2.4.5)\qquad [F(x,z,\xi,y_1)-F(x,z,\xi,y_2)][y_1-y_2]\geq k|y_1-y_2|^2,\quad k>0$$

$$(2.4.6)\qquad \begin{cases} |F(x,z,\xi,y)|\leq c[|y|^{p-1}+|\xi|^{p_1}+|z|^{p_2}+1] \\ |g(x,z,\xi,\eta)|\leq c[|\eta|^{p_3}+|\xi|^{p_3}+|z|^{p_3}+f] \\ p_1<\frac{n+2}{n}(p-1), p_2<(\frac{n+2}{n})^2(p-1), p_3<p-1, f\in L^{p'}(\Omega). \end{cases}$$

$$(2.4.7)\qquad \begin{cases} |g(x,z,\xi,\eta_1)-g(x,z,\xi,\eta_2)|\leq K_1|\eta_1-\eta_2| \\ K_1^2<k^2K^2, k \ as \ in \ (2.4.5),\ K \ as \ in \ (1.4.3). \end{cases}$$

Now, we can give and prove the existence result as follows.

Theorem 2.4.2. Under the hypotheses of (2.4.4)-(2.4.7), $\forall\phi\in H_0^1(\Omega)$ the problem (2.4.3) has a global strong solution

$$u\in L^\infty_{loc}((0,\infty),H_0^1(\Omega))\cap L^p_{loc}((0,\infty),W^{2,p}(\Omega))$$
$$u_t\in L^{p'}_{loc}((0,\infty),L^{p'}(\Omega))$$

Proof. We shall use Theorem 2.1.12 to prove this theorem. To this end, the spaces as in (2.1.27) and (2.1.28) are taken as follows

$$(2.4.8)\qquad \begin{cases} H=L^2(\Omega);\quad H_1=H_0^1(\Omega) \\ X_1=L^p(\Omega);\quad X_2=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) \\ X=\{u\in c^\infty(\overline{\Omega})|u|_{\partial\Omega}=0\} \end{cases}$$

and the mapping $L:X_2\to X_1$ is defined by

$$(2.4.9)\qquad Lu=-\triangle u$$

It is well known that for the operator (2.4.9), the conditions (2.1.32)(2.1.33) in Theorem 2.1.12 are fulfilled.

Define the mapping $G:X_2\to X_1^*$ by

$$<Gu,v>=\int_\Omega[-F(x,u,\nabla u,\triangle u)-g(x,u,\nabla u,D^2u)]vdx$$

$u\in X_2, v\in X_1=L^p(\Omega)$.

We need to check the condition (2.1.34). By (2.4.4),

$$\begin{aligned} <Gu,Lu> &= \int_\Omega[F(x,u,\nabla u,\triangle u)\triangle u+g(x,u,\nabla u,D^2u)\triangle u]dx \\ &\geq \int_\Omega[c_1|\triangle u|^p-c_2-\frac{c_1}{2}|\triangle u|^p-c|g(x,u,\nabla u,D^2u)|^{p'}]dx \\ &\quad (by\ (2.4.6)) \\ &\geq \frac{c_1}{2}\int_\Omega|\triangle u|^pdx-c\int_\Omega\sum_{|\alpha|\leq 2}|D^\alpha u|^{p_3p'}dx-c_3; \end{aligned}$$

Since $p_3 < p-1$, then $p_3p' = p_3p/p-1 < p$. Hence, by Lemma 1.3.10, one can derive

$$< Gu, Lv >\geq \alpha \int_\Omega |\triangle u|^p dx - \beta$$

for some $\alpha, \beta > 0$, and the condition (2.1.34) is verified.

We are now in a position to check the p-coerceiv continuity of $G : X_2 \to X_1^*$. It is clear, by (2.4.6) and Lemma 2.4.1, that G takes a bounded subset $\sum \subset L^p((0,T), W^{2,p}(\Omega)) \cap L^\infty((0,T), H_0^1(\Omega))$ to a bounded subset $G(\sum)$ in $L^{p'}((0,T)\times\Omega)$. Let $u_n \rightharpoonup u_0$ uniformly in $L^p((0,T), X_2), X_2 = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, and $\{u_n\} \subset L^\infty((0,T), H_0^1(\Omega))$ be bounded, furthermore

$$(2.4.10) \qquad 0 = \lim_{n\to\infty} \int_0^T < Gu_n - Gu_0, Lu_n - Lu_0 > dt$$

$$\begin{aligned} = & \lim_{n\to\infty} \int_0^T \int_\Omega \{[F(x,u_n,\nabla u_n,\triangle u_n) - F(x,u_0,\nabla u_0,\triangle u_0)] \times \\ & \times [\triangle u_n - \triangle u_0] + [g(x,u_n,\nabla u_n,D^2u_n) - g(x,u_0,\nabla u_0,D^2u_0)] \times \\ & \times [\triangle u_n - \triangle u_0]\} dxdt \end{aligned}$$

Make the decomposition

$$(2.4.11) \quad \int_0^T \int_\Omega [F(x,u_n,\nabla u_n,\triangle u_n) - F(x,u_0,\nabla u_0,\triangle u_0)][\triangle u_n - \triangle u_0] dxdt$$

$$\begin{aligned} = & \int_0^T \int_\Omega [F(x,u_n,\nabla u_n,\triangle u_n) - F(x,u_n,\nabla u_n,\triangle u_0)][\triangle u_n - \triangle u_0] dxdt \\ & + \int_0^T \int_\Omega [F(x,u_n,\nabla u_n,\triangle u_0) - F(x,u_0,\nabla u_0,\triangle u_0)][\triangle u_n - \triangle u_0] dxdt \end{aligned}$$

and

$$(2.4.12) \quad \int_0^T \int_\Omega [g(x,u_n,\nabla u_n,D^2u_n) - g(x,u_0,\nabla u_0,\triangle u_0)][\triangle u_n - \triangle u_0] dxdt$$

$$\begin{aligned} = & \int_0^T \int_\Omega [g(x,u_n,\nabla u_n,D^2u_n) - g(x,u_n,\nabla u_n,D^2u_0)][\triangle u_n - \triangle u_0] dxdt \\ & + \int_0^T \int_\Omega [g(x,u_n,\nabla u_n,D^2u_0) - g(x,u_0,\nabla u_0,D^2u_0)][\triangle u_n - \triangle u_0] dxdt \end{aligned}$$

Due to Lemma 2.1.20 and Lemma 2.4.1, for any bounded sequence $\{u_n\} \subset L^\infty((0,T), H_0^1(\Omega))$ with $u_n \rightharpoonup u_0$ uniformly in $L^p((0,T), X_2)$ we have

$$\nabla u_n \to \nabla u_0 \ in \ L^{q_1}((0,T)\times\Omega), \quad \forall 0 < T < \infty,$$

$$(2.4.13) \qquad u_n \to u_0 \ in \ L^{q_2}((0,T)\times\Omega), \quad \forall 0<T<\infty.$$

$$q_1 < \frac{n+2}{n}p, \quad q_2 < (\frac{n+2}{n})^2 p.$$

According to Lemma 1.4.3 and (2.4.13), from (2.4.6) we get

$$\lim_{n\to\infty}\int_0^T\int_\Omega [F(x,u_n,\nabla u_n,\triangle u_0)-F(x,u_0,\nabla u_0,\triangle u_0)][\triangle u_n-\triangle u_0]dxdt=0$$

$$\lim_{n\to\infty}\int_0^T\int_\Omega [g(x,u_n,\nabla u_n,D^2u_0)-g(x,u_0,\nabla u_0,D^2u_0)][\triangle u_n-\triangle u_0]dxdt=0$$

Consequently, from (2.4.10)-(2.4.12) we obtain

$$(2.4.14) \quad \lim_{n\to\infty}\int_0^T\int_\Omega [(F(x,u_n,\nabla u_n,\triangle u_n)-F(x,u_n,\nabla u_n,\triangle u_0))(\triangle u_n-\triangle u_0)$$
$$+(g(x,u_n,\nabla u_n,D^2u_n)-g(x,u_n,\nabla u_n,D^2u_0))(\triangle u_n-\triangle u_0)]dxdt=0$$

On the other hand, by (2.4.5) and (2.4.7) we have

$$\begin{aligned}
&\int_0^T\int_\Omega [(F(x,u_n,\nabla u_n,\triangle u_n)-F(x,u_n,\nabla u_n,\triangle u_0))(\triangle u_n-\triangle u_0)\\
&+(g(x,u_n,\nabla u_n,D^2u_n)-g(x,u_n,\nabla u_n,D^2u_0))(\triangle u_n-\triangle u_0)]dxdt\\
\geq\ &\int_0^T\int_\Omega [k|\triangle u_n-\triangle u_0|^2-K_1|D^2u_n-D^2u_0||\triangle u_n-\triangle u_0|]dxdt\\
\geq\ &\int_0^T\int_\Omega [\frac{k}{2}|\triangle u_n-\triangle u_0|^2-\frac{K_1^2}{2k}|D^2u_n-D^2u_0|^2]dxdt\\
\geq\ &\frac{k^2K^2-K_1^2}{2k}\int_0^T\int_\Omega |D^2u_n-D^2u_0|^2dxdt
\end{aligned}$$

Hence from (2.4.14) we get

$$\lim_{n\to\infty}\int_0^T\int_\Omega |D^2u_n-D^2u_0|^2dxdt=0$$

Therefore, from Lemma 1.4.4 and condition (2.4.6) it follows

$$\lim_{n\to\infty} < Gu_n, v > = < Gu_0, v >, \quad \forall v\in X_1=L^p(\Omega).$$

i.e. $G: X_2\to X_1^*$ is p-coerceivly continuous. Thus, by Theorem 2.1.12, the problem (2.4.3) has a global solution

$$u\in L^\infty_{loc}((0,\infty),H_0^1(\Omega))\cap L^p_{loc}((0,\infty),W^{2,p}(\Omega))$$

Finally we show that the solution of (2.4.3) belongs to $W^{1,p'}_{loc}((0,\infty), L^{p'}(\Omega))$. It is known that u satisfies

$$\int_\Omega u(x,t)vdx - \int_0^t \int_\Omega F(x,u,\nabla u, \triangle u)vdxd\tau$$

$$= \int_\Omega \phi(x)vdx + \int_0^t \int_\Omega g(x,u,\nabla u, D^2u)vdxd\tau,$$

$\forall v \in L^p(\Omega)$ and $0 < t < \infty$. Then for any $h > 0$ we have

$$\int_\Omega \frac{u(t+h)-u(t)}{h} vdx = \frac{1}{h}\int_t^{t+h} \int_\Omega F(x,u,\nabla u, \triangle u)vdxd\tau \tag{2.4.15}$$

$$+\frac{1}{h}\int_t^{t+h} \int_\Omega g(x,u,\nabla u, D^2u)vdxd\tau$$

Inserting

$$v = |\frac{u(t+h)-u(t)}{h}|^{p'-2}(\frac{u(t+h)-u(t)}{h})$$

in (2.4.15) one reads

$$\int_\Omega |\frac{u(t+h)-u(t)}{h}|^{p'}dx \le \frac{1}{h}\int_t^{t+h} \int_\Omega |F(x,u,\nabla u, \triangle u)+$$

$$+g(x,u,\nabla u, D^2u)||\frac{u(t+h)-u(t)}{h}|^{\frac{1}{p-1}}dxd\tau$$

$$\le \frac{1}{2}\int_\Omega |\frac{u(t+h)-u(t)}{h}|^{p'}dx + 2^{\frac{1}{p-1}}\frac{1}{h}\int_t^{t+h} \int_\Omega |F+g|^{p'}dxd\tau$$

namely

$$\int_\Omega |\frac{u(t+h)-u(t)}{h}|^{p'}dx \le 2^{p'}\int_0^1 \int_\Omega [|F(x,u,\nabla u, \triangle u(t+sh))$$

$$+g(x,u,\nabla u, D^2u(t+sh))|^{p'}dxds$$

Due to Lemma 2.4.1 and (2.4.6) one gets

$$\int_0^T \int_\Omega |\frac{u(t+h)-u(t)}{h}|^{p'}dxdt \le c\int_0^T \int_0^1 \int_\Omega [|F(x,u,\nabla u, \triangle u(t+sh))|^{p'}+$$

$$+|g(x,u,\nabla u, D^2u(t+sh))|^{p'}]dxdsdt$$

$$\le c\int_0^{t+h} \int_\Omega [|D^2u|^p + |\nabla u|^{\frac{n+2}{n}p} + |u|^{(\frac{n+2}{n})^2 p} + 1]dxdt$$

Hence we obtain that $u_t \in L^{p'}((0,T)\times\Omega)\forall 0<T<\infty$. The proof is complete.

Next, let us investigate the problem given by

$$(2.4.16)\qquad \begin{cases} \frac{\partial u}{\partial t} - F(x,u,\nabla u, D^2 u) = g(x,u,\nabla u), \quad x\in\Omega \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi(x) \end{cases}$$

The following conditions are imposed.

$$(2.4.17)\qquad \int_\Omega F(x,u,\nabla u, D^2u)\triangle u dx \geq c_1 \int_\Omega |\triangle u|^p dx - c - 2,$$

$$\forall u \in C^\infty(\Omega)\cap H_0^1(\Omega).$$

$$(2.4.18)\qquad \int_\Omega [F(x,u,\nabla u_1, D^2 u_1) - F(x,u_2,\nabla u_2, D^2 u_2)]\times$$

$$\times[\triangle u_1 - \triangle u_2]dx \geq 0, \quad \forall u_1,u_2\in c^\infty(\Omega)\cap H_0^1(\Omega).$$

$$(2.4.19)\qquad \begin{cases} |F(x,z,\xi,\eta)| \leq c[|z|^{p_2} + |\xi|^{p_1} + |\eta|^{p-1} + 1] \\ |g(x,z,\xi)| \leq c[|z|^{p_2} + |\xi|^{p_1} + 1] \\ p_1 < \frac{n+2}{n}(p-1), \quad p_2 < (\frac{n+2}{n})^2(p-1) \end{cases}$$

Theorem 2.4.3. Under the assumptions (2.4.17)-(2.4.19), for any $\phi \in H_0^1(\Omega)$ the problem (2.4.16) has a global solution

$$u \in L^\infty_{loc}((0,\infty), H_0^1(\Omega)) \cap L^p_{loc}((0,\infty), W^{2,p}(\Omega))$$

$$u_t \in L^{p'}_{loc}((0,\infty), L^{p'}(\Omega)).$$

The proof of Theorem 2.4.3 is parallel to that of Theorem 2.4.2 by using Theorem 2.1.15 and Remark 2.1.18 instead of Theorem 2.1.12. Here we omit the details of proof.

2.4.3. Global existence and regularity of fully nonlinear second initial boundary value problems

In the following, we shall investigate the global existence of H^3-strong solutions for the second initial boundary value problem given by

$$(2.4.20)\quad \begin{cases} \frac{\partial u}{\partial t} - f(x,u,\nabla u,\triangle u) - g(x,u,\nabla u) = 0 \; mod(\alpha(t), \alpha\in c[0,\infty)) \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \int_\Omega u(x,t)dt = 0, \quad \forall t\geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

We shall apply Theorem 2.1.10 and Theorem 2.1.11 to discuss the problem (2.4.20). The spaces as in (2.1.27) and (2.1.28) is taken as to write

$X = \{u \in c^\infty(\overline{\Omega}) \quad |\frac{\partial \triangle^m u}{\partial n}|_{\partial\Omega} = 0, m = 1, 2, \cdots; \quad \int_\Omega udx = 0\}$

$X_1 = L^p(\Omega)/R;$

X_2=the completion of X with the norm

$$\|u\|_{X_2} = [\int_\Omega |\nabla \triangle u|^2 dx]^{\frac{1}{2}} + \|u\|_{W^{2,p}};$$

$X_3 = \{u \in W^{3,2s}(\Omega) \cap W^{2,2s'(p-1)}(\Omega)| \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \int_\Omega udx = 0\};$

$H = L^2(\Omega)/R; \quad H_1 = H^1(\Omega)/R;$

$H_2 = \{u \in H^2(\Omega)| \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \int_\Omega udx = 0\};$

$H_3 = \{u \in H^{2m}(\Omega)| \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = \cdots = \frac{\partial \triangle^{m-1} u}{\partial n}|_{\partial\Omega} = 0, \quad \int_\Omega udx = 0\}$

with the inner product

$$< u, v >_{H_3} = \int_\Omega \triangle^m u \cdot \triangle^m v dx.$$

By Lemma 1.3.10, the norm of H_3 is equivalent to the H^{2m}-norm, hence there is a $m \geq 2$ such that $H_3 \hookrightarrow X_3$.

The linear mappings $L_1 : H_2 \to H$ and $L_2 : X \to X_1$ are defined by

$$\begin{cases} L_1 u = -\triangle u \\ L_2 u = \triangle^2 u - \triangle u \end{cases} \tag{2.4.21}$$

It is known that the eigenvalue problem

$$\begin{cases} -\triangle u = \lambda u \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \int_\Omega udx = 0 \end{cases}$$

has an infinite eigenvalue sequence $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and the eigenfunctions $\{e_i\} \subset X$ consist of normal orthogonal base of H (Cf.[Sm]). Moreover, it is easy to see that $\{e_i\}$ is also an orthogonal base of H_3. Hence the conditions (2.1.24)(2.1.25) and (2.1.29) are satisfied.

Suppose that $g \in c^1(\overline{\Omega} \times R \times R^n), f \in c^1(\overline{\Omega} \times R \times R^n \times R)$, and

$$\begin{cases} f'_y(x, z, \xi, y) \geq \alpha > 0 \\ f(x, z, \xi, y)y \geq c_1|y|^p - c_2, \quad p \geq 2 \end{cases} \tag{2.4.22}$$

$$\begin{cases} |f(x, z, \xi, y)| \leq c[|z|^{p-1} + |\xi|^{p-1} + |y|^{p-1} + 1] \\ |D_x f| \leq c[|z|^{p_1} + |\xi|^{p_1} + |y|^{p_1} + 1], \quad p_1 < \frac{p}{2}, p > 2 \\ |D_z f| + |D_\xi f| \leq c[|z|^{p_2} + |\xi|^{p_2} + |y|^{p_2} + 1], \quad p_2 < \frac{p}{2} - 1, p > 2 \\ \text{and } p_1 = 1, \quad p_2 = 0 \text{ as } p = 2 \end{cases} \tag{2.4.23}$$

$$(2.4.24) \qquad \begin{cases} |g|+|D_xg| \le c[|z|^{p_1}+|\xi|^{p_1}+1] \\ |D_zg|+|D_\xi g| \le c[|z|^{p_2}+|\xi|^{p_2}+1] \end{cases}$$

Theorem 2.4.4. Let the conditions (2.4.22)-(2.4.23) be satisfied. Then for any $\phi \in H^2(\Omega)/R$ with $\frac{\partial \phi}{\partial n}|_{\partial\Omega} = 0$, the problem (2.4.20) has a global strong solution

$$u \in L^\infty_{loc}((0,\infty), H^2(\Omega)) \cap L^p_{loc}((0,\infty), W^{2,p}(\Omega)) \cap L^2_{loc}((0,\infty), H^3_{loc}(\Omega))$$

$$u_t \in L^{p'}_{loc}((0,\infty), L^{p'}(\Omega)).$$

In addition, in the conditions (2.4.23) and (2.4.24), if we make the below revisions

$$(2.4.25) \qquad |D_zf|+|D_\xi f|+|D_zg|+|D_\xi g| \le c$$

then for any $\phi \in X_3$, the solution of (2.4.20) is unique, which has the further regularity

$$u_t \in L^\infty_{loc}((0,\infty), H^1(\Omega)) \cap L^2_{loc}((0,\infty), H^2(\Omega)).$$

Proof. Define the mapping $G : X_2 \to X_1^*$ by

$$< Gu, v >= \int_\Omega [-f(x,u,\nabla u, \triangle u) - g(x,u,\nabla u)]v dx$$

By Lemma 2.1.20 and Lemma 1.4.4, from (2.4.23) and (2.4.24) it follows that $G : X_2 \to X_1^*$ is p-weakly continuous.

By (2.4.22)-(2.4.24) and Lemma 1.3.10 we can obtain

$$\begin{aligned} < Gu, L_2u > &= \int_\Omega [\nabla f(x,u,\nabla u,\triangle u)\nabla\triangle u + \nabla g(x,u,\nabla u)\nabla\triangle u \\ &\quad + f(x,u,\nabla u,\triangle u)\triangle u + g(x,u,\nabla u)\triangle u]dx \\ &\ge \int_\Omega [\alpha|\nabla\triangle u|^2 + c_1|\triangle u|^p - c_2 - |g||\triangle u| - |D_xf||\nabla\triangle u| \\ &\quad -|D_zf||\nabla\triangle u||Du| - |D_\xi f||D^2u||\nabla\triangle u| - |D_xg||\nabla\triangle u| \\ &\quad -|D_zg||Du||\nabla\triangle u| - |D_\xi g||D^2u||\nabla\triangle u|]dx \\ &\ge \lambda\int_\Omega [|\nabla\triangle u|^2 + |\triangle u|^p]dx - c\int_\Omega [\sum_{|\beta|\le 2} |D^\beta u|^q + 1]dx \end{aligned}$$

for some constants $\lambda > 0, c > 0$ and $q < p$, which implies that the condition (2.1.26) in Theorem 2.1.10 is checked. Consequently by Theorem 2.1.10, the problem (2.4.20) has a global solution

$$u \in L^\infty_{loc}((0,\infty), H^2(\Omega)) \cap L^p_{loc}((0,\infty), W^{2,p}(\Omega)) \cap L^2_{loc}((0,\infty), H^3_{loc}(\Omega))$$

which satisfies

$$\int_\Omega u\cdot vdx = \int_0^t\int_\Omega [f(x,u,\nabla u,\triangle u)+g(x,u,\nabla u)]vdxdt$$
$$+\int_\Omega \phi\cdot vdx; \quad \forall v\in L^p(\Omega)/R, \quad 0\le t<\infty.$$

Then for any $h>0$ we have

$$\int_\Omega \frac{u(t+h)-u(t)}{h}vdx \tag{2.4.26}$$
$$=\frac{1}{h}\int_0^{t+h}\int_\Omega [f(x,u,\nabla u,\triangle u)+g(x,u,\nabla u)]vdx$$

Putting

$$v=|\frac{u(t+h)-u(t)}{h}|^{p'-2}(\frac{u(t+h)-u(t)}{h})-b$$
$$b=\int_\Omega|\frac{u(t+h)-u(t)}{h}|^{p'-2}(\frac{u(t+h)-u(t)}{h})dx$$

in (2.4.26), and notecing that

$$\int_\Omega \frac{u(t+h)-u(t)}{h}\cdot bdx=0, \quad \forall t\ge 0$$

one can deduce, in the same fashion as the proof of Theorem 2.4.2, that $u_t\in L^{p'}((0,T)\times\Omega), \forall 0<T<\infty$.

In the following, we shall use Theorem 2.1.11 to prove the second conclusions. We see that

$$\begin{aligned}<DG(u)v,L_1v> &= \int_\Omega [f'_y(x,u,\nabla u,\triangle u)\triangle v+D_zf(x,u,\nabla u,\triangle u)v\\ &\quad +D_\xi f(x,u,\nabla u,\triangle u)Dv+D_zg(x,u,\nabla u)v\\ &\quad +D_\xi g(x,u,\nabla u)Dv]\triangle vdx\\ &\ge \alpha\int_\Omega|\triangle v|^2dx-\int_\Omega[|D_zf||v||\triangle v|-|D_\xi f||\nabla v||\triangle v|\\ &\quad -|D_zg||v||\triangle v|-|D_\xi g||\nabla v||\triangle v|]dx\\ &\ge \frac{\alpha}{2}\int_\Omega|\triangle v|^2dx-c\int_\Omega[|\nabla v|^2+|v|^2]dx, \ (by\ (2.4.25)).\end{aligned}$$

which implies that the condition (2.1.30) in Theorem 2.1.11 is satisfied. Finally we conclude the proof by verifying the condition (2.1.31). For any $u,v\in X$, we have

$$\begin{aligned}
|<Gu,L_1v>| &= |\int_\Omega \nabla[f(x,u,\nabla u,\triangle u)+g(x,u,\nabla u)]\cdot\nabla v dx| \\
&\leq \frac{1}{2}\int_\Omega |\nabla v|^2dx+\frac{1}{2}\int_\Omega |\nabla f(x,u,\nabla u,\triangle u)+\nabla g(x,u,\nabla u)|^2dx \\
&\leq \frac{1}{2}\int_\Omega |\nabla v|^2dx+c\int_\Omega [|f'_y|^2|\nabla\triangle u|^2+|D_xf|^2+|D_zf|^2|\nabla u|^2 \\
&\quad +|D_\xi f|^2||D^2u|^2+|D_xg|^2+|D_zg|^2|\nabla u|^2+|D_\xi g|^2|D^2u|^2]dx \\
&\leq \frac{1}{2}\int_\Omega |\nabla v|^2dx+c\int_\Omega |D^3u|^{2s}+|D^2u|^{2s'(p-1)}+ \\
&\quad +\sum_{|\alpha|\leq 1}|D^\alpha u|^2+1]dx
\end{aligned}$$

Thus, the condition (2.1.31) is checked. The proof is complete.

2.4.4. Uniform boundedness of fully nonlinear problems

In this subsection, we shall apply Theorem 2.2.3 and Theorem 2.2.6 to discuss the uniform boundedness of global solutions for the fully nonlinear parabolic initial boundary value problem.

First we use Theorem 2.2.6 to investigate the problem given by

$$\begin{cases} \frac{\partial u}{\partial t}-f(x,\triangle u)=g(x), \quad x\in\Omega\subset R^n \\ u|_{\partial\Omega}=0 \\ u(x,0)=\phi(x) \end{cases} \tag{2.4.27}$$

Denote by

$$F(x,y)=\int_0^y f(x,s)ds \tag{2.4.28}$$

Suppose that

$$F(x,y)\geq c_1|y|^p-c_2, \quad p\geq 2 \tag{2.4.29}$$

$$(f(x,y_1)-f(x,y_2))(y_1-y_2)\geq 0 \tag{2.4.30}$$

$$|f(x,y)|\leq c[|y|^{p-1}+1] \tag{2.4.31}$$

The spaces as in (2.1.27) and (2.1.28) are taken by

$$X=\{u\in c^\infty(\overline{\Omega})| \quad \triangle^k u|_{\partial\Omega}=0, k=0,1,\cdots,\}$$

$$X_1 = L^p(\Omega); \quad X_2 = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega); \tag{2.4.32}$$

$$H = L^2(\Omega); \quad H_1 = H_0^1(\Omega); \ and$$

$$H_3 = \{u \in H^{2m}(\Omega) | \quad u|_{\partial\Omega} = 0, \cdots, \triangle^{m-1} u|_{\partial\Omega} = 0\}$$

with the inner product

$$< u, v >_{H_3} = \int_\Omega \triangle^m u \cdot \triangle^m v dx$$

where $m \geq 1$ is taken such that $H_3 \hookrightarrow X_2$.

The linear mapping $L : X_2 \to X_1$ is taken as in (2.4.9).

Theorem 2.4.5. Under the conditions (2.4.29)-(2.4.31), for any $\phi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $g \in L^{p'}(\Omega)$, the problem (2.4.27) has a global strong solution

$$u \in L^\infty((0, \infty), W^{2,p}(\Omega))$$

$$u_t \in L^2((0, \infty), H_0^1(\Omega))$$

moreover, the solutions of (2.4.27) are uniformly bounded in $W^{2,p}(\Omega)$.

Proof. Define the mapping $G : X_2 \to X_1^*$ by

$$< Gu, v >= \int_\Omega [-f(x, \triangle u) - g(x)] v dx, \quad u \in X_2, \quad v \in X_1 \tag{2.4.33}$$

According to the condition (2.4.31), $G : X_2 \to X_1^*$ is bounded and continuous. It is easy to see that for the c^1 functional $F_1 : X_2 \to R$ defined by

$$F_1(u) = \int_\Omega [F(x, \triangle u) + g(x) \triangle u] dx$$

we have

$$\begin{aligned} < Gu, Lu > &= \int_\Omega [f(x, \triangle u) + g(x)] \triangle v dx \\ &= < DF_1 u, v >_{X_2} \end{aligned}$$

Thus, by (2.4.29), the conditions (2.2.21) and (2.2.22) in Theorem 2.2.6 are fulfilled.

Due to (2.4.30) we see that

$$< Gu - Gv, Lu - Lv >= \int_\Omega [f(x, u) - f(x, v)][\triangle u - \triangle v] dx$$

$$\geq 0, \quad \forall u, v \in X_2 = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Hence, by Theorem 2.2.6, this theorem is proved.

Next, we apply Theorem 2.2.3 to consider the uniform boundedness for the problem given by

$$\text{(2.4.34)} \qquad \begin{cases} \frac{\partial u}{\partial t} - f(x, \triangle u) = g(x), & x \in \Omega \\ u|_{\partial\Omega} = 0, \quad \triangle u|_{\partial\Omega} = 0 \\ u(x,0) = \phi \end{cases}$$

We define $F(x,y)$ as in (2.4.28), and assume that $f \in c^1(\overline{\Omega} \times R), f(\cdot, y) \in c^2(\overline{\Omega}) \forall y \in R, f(x,0) = 0$, and

$$\text{(2.4.35)} \qquad f'_y(x,y) \geq \alpha > 0$$

$$\text{(2.4.36)} \qquad \begin{cases} F(x,y) \geq c_1|y|^p - c_2, p \geq 2 \\ f(x,y)y - \triangle_x F(x,y) \geq c_3|y|^p - c_4 \end{cases}$$

$$\text{(2.4.37)} \qquad |f(x,y)| \leq c[|y|^{p-1} + 1]$$

where

$$\triangle_x F(x,y) = \frac{\partial^2 F(x,y)}{\partial x_1^2} + \cdots + \frac{\partial^2 F(x,y)}{\partial x_n^2}.$$

Theorem 2.4.6. Let the conditions (2.4.35)-(2.4.37) are fulfilled. Then for any $\phi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $g \in W_0^{1,2}(\Omega)$ the problem (2.4.34) has an unique global strong solution

$$u \in L^\infty((0,\infty), W^{2,p}(\Omega)) \cap L^2_{loc}((0,\infty), H^3(\Omega))$$

$$u_t \in L^2((0,\infty), H_0^1(\Omega))$$

moreover, the solutions of (2.4.34) are uniformly bounded in $W^{2,p}(\Omega)$.

Proof. The spaces as in (2.2.11) and (2.2.12) are taken by (2.4.32) and

$$\text{(2.4.38)} \qquad \begin{cases} X_3 = \{u \in H^3(\Omega) \cap W^{2,p}(\Omega) | \quad u|_{\partial\Omega} = 0, \triangle u|_{\partial\Omega} = 0\} \\ H_2 = H^2(\Omega) \cap H_0^1(\Omega) \end{cases}$$

and $m \geq 2$ in (2.4.32) is taken such that $H_3 \hookrightarrow X_3$.

The linear mappings $L_1 : X_2 \to X_1$ and $L_2 : X \to X_1$ are define as (2.4.21). The mapping $G : X_3 \to X_1^*$ is defined as (2.4.33) for $u \in X_3, v \in X_1 = L^p(\Omega)$.

Let the functional $\widetilde{F} : X_2 \to R$ be defined by

$$\widetilde{F}(u) = \int_\Omega [F(x, \triangle u) + g(x)\triangle u]dx$$

By the condition (2.4.37), it is clear that $G : X_3 \to X_1^*$ is weakly continuous and $\widetilde{F} : X_2 \to R$ is c^2.

By the condition (2.4.36) it is easy to see that the condition (2.2.16) and (2.2.17) in Theorem 2.2.3 are fulfilled. We are now in a position to check the condition (2.2.18).

Because $f(x, 0) = 0$ and $g \in W_0^{1,2}(\Omega), \forall u \in X$, we have

$$\text{(2.4.39)} \qquad \int_0^t < Gu, L_2 u > dt = \int_0^t \int_\Omega [-f(x, \triangle u) - g][\triangle^2 u - \triangle u]dxdt$$

$$= \int_0^t \int_\Omega [f'_y(x, u)|\nabla\triangle u|^2 + f(x, \triangle u)\triangle u +$$
$$+\nabla_x f(x, \triangle u)\triangle u + \nabla g \cdot \nabla\triangle u + g \cdot \triangle u]dxdt$$

On the other hand, $\forall u \in X$

$$\int_\Omega \nabla_x f(x, \triangle u)\nabla\triangle u dx = \int_\Omega \sum_{i=1}^n \frac{\partial f(x, y)}{\partial x_i}\frac{\partial \triangle u}{\partial x_i}|_{y=\triangle u} dx$$

$$= \int_\Omega [div G(x) - \triangle x F(x, \triangle u)]dx$$

where

$$G(x) = \{G_1(x), \cdots, G_n(x)\},$$
$$G_i(x) = \int_0^{\triangle u(x)} \frac{\partial f(x, z)}{\partial x_i}dz$$

Because $\triangle u|_{\partial\Omega} = 0, G|_{\partial\Omega} = 0$, we have

$$\int_\Omega D_x f(x, \triangle u)\nabla\triangle u dx = -\int_\Omega \triangle_x F(x, \triangle u)dx$$

Consequently, from (2.4.39) it follows

$$\int_0^t < Gu, L_2 u > dt = \int_0^t \int_\Omega [f'_y(x, \triangle u)|\nabla\triangle u|^2 +$$
$$+f(x, \triangle u)\triangle u - \triangle_x F(x, \triangle u) + \nabla g \cdot \nabla\triangle u + g \cdot \triangle u]dxdt$$
$$\geq \int_0^t \int_\Omega [\frac{\alpha}{2}|\nabla\triangle u|^2 + \frac{c_3}{2}|\triangle u|^p - \frac{2}{\alpha}|\nabla g|^2 - c|g|^{p'} - c_4]dxdt$$
$$\geq c_1 \int_0^t \int_\Omega [|\nabla\triangle u|^2 + |\triangle u|^p]dxdt - c_2$$

which implies, by Lemma 1.3.10 and Lemma 1.4.1, that the condition (2.2.23) is satisfied. By Theorem 2.2.6, it remains to verify the uniqueness, and which can be derived from (2.4.34) and (2.4.35) in the some fashion as the proof of Theorem 2.1.9. The proof is complete.

2.4.5. Further regularity and uniform boundedness

Let us continue to discuss the further regularity and uniform boundedness of fully nonlinear parabolic initial boundary value problems.

We first consider the regularity of global solutions of the problem given by

$$\begin{cases} \frac{\partial u}{\partial t} - f(x, \triangle u) = g(x, u, \nabla u), \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \\ u(x, 0) = \phi(x) \end{cases} \tag{2.4.40}$$

Let $F(x, y)$ be defined as (2.4.28). Assume that $f \in c^1(\overline{\Omega} \times R), g \in c^1(\overline{\Omega} \times R \times R^n), f(x, 0) = 0$, and

$$g(x, u, \nabla u)|_{\partial\Omega} = 0, \quad \forall u \in H_0^1(\Omega) \tag{2.4.41}$$

$$\begin{cases} f'_y(x, y) \geq \alpha > 0 \\ F(x, y) \geq c_1|y|^p - c_2, \quad p \geq 2 \end{cases} \tag{2.4.42}$$

$$\begin{cases} |f(x, y)| \leq c[|y|^{p-1} + 1] \\ |D_x f(x, y)| \leq c[|y|^{\frac{p}{2}} + 1] \end{cases} \tag{2.4.43}$$

$$\begin{cases} |g| + |D_x g| \leq c[|z|^{\frac{p}{2}} + |\xi|^{\frac{p}{2}} + 1] \\ |D_z g| + |D_\xi g| \leq c[|z|^{\frac{p}{2}-1} + |\xi|^{\frac{p}{2}-1} + 1] \end{cases} \tag{2.4.44}$$

Theorem 2.4.7. Let the conditions (2.4.41)-(2.4.44) hold. Then for any $\phi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, the problem (2.4.40) has a global solution

$$u \in L_{loc}^\infty((0, \infty), W^{2,p}(\Omega)) \cap L_{loc}^2((0, \infty), H^3(\Omega))$$

$$u_t \in L_{loc}^2((0, \infty), H_0^1(\Omega))$$

Furthermore, if the condition (2.4.44) is strenthened by

$$\begin{cases} |g| + |D_x g| \leq c[|z| + |\xi| + 1] \\ |D_z g| + |D_\xi g| \leq c \end{cases} \tag{2.4.44$'$}$$

then for any $\phi \in W^{3,2s}(\Omega) \cap W^{2,2s'(p-2)}(\Omega) \cap H_0^1(\Omega) \quad (1 < s < \infty) \quad$ (as $p = 2, \quad \phi \in H^3(\Omega) \cap H_0^1(\Omega)$), the solution of (2.4.40) is unique which has the further regularity

$$u_t \in L^\infty_{loc}((0,\infty), H_0^1(\Omega)) \cap L^2_{loc}((0,\infty), H^2(\Omega).$$

Proof. We shall use Theorem 2.2.6 and Theorem 2.2.7 to prove this theorem. To this end, the spaces as in (2.1.27) and (2.1.28) are taken as (2.4.32) and

$$\begin{cases} H_3 = H^2(\Omega) \cap H_0^1(\Omega) \\ X_3 = \{u \in W^{3,2s}(\Omega) \cap W^{2,2s'(p-2)}(\Omega) | \quad u|_{\partial\Omega} = 0, \triangle u|_{\partial\Omega} = 0\} \\ \ as\ p = 2, X_3 = \{u \in H^3(\Omega) | \quad u|_{\partial\Omega} = 0, \triangle u|_{\partial\Omega} = 0\} \end{cases}$$

The linear mapping $L : X_2 \to X_1$ is defined as (2.4.9) and the mapping $G = A + B : X_2 \to X_1^*$ is defined by

$$< Au, v >= \int_\Omega -f(x, \triangle u) v dx$$

$$< Bu, v >= \int_\Omega -g(x, u, \nabla u) v dx.$$

We define the functional $F_1 : X_2 \to R$ by

$$F_1(u) = \int_\Omega F(x, \triangle u) dx$$

where $F(x, y)$ is as in (2.4.28).

By (2.4.42), the conditions (2.2.21)(2.2.22) and (2.2.24) is readily checked. And by (2.4.44) it is east to see that $B : X_2 \to X_1^*$ is a compact mapping, hence for any $u_n \rightharpoonup^* u_0$ in $L^\infty((0,T), W^{2,p}(\Omega))$ we have

$$\lim_{n\to\infty} \int_0^T \int_\Omega g(x, u_n, \nabla u_n) \triangle u_n dx = \int_0^T \int_\Omega g(x, u_0, \nabla u_0) \triangle u_0 dx$$

$$\lim_{n\to\infty} \int_0^T \int_\Omega g(x, u_n, \nabla u_n) v dx = \int_0^T \int_\Omega g(x, u_0, \nabla u_0) v dx$$

$\forall v \in L^p(\Omega)$. We now need to check the condition (2.2.23). By (2.4.41) we get

$$\begin{aligned} & | < Bu, Lv > | = | \int_\Omega \nabla g(x, u, \nabla u) \nabla v dx | \\ \leq \ & \frac{1}{2} \int_\Omega |\nabla v|^2 dx + c \int_\Omega [|D_x g|^2 + |D_z g|^2 ||\nabla u|^2 + |D_\xi g|^2 |D^2 u|^2] dx \\ \leq \ & \frac{1}{2} \|v\|_{H_1}^2 + c \int_\Omega |D^2 u|^p dx + c \end{aligned}$$

which implies, by (2.4.42), that the condition (2.2.23) is satisfied. Thus, by Theorem 2.2.6, the problem (2.4.40) has a global solution

$$u \in L^\infty_{loc}((0,\infty), W^{2,p}(\Omega))$$
$$u_t \in L^2_{loc}((0,\infty), H^1_0(\Omega))$$

which satisfies

$$(2.4.45) \qquad \begin{cases} -f(x,\triangle u) = g(x,u,\nabla u) - \frac{\partial u}{\partial t}, & a.e(t,x) \in [0,\infty)\times\Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

In the following, we show that $u \in L^2_{loc}((0,\infty), H^3(\Omega))$. Let $|h| > 0$ be small enough, and denote by

$$D^h_i u = \frac{u(x+he_i) - u(x)}{h}, \quad e_i = \{\delta_{i1},\cdots,\delta_{in}\}$$

where δ_{ij} is the Kronecker symbal. From (2.4.45) we get

$$(2.4.46) \qquad -f_h(x,\triangle u)\triangle D^h_i u = D^h_i g(x,u,\nabla u) + f^h_i(x,\triangle u) - \frac{\partial D^h_i u}{\partial t}$$

$\forall x \in \widetilde{\Omega}$, here $\widetilde{\Omega} \subset\subset \Omega$ and $\text{dist}(\widetilde{\Omega},\partial\Omega) \geq h$, and

$$f_h(x,\triangle u) = \int_0^1 f'_y(x, s\triangle u(x+he_i) + (1-s)\triangle u(x))ds$$
$$f^h_i(x,\triangle u) = \frac{f(x+he_i,\triangle u) - f(x,\triangle u)}{h}$$

Multiplying the both side of (2.4.46) by $-\triangle D^h_i u$, and integrating it on $(0,T)\times\widetilde{\Omega}$, from (2.4.41) it follows

$$\begin{aligned} & \alpha\int_0^T\int_{\widetilde{\Omega}} |\triangle D^h_i u|^2 dxdt \leq \int_0^T\int_{\widetilde{\Omega}} [|\frac{\partial D^h_i u}{\partial t}||\triangle D^h_i u| + \\ & +|D^h_i g||\triangle D^h_i u| + |f^h_i|\cdot|\triangle D^h_i u|]dxdt \\ \leq\ & \frac{\alpha}{2}\int_0^T\int_{\widetilde{\Omega}} |\triangle D^h_i u|^2 dxdt + c\int_0^T\int_{\widetilde{\Omega}} [|\frac{\partial D^h_i u}{\partial t}|^2 \\ & +|D^h_i g|^2 + |f^h_i(x,\triangle u)|^2]dxdt \end{aligned}$$

According to (2.4.43)(2.4.44), we have

$$\begin{aligned} & \frac{\alpha}{2}\int_0^T\int_{\widetilde{\Omega}} |\triangle D^h_i u|^2 dxdt \leq c\int_0^T\int_{\Omega} [|\frac{\partial^2 u}{\partial t\partial x_i}|^2 + \\ & +|D_x g|^2 + |D_z g|^2|Du|^2 + |D_\xi g|^2|D^2u|^2 + |D_x f|^2]dxdt \\ \leq\ & c\int_0^T\int_\Omega [|\frac{\partial}{\partial t}\frac{\partial u}{\partial x_i}|^2 + \sum_{|\alpha|\leq 2}|D^\alpha u|^p + 1]dxdt \end{aligned}$$

where the constant $c > 0$ is independent of $\widetilde{\Omega}$ and h. Thus we obtain

$$\int_0^T\int_\Omega |\nabla\triangle u|^2 dxdt \leq c\int_0^T\int_\Omega [|\frac{\partial}{\partial t}\nabla u|^2 + \sum_{|\alpha|\leq 2}|D^\alpha u|^p + 1]dxdt$$

From Lemma 1.4.1, we derive the first one conclusion.

The proof of the second conclusions is parallel to that of Theorem 2.4.4 by using Theorem 2.2.7 instead of Theorem 2.1.11, here we omit the details. The proof is complete.

By using the same method, for the below problem, which is a special form of (2.4.20)

$$\text{(2.4.47)} \qquad \begin{cases} \frac{\partial u}{\partial t} - f(x, \triangle u) = g(x, u, \nabla u), & mod(\alpha(t)) \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \int_\Omega u dx = 0 \\ u(x, 0) = \phi(x) \end{cases}$$

we can also obtain the further regularity theorem.

Theorem 2.4.8. Under the conditions (2.4.42)-(2.4.44), for any $\phi \in X_2 = \{u \in W^{2,p}(\Omega)| \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \int_\Omega u dx = 0\}$, the solution of (2.4.27) has the further regularity

$$u \in L^\infty_{loc}((0,\infty), W^{2,p}(\Omega))$$
$$u_t \in L^2_{loc}((0,\infty), H^1(\Omega)).$$

Next, we shall discuss the H^3-uniform boundedness of global solutions of (2.4.34). We make the further assumptions as follows

$$\text{(2.4.48)} \qquad \begin{cases} |f'_y(x, y)| \leq c[|y|^{p-2} + 1] \\ |D_x f(x, y)| \leq c[|y|^{\frac{p}{2}} + 1] \end{cases}$$

Denote by

$$X_4 = \begin{cases} \{v \in W^{3,2s}(\Omega) \cap W^{2,2s(p-2)}(\Omega)| \quad v|_{\partial\Omega} = \triangle v|_{\partial\Omega} = 0\}, \ as\ p > 2, \\ \{v \in H^3(\Omega)| \quad v|_{\partial\Omega} = \triangle v|_{\partial\Omega} = 0\}, \ as\ p = 2. \end{cases}$$

Theorem 2.4.9. Under the hypotheses of Theorem 2.4.6, if (2.4.48) is also satisfied, then for any $\phi \in X_4$ and $g \in W_0^{1,2}(\Omega)$, the solution of (2.4.34) has the further regularity

$$u \in L^\infty((0,\infty), H^3(\Omega))$$

$$u_t \in L^\infty((0,\infty), H_0^1(\Omega)) \cap L^2((0,\infty), H^2(\Omega))$$

moreover, for any bounded subset $\sum \subset X_4$, the solutions of (2.4.34) are uniformly bounded for (u_t, u) in $H_0^1(\Omega) \times [H^3(\Omega) \cap W^{2,p}(\Omega)]$ provided $\phi \in \sum$.

Proof. By the conditions (2.4.35) and (2.4.48), it is easy to see that the conditions of Theorem 2.2.4 are satisfied. Hence, by Theorem 2.2.4, the solutions of (2.4.34) have the regularity

$$u_t \in L^\infty((0,\infty), H_0^1(\Omega)) \cap L^2((0,\infty), H^2(\Omega))$$

and (u_t, u) are uniformly bounded in $H_0^1(\Omega) \times W^{2,p}(\Omega)$. Then, in the same manner as the proof of Theorem 2.4.7, from the following equation

$$-f(x, \triangle u) = g(x) - \frac{\partial u}{\partial t},\ a.e(t,x) \in [0,\infty) \times \Omega$$

one can obtain the uniform boundedness of u in $H^3(\Omega)$. The proof is complete.

Similarly, for the below second initial boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} - f(\triangle u) = g(x),\ mod(\alpha(t)) \\ u|_{\partial\Omega} = 0, \quad \int_\Omega u dx = 0 \\ u(x,0) = \phi \end{cases}$$

under the conditions (2.4.35)-(2.4.37) and (2.4.48) one can also derive the uniform boundedness of (u_t, u) in $H^1(\Omega) \times [W^{2,p}(\Omega) \cap \widetilde{H}^3(\Omega)]$ provided $\phi \in \sum \subset X_4$, where $\widetilde{H}^3(\Omega)$ is the completion of the space

$$N = \{u \in H^3(\Omega)| \quad \frac{\partial u}{\partial n}_{\partial\Omega} = 0, \int_\Omega u dx = 0\}$$

with the norm

$$[\int_\Omega (|\nabla \triangle u|^2 + |\triangle u|^2) dx]^{\frac{1}{2}}$$

and $X_4 = \{u \in W^{3,2s}(\Omega) \cap W^{2,2s'(p-2)}(\Omega)| \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \int_\Omega u dx = 0\}$.

2.4.6. Uniform boundedness and regularity of quasilinear equations

Let us consider the quasilinear problem given by

$$\begin{cases} \frac{\partial u}{\partial t} - D_i A_i(x, u, \nabla u) + A_0(x, u, \nabla u) = g(x), \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi(x) \end{cases} \tag{2.4.49}$$

Suppose that there is a c^1 function $F(x, z, \xi)$ such that

$$\left\{ \begin{array}{l} \frac{\partial F(x,z,\xi)}{\partial \xi_i} = A_i(x, z, \xi) \\ \frac{\partial F(x,z,\xi)}{\partial z} = A_0(x, z, \xi) \end{array} \right. \tag{2.4.50}$$

and

$$F(x, z, \xi) \geq c_1|\xi|^p - c_2, \quad p \geq 2 \tag{2.4.51}$$

In addition, we assume that

$$[A_i(x, z, \xi_1) - A_i(x, z, \xi_2)][\xi_{i1} - \xi_{i2}] \geq c|\xi_1 - \xi_2|^2 \tag{2.4.52}$$

$$\left\{ \begin{array}{l} |A_i(x, z, \xi)| \leq c|\xi|^{p-1} + \mu_1(|z|) \\ |A_0(x, z, \xi)| \leq c|\xi|^{p_1} + \mu_2(|z|) \end{array} \right. \tag{2.4.53}$$

where

$$p_1 = \left\{ \begin{array}{ll} < p + \frac{p}{n} - 1, & as\ p < n \\ < p, & sa\ p = n \\ = p, & as\ p > n. \end{array} \right.$$

$$\mu_1(|z|) = \left\{ \begin{array}{ll} c[|z|^{q_1} + 1] \left\{ \begin{array}{ll} q_1 < \frac{n(p-1)}{n-p}, & as\ p < n \\ q_1 < \infty, & as\ p = n \end{array} \right. & \\ \mu_1 \in c[0, \infty), & as\ p > n \end{array} \right.$$

$$\mu_2(|z|) = \left\{ \begin{array}{ll} c[|z|^{q_2} + 1] \left\{ \begin{array}{ll} q_2 < \frac{np}{n-p} - 1 & as\ p < n \\ q_2 < \infty, & as\ p = n \end{array} \right. & \\ \mu_2 \in c[0, \infty), & as\ p > n. \end{array} \right.$$

We shall apply Theorem 2.2.5 to discuss the uniform boundedness of global weak solutions of (2.4.49). The spaces as in (2.1.27) and (2.1.28) are taken as follows

$$\begin{array}{l} X = \{u \in c^\infty(\overline{\Omega})| \quad u|_{\partial\Omega} = 0\}, \\ X_1 = X_2 = W_0^{1,p}(\Omega) \\ H = H_1 = L^2(\Omega) \\ H_3 = H^m(\Omega) \cap H_0^1(\Omega), \quad m \geq 1\ such\ that\ H_3 \hookrightarrow W_0^{1,p}(\Omega) \end{array}$$

The linear operator $L = id : X_2 \to X_1$ is an identity mapping, and by Lemma 2.1.7, H_3 and $L^2(\Omega)$ have a common orthogonal base.

We say $u \in L^p_{loc}((0, \infty), W_0^{1,p}(\Omega))$ is a global weak solution of (2.4.49), if for any $v \in W_0^{1,p}(\Omega), u$ satisfies

$$\int_\Omega u \cdot v dx + \int_0^t \int_\Omega [A_i(x, u, \nabla u) D_i v + A_0(x, u\nabla u) v] dx dt$$

$$= \int_0^t \int_\Omega g(x) \cdot v dx + \int_\Omega \phi \cdot v dx, \quad \forall 0 < t < \infty.$$

Theorem 2.4.10. Under the conditions (2.4.50)-(2.4.53), for any $\phi \in W_0^{1,p}(\Omega)$ and $g \in L^{p'}(\Omega)$, the problem (2.4.49) has a global weak solution

$$u \in L^\infty((0,\infty), W_0^{1,p}(\Omega))$$

$$u_t \in L^2((0,\infty) \times \Omega)$$

moreover, the solutions of (2.4.49) are uniform bounded in $W_0^{1,p}(\Omega)$.

Proof. Define the mapping $G : X_1 \to X_1^*$ by

$$< Gu, v >= \int_\Omega [A_i(x,u,\nabla u) D_i u + A_0(x,u,\nabla u) v - g \cdot v] dx$$

The conditions (2.4.50) and (2.4.51) implies that (2.2.21) and (2.2.22) are satisfied. It remains to verify that $G : X_1 \to X_1^*$ is coerceivly continuous. Let $u_n \rightharpoonup u_0$ in $L^\infty((0,\infty), W_0^{1,p}(\Omega))$, and

$$\lim_{n\to\infty} \int_0^t \int_\Omega [(A_i(x,u_n,\nabla u_n) - A_i(x,u_0,\nabla u_0))(D_i u_n - D_i u_0)$$

$$+(A_0(x,u_n,\nabla u_n) - A_0(x,u_0,\nabla u_0))(u_n - u_0)] dxdt = 0 \tag{2.4.54}$$

By the compact embedding theorems, $\forall 0 < T < \infty, u_n$ converges to u_0 in measure on $\Omega \times (0,T)$, which means

$$u_n \to u_0 \ in \begin{cases} L^q((0,T)\times\Omega), & q < \frac{np}{n-p}, \quad n > p \\ L^q((0,T)\times\Omega), & q < \infty, \quad n = p \\ c(\overline{\Omega}), \ a.e. \ t \geq 0, \ as \ p > n \end{cases} \tag{2.4.55}$$

Due to (2.4.53), as $p \leq n$, we have

$$|\int_0^t \int_\Omega A_0(x,u_n,\nabla u_n)(u_n - u_0) dxdt|$$

$$\leq \int_0^t \int_\Omega c[|\nabla u_n|^{p_1} + \mu_2(|u_n|)] |u_n - u_0| dxdt \tag{2.4.56}$$

$$\leq c[\int_0^t \int_\Omega |u_n - u_0|^{s_1} dxdt]^{\frac{1}{s_1}} [\int_0^t \int_\Omega |\nabla u_n|^{p_1 s_1'} dxdt]^{\frac{1}{s_1'}}$$

$$+c[\int_0^t\int_\Omega |u_n-u_0|^{s_2}dxdt]^{\frac{1}{s_2}}[\int_0^t\int_\Omega [|u_n|^{s_2' q_2}+1]dxdt]^{\frac{1}{s_2'}}$$

where $s_2' p_1 = p$ and $s_2' q_2 = np/n-p \quad (s_2' q_2 < \infty$ as $n = p)$, hence $s_1, s_2 < \frac{np}{n-p}$. Because $\{u_n\} \subset L^\infty((0,\infty), W_0^{1,p}(\Omega))$ is bounded, from (2.4.55) and (2.4.56) it follows

$$\lim_{n\to\infty}\int_0^t\int_\Omega A_0(x,u_n,\nabla u_n)(u_n-u_0)dxdt = 0, \ as\ p \le n.$$

As $p > n$, we have

$$\begin{aligned} & |\int_0^t\int_\Omega A_0(x,u_n,\nabla u_n)(u_n-u_0)dxdt| \\ \le\ & c\sup_{0\le\tau\le t}\int_\Omega |\nabla u_n(x,\tau)|^p dx\cdot\int_0^t \sup_\Omega |u_n(x,\tau)-u_0(x,\tau)|d\tau \\ & +c\sup_{\Omega\times[0,t]}\mu(|u_n|)\int_0^t\int_\Omega |u_n-u_0|dxdt \end{aligned}$$

By (2.4.55) and the boundedness below

$$\|u_n\|_{c^0} \le c_1\|u_n\|_{W^{1,p}} \le c, \quad \forall t \ge 0$$

and

$$\sup_\Omega |u_n-u_0| \to 0, \quad n\to\infty, \quad a.e.\ t\ge 0$$

it follows that

$$\lim_{n\to\infty}\int_0^t\int_\Omega A_0(x,u_n,\nabla u_n)(u_n-u_0)dxdt = 0, \quad as\ p > n.$$

In the same way, we can derive

$$\lim_{n\to\infty}\int_0^t\int_\Omega A_0(x,u_0,\nabla u_0)(u_n-u_0)dxdt = 0.$$

Thus from (2.4.54) we obtain

$$\lim_{n\to\infty}\int_0^t\int_\Omega [A_i(x,u_n,\nabla u_n)-A_i(x,u_0,\nabla u_0)][D_iu_n-D_iu_0]dxdt = 0 \tag{2.4.57}$$

By Lemma 1.4.3, from (2.4.53) and (2.4.55) it follows

$$A_i(x,u_n,\nabla u_0) \to A_i(x,u_0,\nabla u_0)\ in\ L^{p'}((0,T)\times\Omega)$$

Consequently, we have

$$\lim_{n\to\infty}\int_0^t\int_\Omega [A_i(x,u_n,\nabla u_0)-A_i(x,u_0,\nabla u_0)][D_iu_n-D_iu_0]dxdt=0$$

From (2.4.57) we get

$$\lim_{n\to\infty}\int_0^t\int_\Omega [A_i(x,u_n,\nabla u_n)-A_i(x,u_n,\nabla u_0)][D_iu_n-D_iu_0]dxdt=0$$

By (2.4.52) we derive

$$\lim_{n\to\infty}\int_0^t\int_\Omega |\nabla u_n-\nabla u_0|^2dxdt=0$$

namely, ∇u_n converges to ∇u_0 in measure on $(0,t)\times\Omega\forall 0<t<\infty$. Thus, from (2.4.53) and (2.4.55), one can infer that $G: X_1\to X_1^*$ is coerceivly continuous. The proof is complete.

If the further assumptions are imposed that the function $F(x,z,\xi)=F(z,\xi)\in c^2(R\times R^n)$, and

$$\text{(2.4.58)}\qquad \begin{cases} \frac{\partial^2F(z,\xi)}{\partial\xi_i\partial\xi_j}\eta_i\eta_j\geq\lambda|\eta|^2, \quad \lambda>0, \quad \forall\eta\in R^n,\\ \frac{\partial^2F(z,\xi)}{\partial z^2}\geq -c_1,\\ |\frac{\partial^2F(z,\xi)}{\partial z\partial\xi}|\leq c_2\end{cases}$$

then by virtue of Theorem 2.2.7 we can obtain the below regularity theorem.

Theorem 2.4.11. Under the hypotheses of Theorem 2.4.10, if (2.4.58) are satisfied, then for any $\phi\in W^{2,2s}(\Omega)\cap W^{1,2s'(p-2)}(\Omega)$ (as $p=2,\phi\in H^2(\Omega)\cap H_0^1(\Omega)$), the solution of (2.4.49) is unique, which has the regularity

$$u\in L^\infty_{loc}((0,\infty),H^2_{loc}(\Omega))$$

$$u_t\in L^\infty((0,\infty),L^2(\Omega))\cap L^2_{loc}((0,\infty),H^1(\Omega))$$

and the solutions (u_t,u) are uniformly bounded in $L^2(\Omega)\times W_0^{1,p}(\Omega)$ provided ϕ in a bounded subset $\sum\subset W^{2,2s}(\Omega)\cap W_0^{1,2s'(p-2)}(\Omega)$.

Proof. By (2.4.58), the conditions $(2.1.30)'$ and $(2.1.31)'$ in Theorem 2.2.7 are easily checked. Hence $u_t\in L^\infty((0,\infty),L^2(\Omega))\cap L^2_{loc}((0,\infty),H^1(\Omega))$, and u_t is uniformly bounded in $L^2(\Omega)$. By the H^2-interior regularity of equasilinear elliptic equations, from the equation

$$\begin{cases} -D_iA_i(u,\nabla u)+A_0(u,\nabla u)=g(x)-\frac{\partial u}{\partial t}\\ u|_{\partial\Omega}=0\end{cases}$$

with $g \in L^2(\Omega)$, $\frac{\partial u}{\partial t} \in L^2(\Omega) \forall 0 < t < \infty$, one can derive the regularity of $u \in L^{\infty}_{loc}((0,\infty), H^2_{loc}(\Omega))$ provided the conditions (2.4.58). The proof is complete.

2.5. Nonlinear and Fully Nonlinear Hyperbolic Equations

2.5.1. Nonlinear wave equations with gradient terms

In this subsection, we investigate the global existence of strong solution for the nonlinear wave equations with gradient terms.

Let us begine with the Neumann boundary condition of the below nonlinear wave equations

$$
(2.5.1) \qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} + k_1 \frac{\partial u}{\partial t} - \triangle u + k_2 u + f(x, u, \nabla u) = 0, \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0 \\ u(x,0) = \phi, \quad u_t(x,0) = \psi \end{cases}
$$

where $k_1 \geq 0, k_2 > 0$ are constants.

Suppose that $f \in c^1(\overline{\Omega} \times R \times R^n)$, and

$$
(2.5.2) \qquad \begin{cases} |\nabla_x f(x,z,\xi)| + |f(x,z,\xi)| \leq c[|z| + |\xi| + 1] \\ |D_z f(x,z,\xi)| + |\nabla_\xi f(x,z,\xi)| \leq c \end{cases}
$$

Applying Theorem 2.3.3, for the problem (2.5.1) we can obtain the global existence theorem as follows

Theorem 2.5.1. Let the condition (2.5.2) hold. Then for any $(\phi, \psi) \in H^2(\Omega) \times H^1(\Omega)$ with $\frac{\partial \phi}{\partial n}|_{\partial\Omega} = 0$, the problem (2.5.2) has a global strong solution

$$
u \in W^{2,2}_{loc}((0,\infty), L^2(\Omega)) \cap W^{1,\infty}_{loc}((0,\infty), H^1(\Omega)) \cap L^{\infty}_{loc}((0,\infty), H^2(\Omega)).
$$

Proof. The spaces as in (2.3.1) are taken as to read

$$
\begin{aligned}
& X = \{u \in c^{\infty}(\overline{\Omega})| \quad \frac{\partial \triangle^m u}{\partial n}|_{\partial\Omega} = 0, m = 0, 1, \cdots, \}. \\
& H = X_1 = L^2(\Omega), H_1 = H^1(\Omega) \\
& H_2 = X_2 = \{u \in H^2(\Omega)| \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}
\end{aligned}
$$

with the inner product

$$< u, v >_{H_2} = \int_\Omega (-\triangle u + au)(-\triangle v + av)dx; \quad a > 0. \tag{2.5.3}$$

It is known that the norm of H_2 defined by the inner product (2.5.3) is equivalent to the H^2-norm.

The linear mapping $L : X \to X_1$ is defined by

$$Lu = -\triangle u + au \tag{2.5.4}$$

where $a > 0$ is the constant as in (2.5.3). It is well known that for the operator (2.5.4), the conditions (2.3.2) and (2.3.3) in Theorem 2.3.3 are satisfied.

We define the mapping $G = A + B : X_2 \to X_1^*$ as follows

$$< Au, v > = \int_\Omega [-\triangle u + k_2 u]v dx$$

$$< Bu, v > = \int_\Omega f(x, u, \nabla u)v dx$$

$\forall u \in X_2, v \in X_1$. Denote $F : X_2 \to R$ by

$$F(u) = \frac{1}{2}\int_\Omega [|\triangle u|^2 + (k_2 + a)|\nabla u|^2 + ak_2 u^2]dx$$

It is easy to see that

$$\begin{aligned} < Au, Lv > &= \int_\Omega [-\triangle u + k_2 u][-\triangle v + av]dx \\ &= \int_\Omega [\triangle u \cdot \triangle v + (k_2 + a)\nabla u \cdot \nabla v + ak_2 u \cdot v]dx \\ &= < DFu, v >_{X_2}, \quad \forall u, v \in X. \end{aligned}$$

and

$$\|u\|_{X_2} \to \infty \Leftrightarrow F(u) \to \infty.$$

Therefore the conditions (2.3.5) and (2.3.6) are checked.

By virtue of the compact embedding theorems and the condition (2.5.2), it is easy to see that $G = A + B : X_2 \to X_1^*$ is weakly continuous. It remains to check the conditions (2.3.7) and (2.3.8).

By (2.3.2) we have

$$| < Bu, Lv > | = | \int_\Omega f(x, u, \nabla u)(-\triangle v + av)dx|$$

$$= \int_\Omega [\nabla_x f(x, u, \nabla u)\nabla v + D_z f(x, u, \nabla v)\nabla v \cdot \nabla v$$

$$
\begin{aligned}
&+D_{\xi_j} f(x,u,\nabla u)D_{ij}uD_iv + af(x,u,\nabla u)v]dx| \\
\leq\ & c\int_\Omega [|D^2u|^2 + |\nabla u|^2 + |u|^2]dx + c\int_\Omega [|\nabla v|^2 + |v|^2]dx + c \\
\leq\ & c[F(u) + \|v\|_{H_1}^2 + 1]
\end{aligned}
$$

and

$$
\begin{aligned}
| < Gu, v > | &= |\int_\Omega (-\triangle u + k_2 u + f(x,u,\nabla u))v dx| \\
&\leq \frac{1}{2}\int_\Omega |v|^2 dx + \frac{1}{2}\int_\Omega |-\triangle u + k_2 u + f(x,u,\nabla u)|^2 dx \\
&\leq \frac{1}{2}\|v\|_H^2 + cF(u) + c
\end{aligned}
$$

Thus the conditions (2.3.7) and (2.3.8) are verified. Hence by Theorem 2.3.3, this theorem is proved.

Next, we consider the nonlinear wave equations with the Dirichlet boundary condition

$$
(2.5.5) \qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} - \triangle u + f(x,u,\nabla u) = 0, \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi, u_t(x,0) = \psi \end{cases}
$$

where $k \geq 0$ is a constant.

Theorem 2.5.2. Under the condition (2.5.2), for any $(\phi,\psi) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, the problem (2.5.5) has a global strong solution

$$
\begin{aligned}
&u \in L^\infty_{loc}((0,\infty), H^2(\Omega)) \\
&u_t \in L^\infty_{loc}((0,\infty), H_0^1(\Omega)) \\
&u_{tt} \in L^2_{loc}((0,\infty), L^2(\Omega).
\end{aligned}
$$

Proof. The spaces as in (2.3.1) are taken as follows

$$
\begin{aligned}
&X = \{u \in c^\infty(\overline{\Omega})| \quad \triangle^m u|_{\partial\Omega} = 0, m = 0,1,\cdots,\} \\
&X_1 = L^2(\Omega), X_2 = H^2(\Omega) \cap H_0^1(\Omega) \\
&H = L^2(\Omega), H_1 = H_0^1(\Omega), \\
&H_2 = \{u \in H^2(\Omega)| \quad u|_{\partial\Omega} = 0\},\ \textit{with the inner product} \\
&< u, v >_{H_2} = \int_\Omega \triangle u \cdot \triangle v dx
\end{aligned}
$$

And the linear operator $L : X \to X_1$ is as (2.5.4).

The mapping $G = A + B : X_2 \to X_1^*$ is defined by

$$\begin{cases} Au = -\triangle u \\ Bu = f(x, u, \nabla u) \end{cases}$$

and the functional $F : X_2 \to R$ is defined by

$$F(u) = \frac{1}{2}\int_\Omega [|\triangle u|^2 + a|\nabla u|^2]dx \tag{2.5.6}$$

The main difference between the both proofs of Theorem 2.5.1 and Theorem 2.5.2 is that the integral by part cann't work anymore for the sase

$$\int_\Omega f(x, u, \nabla u)\triangle v dx, \quad u, v \in H^2(\Omega) \cap H_0^1(\Omega)$$

Hence we shall check the relaxed condition (2.3.18) in Remark 2.3.4 instead of (2.3.7).

We see that for any $u \in c^1([0, \infty), X), u(0) = \phi$,

$$\begin{aligned} & |\int_0^t < Bu, \frac{d}{dt}Lu > dt| = |\int_0^t \int_\Omega f(x, u, \nabla u)(-\triangle u_t + au_t)dxdt| \\ \leq \ & |\int_\Omega f(x, u(t), \nabla u(t))\triangle u(t)dx| + |\int_\Omega f(x, \phi, \nabla\phi)\triangle\phi dx| \\ & + \int_0^t \int_\Omega a|f(x, u, \nabla u)||u_t|dxdt + \\ & + \int_0^t \int_\Omega [|D_z f(x, u, \nabla u)||u_t||\triangle u| + |\frac{\partial f}{\partial \xi_i}||\frac{\partial}{\partial t}\frac{\partial u}{\partial x_i}||\triangle u|] \\ \leq \ & \frac{1}{4}\int_\Omega [|\triangle u|^2 + 4|f(x, u, \nabla u)|^2]dx + \int_\Omega |f(x, \phi, \nabla\phi)|\triangle\phi|dx \\ & + \int_0^t \int_\Omega c(|\triangle u|^2 + |\nabla u|^2 + |u|^2)dxdt \\ & + c\int_0^t \int_\Omega [|\nabla u_t|^2 + |u_t|^2]dxdt + c \end{aligned}$$

By the condition (2.5.2), as the constant $a > 0$ in (2.5.4) is taken large enough, then from (2.5.6) we can see that

$$\frac{1}{4}\int_\Omega [|\triangle u|^2 + 4|f(x, u, \nabla u)|^2]dx \leq \frac{1}{2}F(u)$$

Therefore we get

$$|\int_0^t < Bu, \frac{d}{dt}Lv > dt| \leq \frac{1}{2}F(u) + c +$$

$$+\int_0^t [cF(u)+c\|\frac{du}{dt}\|^2_{H^1}+c]dt$$

Thus the condition (2.3.18) is checked. The proof is complete.

Example 2.5.3. We give the below example

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \triangle u + ku + \frac{|\nabla u|^3}{1+|\nabla u|^2} = g(x), \quad g \in L^2(\Omega) \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0 \ (or \ u|_{\partial\Omega} = 0) \\ u(x,0) = \phi, \quad u_t(x,0) = \psi \end{cases} \tag{2.5.7}$$

It is easy to see that the condition (2.5.2) is satisfied. Hence the problem (2.5.7) has a global strong solution

$$u \in W^{2,2}_{loc}((0,\infty), L^2(\Omega)) \cap W^{1,\infty}_{loc}((0,\infty), H^1(\Omega)) \cap L^\infty_{loc}((0,\infty), H^2(\Omega)).$$

2.5.2. Uniform boundedness of nonlinear vibration relating to thin plates

Let us consider the uniform boundedness of global weak solutions of the below nonlinear biharmonic virbration equations, which are related with the virbration of beams and thin plates.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \triangle^2 u - f(x, \triangle u) = g(x), \quad x \in \Omega \\ u|_{\partial\Omega} = 0, \quad \triangle u|_{\partial\Omega} = 0 \\ u(x,0) = \phi, \quad u_t(x,0) = \psi \end{cases} \tag{2.5.8}$$

Let $F(x,y)$ be defined as in (2.4.28). Suppose that

$$F(x,y) \geq c_1|y|^p - c_2 \tag{2.5.9}$$

$$|f(x,y)| \leq c[|y|^{p-1}+1] \tag{2.5.10}$$

We shall use Theorem 2.3.3 to discuss (2.5.8). The spaces as in (2.3.1) are taken as follows

$$\begin{aligned} &X = \{u \in c^\infty(\overline{\Omega})| \quad \triangle^k u|_{\partial\Omega} = 0, k = 0, 1, \cdots, \}, \\ &X_1 = H^1_0(\Omega) \cap L^p(\Omega) \\ &X_2 = \{u \in H^3(\Omega) \cap W^{2,p}(\Omega)| \quad u|_{\partial\Omega} = 0, \triangle u|_{\partial\Omega} = 0\} \\ &H = L^2(\Omega), \quad H_1 = H^1_0(\Omega) \\ &H_2 = \{u \in H^{2m}(\Omega)| \quad u|_{\partial\Omega} = 0, \cdots, \triangle^{m-1} u|_{\partial\Omega} = 0\} \end{aligned}$$

with the inner product

$$< u, v >_{H_2} = \int_\Omega \triangle^m u \cdot \triangle^m v dx$$

where $m \geq 2$ is taken such that $H_2 \hookrightarrow X_2$.

We say that $u \in W^{1,2}_{loc}((0,\infty), L^2(\Omega)) \cap L^\infty_{loc}((0,\infty), X_2)$ is a weak solution of (2.5.8), if $\forall v \in H^1_0(\Omega) \cap L^p(\Omega)$,

$$\int_\Omega \frac{\partial u}{\partial t} \cdot v dx = \int_0^t \int_\Omega [\nabla \triangle u \cdot \nabla v + f(x, \triangle u)v + gv]dx$$

$$- \int_\Omega \psi \cdot v dx, \quad \forall 0 \leq t < \infty.$$

Theorem 2.5.4. Under the conditions (2.5.9) and (2.5.10), for any $g \in L^{p'}(\Omega)$ and $(\phi, \psi) \in X_2 \times H^1_0(\Omega)$, the problem (2.5.8) has a global weak solution

$$u \in W^{1,\infty}((0,\infty), H^1_0(\Omega)) \cap L^\infty((0,\infty), X_2)$$

and the solutions of (2.5.8) are uniformly bounded in $[H^3(\Omega) \cap W^{2,p}(\Omega)] \times H^1_0(\Omega)$.

Proof. Let the linear mapping $L : X \to X_1$ be defined by

$$Lu = -\triangle u$$

and $G : X_2 \to X_1^*$ be defined by

$$< Gu, v > = - \int_\Omega [\nabla \triangle u \cdot \nabla v + f(x, \triangle u)v + g \cdot v]dx$$

It is easy to see that for the functional $F_1 : X_2 \to R$ defined by

$$F_1(u) = \int_\Omega [\frac{1}{2}|\nabla \triangle u|^2 + F(x, \triangle u) + g(x)\triangle u]dx$$

we have

$$< Gu, Lv > = < DFu, v >_{X_2}, \quad \forall u, v \in X_2$$

By (2.5.9), F satisfies the condition (2.3.6), and from (2.5.10) we can see that $G : X_2 \to X_1^*$ is weakly continuous. Therefore the theorem follows from Theorem 2.4. The proof is complete.

Remark 2.5.5. More generally, by using Theorem 2.3.3 we can discuss the uniform boundedness of global weak solutions for the below nonlinear equations

$$(2.5.11)\qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} + (-1)^m \triangle^m u + (-1)^k f(x, \triangle^k u) = g(x), \\ u|_{\partial\Omega} = 0, \cdots, \triangle^{m-1} u|_{\partial\Omega} = 0 \\ u(x,0) = \phi, \quad u_t(x,0) = \psi \end{cases}$$

where $0 \leq k < m$. Under the conditions (2.5.9) and (2.5.10), the problem (2.5.11) has a global weak solution

$$u \in W^{1,\infty}((0,\infty), H^k(\Omega)) \cap L^\infty((0,\infty), H^{m+k}(\Omega) \cap W^{2k,p}(\Omega))$$

and the solutions are uniformly bounded in $H^k(\Omega) \times [H^{m+k}(\Omega) \cap W^{2k,p}(\Omega)]$.

2.5.3. Fully nonlinear and quasilinear equations with a strong damping term

First, let us consider the problem given by

$$(2.5.12)\qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} - k\frac{\partial \triangle u}{\partial t} - f(x, \triangle u) = g(x, u, \nabla u, D^2 u) \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi, \quad u_t(x,0) = \psi \end{cases}$$

Let $F(x,y)$ be the function as in (2.4.28). Suppose that

$$(2.5.13)\qquad \{\ F(x,y) \geq c_1|y|^p - c_2, \quad p \geq 2\ |f(x,y)| \leq c[|y|^{p-1} + 1]$$

$$(2.5.14)\qquad [f(x,y_1) - f(x,y_2)][y_1 - y_2] \geq \alpha|y_1 - y_2|^2, \quad \alpha > 0$$

$$(2.5.15)\qquad |g(x,z,\xi,\eta)| \leq c[|z|^{\frac{p}{2}} + |\xi|^{\frac{p}{2}} + |\eta|^{\frac{p}{2}} + 1]$$

$$(2.5.16)\qquad \begin{cases} |g(x,z,\xi,\eta_1) - g(x,z,\xi,\eta_2)| \leq K_1|\eta_1 - \eta_2| \\ K_1^2 < \alpha^2 K^2, \quad \alpha \ as\ in\ (2.5.14), \quad K\ as\ in\ (1.4.3) \end{cases}$$

We shall apply Theorem 2.3.6 to discuss the global existence of strong solutions of (2.5.12). The spaces as in (2.3.19) are taken as to write

$$\begin{aligned} &X = \{u \in c^\infty(\overline{\Omega})| \quad \triangle^k u|_{\partial\Omega} = 0, k = 0, 1, \cdots, \} \\ &X_1 = L^p(\Omega); \quad X_2 = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \\ &H = L^2(\Omega); \quad H_1 = H_0^1(\Omega); \\ &H_2 = H^2(\Omega) \cap H_0^1(\Omega),\ with\ < u, v >_{H_2} = \int_\Omega \triangle u \cdot \triangle v dx \\ &H_3 = \{u \in H^{2m}(\Omega)| \quad u|_{\partial\Omega} = \cdots = \triangle^{m-1} u|_{\partial\Omega} = 0\}, \\ &< u, v >_{H_3} = \int_\Omega \triangle^m u \cdot \triangle^m v dx \end{aligned}$$

The linear mapping $L : X_2 \to X_1$ and $\mathcal{L} : X_2 \to X_1$ are as follows

$$Lu = \mathcal{L}u = -\triangle u.$$

It is clear that the conditions (2.3.20)(2.3.21) and (2.3.23) are satisfies.

Theorem 2.5.6. Under the conditions (2.5.13)-(2.5.16), for any $(\phi, \psi) \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)] \times H_0^1(\Omega$, the problem (2.5.12) has a global strong solution

$$u \in L^\infty_{loc}((0,\infty), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$$

$$u_t \in L^\infty_{loc}((0,\infty), H_0^1(\Omega)) \cap L^2_{loc}((0,\infty), H^2(\Omega)), \textit{ and}$$

$$u_{tt} \in L^{p'}((0,T) \times \Omega), \quad \forall 0 < T < \infty.$$

Proof. Define the mapping $G = A + B : X_2 \to X_1^*$ by

$$< Au, v >= -\int_\Omega f(x, \triangle u) v dx$$

$$< Bu, v >= -\int_\Omega g(x, u, \nabla u, D^2 u) v dx$$

In the same fashion as in the proof of Theorem 2.4.2, one can check, from the conditions (2.5.13)-(2.5.16), that $G : X_2 \to X_1^*$ is coerceivly continuous.

Denote by

$$F_1(u) = \int_\Omega F(x, \triangle u) dx$$

It is easy to see that the conditions (2.3.5) and (2.3.6) are fulfilled. We are now in a position to check the condition (2.3.24). We see that

$$\begin{aligned} | < Bu, Lv > | &\leq \int_\Omega |g(x, u, \nabla u, D^2 u)||\triangle v| dx \\ &\leq \frac{k}{2} \int_\Omega |\triangle v|^2 dx + \frac{2}{k} \int_\Omega |g(x, u, \nabla u, D^2 u)|^2 dx \\ &\leq \frac{k}{2} \|v\|^2_{H_2} + c \int_\Omega [|D^2 u|^p + |\nabla u|^p + |u|^p + 1] dx \\ &\leq \frac{k}{2} \|v\|^2_{H_2} + cF_1(u) + c. \end{aligned}$$

Thus, by Theorem 2.3.6, the problem (2.5.12) has a solution

$$u \in L^\infty((0,T), W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)), \quad \forall 0 < T < \infty,$$

$$u_t \in L^\infty((0,T), H_0^1(\Omega)) \cap L^2((0,T), H^2(\Omega)), \quad \forall 0 < T < \infty$$

which satisfies

$$\int_\Omega \frac{\partial u}{\partial t} v dx - k\int_\Omega \triangle u \cdot v dx - \int_0^t \int_\Omega f(x, \triangle u) v dx dt \tag{2.5.17}$$

$$= \int_0^t \int_\Omega g(x, u, \nabla u, D^2 u) v dx dt + \int_\Omega \psi \cdot v dx - k \int_\Omega \triangle \psi \cdot v dx$$

In the same manner as in the proof of Theorem 2.4.2, from (2.5.17) we can obtain the regularity

$$u_{tt} \in L^{p'}((0,T) \times \Omega), \quad \forall 0 < T < \infty.$$

The proof is complete.

Next, we investigate the problem given by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - k\frac{\partial \triangle u}{\partial t} + f(x, \triangle^2 u) = g(x, u, \nabla u, D^2 u, D^3 u) \\ u|_{\partial\Omega} = 0, \quad \triangle u|_{\partial\Omega} = 0 \\ u(x,0) = \phi, \quad u_t(x,0) = \psi \end{cases} \tag{2.5.18}$$

Besides the condition (2.5.13), we assume that $g \in c^1(\overline{\Omega} \times R^N)$, and

$$(f(x, y_1) - f(x, y_2))(y_1 - y_2) \geq 0 \tag{2.5.19}$$

$$\begin{cases} g(x, u, \nabla u, D^2 u, D^3 u)|_{\partial\Omega} = 0 \\ \forall u \in c^\infty(\overline{\Omega}) \ with \ u|_{\partial\Omega} = 0, \quad \triangle u|_{\partial\Omega} = 0 \end{cases} \tag{2.5.20}$$

$$\begin{cases} |g(x,\xi)| + |D_x g(x,\xi)| \leq c[\sum_{|\alpha| \leq 3} |\xi_\alpha|^{\frac{p}{2}} + 1] \\ \sum_{|\alpha| \leq 3} |D_{\xi_\alpha} g(x,\xi)| \leq c[\sum_{|\beta| \leq 3} |\xi_\beta|^{\frac{p}{2}-1} + 1] \\ \xi = \{\xi_\alpha | \alpha = \{\alpha_1, \cdots, \alpha_n\}, \quad |\alpha| \leq 3\} \end{cases} \tag{2.5.21}$$

The spaces as in (2.3.19) are takes as that X, X_1 and H are the same as in Theorem 2.5.6, and

$$X_2 = \{u \in W^{4,p}(\Omega) | \quad u|_{\partial\Omega} = 0, \triangle u|_{\partial\Omega} = 0\}$$

$$H_1 = H^2(\Omega) \cap H_0^1(\Omega), \ with \ < u, v >_{H_1} = \int_\Omega \triangle u \cdot \triangle v dx$$

$$H_2 = \{u \in H^3(\Omega) | \quad u|_{\partial\Omega} = 0, \triangle u|_{\partial\Omega} = 0\}$$

$$< u, v >_{H_2} = \int_\Omega \nabla \triangle u \cdot \nabla \triangle v dx$$

$$H_3 = \{u \in H^{4m}(\Omega) | \quad u|_{\partial\Omega} = 0, \cdots, \triangle^{2m-1} u|_{\partial\Omega} = 0\}$$

$$< u, v >_{H_3} = \int_\Omega \triangle^{2m} u \cdot \triangle^{2m} v dx$$

and the linear mapping $\mathcal{L}: X_2 \to X_1$ and $L: X_2 \to X_1$ are respectively as follows

$$\begin{cases} \mathcal{L}u = -\triangle u \\ Lu = \triangle^2 u \end{cases}$$

By Lemma 1.4.1 and the Poincare inequality, we get that $H_2 \hookrightarrow H_1$ is compact. It is known that the eigenvalue problem

$$\begin{cases} \triangle^2 u = \lambda u \\ u|_{\partial\Omega} = 0, \quad \triangle u|_{\partial\Omega} = 0 \end{cases}$$

has an infinite eigenvalue sequence, and the eigenfunctions $\{e_n\}$ consist of a normal orthogonal base of $L^2(\Omega)$. By the definition of H_3, it is clear that $\{e_n\}$ is also an orthogonal base of H_3. Hence the conditions (2.3.20)(2.3.21) and (2.3.23) are fulfilled.

Theorem 2.5.7. Let the conditions (2.5.13) and (2.5.19)-(2.5.21) be satisfied. Then for any $(\phi, \psi) \in X_2 \cap [H^2(\Omega) \cap H_0^1(\Omega)]$, the problem (2.5.18) has a global strong solution

$$u \in L^\infty_{loc}((0,\infty), W^{4,p}(\Omega))$$

$$u_t \in L^\infty_{loc}((0,\infty), H^2(\Omega)) \cap L^2_{loc}((0,\infty), H^3(\Omega))$$

$$u_{tt} \in L^{p'}((0,T) \times \Omega), \quad \forall 0 < T < \infty.$$

Proof. We use Theorem 2.3.7 to verify this theorem. Obviously, the mapping $B: X_2 \to X_1^*$ defined by

$$< Bu, v >= -\int_\Omega g(x, u, \nabla u, D^2u, D^3u) v dx$$

is a compact operator. Hence the conditions in claim ii) of Theorem 2.3.7 are satisfied.

It suffices to check the condition (2.3.24), and the proof is similar to that of Theorem 2.5.6.

By (2.5.20), we have

$$\begin{aligned} | < Bu, Lv > | &= |\int_\Omega \nabla g(x, u, \nabla u, D^2u, D^3u) \cdot \nabla \triangle v dx| \\ &\le \frac{k}{2}\int_\Omega |\nabla \triangle v|^2 dx + \frac{2}{k}\int_\Omega |\nabla g(x, u, \nabla u, D^2u, D^3u)|^2 dx \\ &\le \frac{k}{2}\|v\|^2_{H_2} + c\int_\Omega [|D_x g|^2 + \sum_{i=1}^n \sum_{|\alpha|\le 3} |D_{\xi_\alpha} g|^2 |D_i D^\alpha u|^2] dx \end{aligned}$$

$$\leq \quad \frac{k}{2}\|v\|_{H_2}^2 + c\int_\Omega [\sum_{|\alpha|\leq 4} |D^\alpha u|^p + 1]dx, \ (by \ (2.5.21)).$$

Thus, the condition (2.3.24) is checked. The proof is complete.

Finally, we conclude section 2.5 by considering the following problem

$$(2.5.22) \qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} - k\frac{\partial \triangle u}{\partial t} + \sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha A_\alpha(x,u,\cdots,D^m u) \\ = g(x,u,\cdots,D^m u) \\ u|_{\partial\Omega} = 0,\cdots,D^{m-1}u|_{\partial\Omega} = 0 \\ u(x,0) = \phi, \quad u_t(x,0) = \psi \end{cases}$$

where $m \geq 1$

We say that $u \in W^{1,2}_{loc}((0,\infty),L^2(\Omega)) \cap L^\infty_{loc}((0,\infty),W_0^{m,p}(\Omega))$ is a global weak solution of (2.5.22), if $\forall v \in c_0^\infty(\Omega)$,

$$\int_\Omega \frac{\partial u}{\partial t} v dx + k\int_\Omega \nabla u \nabla v dx + \int_0^t \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x,u,\cdots,D^m u)D^\alpha v dx dt$$

$$= \int_0^t \int_\Omega g(x,u,\cdots,D^m u) v dx dt + \int_\Omega \psi v dx + k\int_\Omega \nabla\psi\cdot\nabla v dx$$

Suppose that there is a function $F(x,\xi), \xi = \{\xi_\alpha| \quad \alpha = \{\alpha_1,\cdots,\alpha_n\}, |\alpha| \leq m\}$, which satisfies

$$(2.5.23) \qquad A_\alpha(x,\xi) = \frac{\partial F(x,\xi)}{\partial \xi_\alpha}$$

Moreover, we assume that

$$(2.5.24) \qquad F(x,\xi) \geq c_1 \sum_{|\beta|=m} |\xi_\beta|^p - c_2, \quad p \geq 2$$

$$(2.5.25) \qquad \begin{cases} \sum_{|\alpha|=m}[A_\alpha(x,\eta,\tau_1) - A_\alpha(x,\eta,\tau_2)][\tau_{1\alpha} - \tau_{2\alpha}] \\ \geq \lambda|\tau_1 - \tau_2|^2, \quad \lambda > 0 \\ \tau = \{\tau_\alpha| \quad \alpha = (\alpha_1,\cdots,\alpha_n), \quad |\alpha| = m\}, \\ \eta = \{\eta_\beta| \quad \beta = (\beta_1,\cdots,\beta_n), \quad |\beta| \leq m-1\} \end{cases}$$

$$(2.5.26) \qquad \begin{cases} |A_\alpha(x,\xi)| \leq c[\sum_{|\beta|\leq m}|\xi_\beta|^{p-1} + 1] \\ |g(x,\xi)| \leq c[\sum_{|\beta|\leq m}|\xi_\beta|^{\frac{p}{2}} + 1] \end{cases}$$

By applying Theorem 2.3.6, we can obtain the below result.

Theorem 2.5.8. Let the conditions (2.5.23)-(2.5.26) be satisfied. Then for any $(\phi, \psi) \in W_0^{m,p}(\Omega) \times L^2(\Omega)$, the problem (2.5.22) has a global weak solution

$$u \in L^\infty_{loc}((0,\infty), W_0^{m,p}(\Omega))$$

$$u_t \in L^\infty_{loc}((0,\infty), L^2(\Omega)) \cap L^2_{loc}((0,\infty), H_0^1(\Omega)).$$

Proof. Let the spaces be taken as follows

$$X = c_0^\infty(\Omega); \quad X_2 = X_1 = W_0^{m,p}(\Omega)$$

$$H = H_1 = L^2(\Omega); \quad H_2 = H_0^1(\Omega)$$

and the linear operator $L : X_2 \to X_1$ be the identity mapping $L = id$. Define $G = A + B : X_2 \to X_1^*$ by

$$< Au, v >= \int_\Omega \sum_{|\alpha| \leq m} A_\alpha(x, u, \cdots, D^m u) D^\alpha v dx$$

$$< Bu, v >= - \int_\Omega g(x, u, \cdots, D^m u) v dx$$

From the conditions (2.5.23) and (2.5.24), one can derive the condition (2.3.5) and (2.3.6). As the proof of Theorem 2.4.10, from (2.5.25) and (2.5.26) we can refer that $G = A + B : X_2 \to X_1^*$ is coerceivly continuous. Finally, from (2.5.26) we can derive that the condition (2.3.24) is satisfied. Therefore, by Theorem 2.3.6, this theorem is proved.

2.6. Systems of Nonlinear Parabolic Equations

2.6.1. Global weak existence of quasilinear systems

We shall apply the abstract theorems on the upward weakly continuous and horizontal coerceivly continuous operators to investigate the global weak existence of quasilinear systems.

First, we consider the quasilinear parabolic systems given by

$$\text{(2.6.1)} \qquad \begin{cases} \frac{\partial u_k}{\partial t} - D_i[a_{ij}^{kl}(x,u) D_j u_l + b_i^k(x,u)] + h_i^{kl}(x,u) D_i u_l \\ u|_{\partial\Omega} = 0, \\ u(x,0) = \phi(x) \end{cases}$$

where $u=\{u_1,\cdots,u_m\}$. Suppose that

$$(2.6.2)\qquad \begin{cases} \lambda|\xi|^2\le a_{ij}^{kl}\xi_{ki}\xi_{lj}, & \forall(x,z)\in\Omega\times R^m,\xi\in R^{mn}\\ \lambda>0 \text{ a constant} \end{cases}$$

$$\int_\Omega[b_i^k(x,u)D_iu_k+h_i^{kl}(x,u)u_kD_iu_l+c^k(x,u)\cdot u_k]dx$$

$$(2.6.3)\qquad \ge\int_\Omega\alpha^k|u_k|^{p_k}dx-\alpha[\int_\Omega|u|^2dx+1];\quad \forall u\in c_0^\infty(\Omega,R^m)$$

where $p_k\ge 2,\alpha^k,\alpha>0$ are constants.

$$(2.6.4)\qquad \begin{cases} |a_{ij}^{rl}(x,z)|,|h_i^{rl}(x,z)|\le\beta^k|z_k|^{q_k}+\beta\\ |c^l(x,z)|,|b_i^l(x,z)|\le\beta^k|z_k|^{\widetilde{q}_k}+\beta\\ 0\le q_k<\max\{p_k/2,\frac{n+2}{n}\};0\le\widetilde{q}_k<\max\{p_k,\frac{2(n+2)}{n}\} \end{cases}$$

We use Theorem 2.1.8 to discuss the problem (2.6.1). Let

$$\begin{aligned} &X=c_0^\infty(\Omega,R^m),H=L^2(\Omega,R^m),X_2=c_0^1(\Omega,R^m)\\ &X_1=\text{ the completion of } X \text{ under the norm}\\ &\|u\|_{X_1}=[\int_\Omega|\nabla u|^2dx]^{\frac{1}{2}}+\sum_{k=1}^m[\int_\Omega|u_k|^{p_k}dx]^{\frac{1}{p_k}} \end{aligned}$$

We say $u\in L^p_{loc}((0,\infty),X_1)(p=\{2,p_1,\cdots,p_m\}),L^p((0,\infty),X_1)$ is difinede as in Subsection 2.1.1, is a global weak solution of (2.6.1) if for any $v\in X_2,u$ satisfies that

$$\int_\Omega u\cdot vdx+\int_0^t\int_\Omega[a_{ij}^{kl}(x,u)D_ju_lD_iv_k+b_i^k(x,u)v_k+h_i^{kl}(x,u)D_iu_lv_k$$
$$+c^k(x,u)v_k-f^k(x,t)v_k]dx=\int_\Omega\phi\cdot vdx$$

Theorem 2.6.1. Under the conditions (2.6.2)-(2.6.4), if $f\in L^2_{loc}((0,\infty),L^2(\Omega,R^m))$ and $\phi\in L^2(\Omega,R^m)$, then the problem (2.6.1) has a global weak solution

$$u\in L^\infty_{loc}((0,\infty),L^2(\Omega,R^m))\cap L^p_{loc}((0,\infty),X_1)$$

where $p=\{2,p_1,\cdots,p_m\},p_k(1\le k\le m)$ defined as in (2.6.3).

Proof. We define the mapping $G : R^+ \times X_1 \to X_2^*$ by

$$< Gu, v >= \int_\Omega [a_{ij}^{kl}(x,u) D_j u_l D_i v_k + b_i^k(x,u) v_k$$

$$+ h_i^{kl}(x,u) D_i u_l v_k + c^k(x,u) v_k - f^k(x,t) v_k] dx$$

We need to check the conditions (A_1) and (A_2) in Theorem 2.1.8, and show that $G : R^+ \times X_1 \to X_2^*$ is p-weakly continuous.

The hypotheses (2.6.2) and (2.6.3) means the condition (A_1) is satisfies, and parallel to the proof as used in Example 2.1.21, from (2.6.4) one can derive the condition (A_2).

Let $u_n \rightharpoonup u_0$ uniformly in $L^p((0,T), X_1) \forall 0 < T < \infty$, i.e.

$$u_n \rightharpoonup u_0 \ in \ L^p((0,T), X_1), \ and$$

$$\lim_{n\to\infty} \int_0^T [\int_\Omega (u_n - u_0) v dx]^2 dt = 0, \quad \forall v \in X$$

By Lemma 2.1.20, we have

$$u_n \to u_0 \ in \ L^2((0,T) \times \Omega, R^m)$$

Thus, in the same method as used in Theorem 1.5.1, from Lemma 2.4.1 we can get

$$\lim_{n\to\infty} \int_0^T < Gu_n, v > dt = \int_0^T < Gu_0, v > dt, \quad \forall v \in X_2, T < \infty.$$

By Theorem 2.1.8 we obtain this theorem. The proof is complete.

Next, we consider the quasilinear parabolic system as follows

$$(2.6.5) \quad \begin{cases} \frac{\partial u_k}{\partial t} - D_i A_i^k(x,u,\nabla u) + B^k(x,u,\nabla u) = f^k(x,t), \quad 1 \le k \le m \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Suppose that

$$(2.6.6) \quad \begin{cases} \lambda |\xi|^p \le A_i^k(x,z,\xi)\xi_{ki}, \quad \forall (x,z) \in \Omega \times R^m, \xi \in R^{mn} \\ \lambda > 0, p > 1 \ are \ cnstants \end{cases}$$

$$(2.6.7) \quad \int_\Omega B^k(x,u,\nabla u) u_k dx \ge -\alpha [\int_\Omega |u|^2 dx + 1], \quad \alpha \ge 0$$

$$[A_i^k(x,z,\xi) - A_i^k(x,z,\eta)][\xi_{ki} - \eta_{ki}]$$

$$(2.6.8) \qquad \geq \lambda_1|\xi-\eta|^2; (\lambda_1>0) \quad \forall (x,z)\in\Omega\times R^m, \xi,\eta\in R^{mn}$$

$$(2.6.9) \qquad \begin{cases} |A_i^l(x,z,\xi)| \leq \beta[|z|^{p_1}+|\xi|^{p-1}+1] \\ |B^l(x,z,\xi)| \leq \beta[|z|^{p_2}+|\xi|^{p_3}+1] \end{cases}$$

where

$$p_1 < \frac{n+2}{n}(p-1)$$
$$p_2 = \begin{cases} < \frac{n+2}{n}(p-1)+\frac{n+2}{n^2}p, & n>p \\ < \frac{n+2}{n}p, & p\geq n \end{cases}$$
$$p_3 = \begin{cases} < \frac{n+1}{n}p-1, & n>p \\ < p, & n\leq p \end{cases}$$

Theorem 2.6.2. Under the hypotheses (2.6.6)-(2.6.9), if $f\in L_{loc}^{p'}((0,\infty),L^{p'}(\Omega,R^m))$, $\phi\in L^2(\Omega,R^m)$, then the problem (2.6.5) has a global weak solution

$$u\in L_{loc}^{\infty}((0,\infty),L^2(\Omega,R^m))\cap L_{loc}^p((0,\infty),W_0^{1,p}(\Omega,R^m))$$

Proof. We use Theorem 2.1.12 to prove this theorem. Let

$$X=c_0^\infty(\Omega,R^m), H=H_1=L^2(\Omega,R^m),$$
$$X_2=X_1=W_0^{1,p}(\Omega,R^m), L=id: X_2\to X_1$$

Define the mapping $G: R^+\times X_1\to X_1^*$ by

$$<Gu,v>=\int_\Omega [A_i^k(x,u,\nabla u)D_iv_k+B^k(x,u,\nabla u)v_k-f^kv_k]dx$$

Obviously, the conditions (2.1.32)-(2.1.34) in Theorem 2.1.12 are fulfiled. And the proof that $G: R^+\times X_1\to X_1^*$ is p-coerceivly continuous is parallel to that of Theorem 2.4.2, we omit it here. This proof is complete.

2.6.2. Regularity and uniqueness

Let us consider the $W^{2,p}$-strong solutions of the semilinear parabolic systems given by

$$(2.6.10) \qquad \begin{cases} \frac{\partial u_k}{\partial t}-D_i(a_{ij}^{kl}(x)D_ju_l)+b_i^{kl}(x)D_iu_l+c^k(x,u)=f^k(x,t) \\ u|_{\partial\Omega}=0 \\ u(x,0)=\phi(x) \end{cases}$$

where $a_{ij}^{kl}, b_i^{kl} \in c^1(\bar{\Omega})$, and

$$\lambda|\xi|^2 \le a_{ij}^{kl}(x)\xi_{ki}\xi_{lj}, \quad \forall x \in \bar{\Omega}, \xi \in R^{mn} \tag{2.6.11}$$

We also assume that

$$\begin{cases} c^k(x,z)z_k \ge \alpha^k|z_k|^{p_k} - \alpha|z|^2 - \alpha \\ p_k \ge 2, \alpha^k > 0, \alpha \ge 0 \end{cases} \tag{2.6.12}$$

$$\frac{\partial}{\partial z_j}c^k(x,z)y_jy_k \ge -c|y|^2, \quad c \ge 0 \tag{2.6.13}$$

$$\begin{cases} |c^k(x,z)| \le \beta^k|z_k|^{q_k} + \beta \\ 0 \le q_k < \max\{p_k, \frac{2(n+2)}{n}\} \end{cases} \tag{2.6.14}$$

Let X, X_1, X_2 and H be as in Theorem 2.6.1, and

$$r_k = \max\{p_k, \frac{2(n+2)}{n}\}, \quad s_k = \min\{2, r_k \cdot q_k^{-1}\}$$
$$s = \min_{1\le k\le m}\{s_k\}, \quad (1 < s \le 2).$$

Theorem 2.6.3. Let the conditions (2.6.11)-(2.6.14) be valid. If $f \in L^2_{loc}((0,\infty), L^2(\Omega, R^m)), \phi \in W^{2,2}(\Omega, R^m) \cap H^1_0(\Omega, R^m) \cap L^q(\Omega, R^m), q \ge \max_{1\le k\le m}\{2q_k\}$, then the system (2.6.10) has an unique strong solution

$$u \in L^p_{loc}((0,\infty), X_1) \cap L^s_{loc}((0,\infty), W^{2,s}(\Omega, R^m))$$
$$u_t \in L^\infty_{loc}((0,\infty), L^2(\Omega, R^m))$$

where $p = \{2, p_1, \cdots, p_k\}$.

Proof. We apply Theorem 2.1.9 to prove this theorem. Let

$$< Gu, v >= \int_\Omega [a_{ij}^{kl} D_j u_l D_i v_k + b_i^{kl} D_i u_l v_k + c^k v_k - f^k v_k] dx$$

By (2.6.11) and (2.6.13), we have

$$\begin{aligned} < DG(u)v, v > &= \int_\Omega [a_{ij}^k D_j v_l D_i v_k + b_i^{kl} D_i v_l v_k + \\ &\quad + \frac{\partial}{\partial z_j} c^k(x,u) v_j v_k] dx \\ &\ge \int_\Omega [\lambda|\nabla v|^2 - c|\nabla v| \cdot |v| - c|v|^2] dx \\ &\ge -c_1 \int_\Omega |v|^2 dx; \quad c_1 \ge 0, \forall u, v \in X \end{aligned}$$

and

$$| < Gu, v > | \le c \int_\Omega [|\nabla^2 u|^2 + |\nabla u|^2 + |u|^q + |f|^2] dx +$$
$$+ \frac{1}{2} \int_\Omega |v|^2 dx$$

Let X_3 be the space defined as follows

$$X_3 = \{u \in L^2(\Omega, R^m) | \int_\Omega [|\nabla^2 u|^2 + |\nabla u|^2 + |u|^q] dx < \infty\}$$

Thus, the conditions (2.1.13) and (2.1.14) in Theorem 2.1.9 are satisfied. Therefore, $\forall \phi \in X_3$, the solution $u \in L^p_{loc}((0,\infty), X_1)$ of (2.6.10) is unique which has the regularity

$$u_t \in L^\infty_{loc}((0,\infty), L^2(\Omega, R^m)) \tag{2.6.15}$$

By the L^p-estimates of elliptic systems, from (2.6.10) and (2.6.15) we obtain

$$\|u\|_{W^{2,s}} \le c[\|u_t\|_{L^2} + \|f\|_{L^2} + \|c(x,u)\|_{L^s}]$$

It means that

$$\begin{aligned} \int_0^T \|u\|^s_{W^{2,s}} dt &\le c \int_0^T [|u_t|^2 + |f|^2 + |c(x,u)|^s] dx dt \\ &\le c \int_0^T \int_\Omega [|u_t|^2 + |f|^2 + \beta^k |u_k|^{r_k} + \beta] dx dt \end{aligned}$$

Because $u \in L^p_{loc}((0,\infty), X_1)$, this theorem is proven.

We also consider the semilinear systems as follows

$$\begin{cases} \frac{\partial u_k}{\partial t} - D_i(a^{kl}_{ij}(x) D_j u_i) + B^k(x, u, \nabla u) = f^k(x,t) \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi(x) \end{cases} \tag{2.6.16}$$

$a^{kl}_{ij} \in c^1(\bar{\Omega})$ satisfy (2.6.11), and

$$\int_\Omega B^k(x,u,\nabla u) u_k dx \ge -c \int_\Omega |u|^2 dx - c, \quad c \ge 0 \tag{2.6.17}$$

$$\int_\Omega [\frac{\partial B^k(x,u,\nabla u)}{\partial z_j} v_j v_k + \frac{\partial B^k(x,u,\nabla u)}{\partial \xi_{lj}} D_j v_l v_k] dx$$

$$\ge -c \int_\Omega |v|^2 dx, \quad \forall u, v \in c^\infty_0(\Omega, R^m) \tag{2.6.18}$$

(2..19) $$|B^k(x,z,\xi)| \le c[|z|^{p_1} + |\xi|^{p_2} + 1]$$

where

$$p_1 = \begin{cases} < 1 + \frac{4(n+1)}{n^2}, & n > 2 \\ < 4, & n \le 2 \end{cases}$$

$$p_2 = \begin{cases} < 1 + \frac{2}{n}, & n > 2 \\ < 2, & n \le 2 \end{cases}$$

By using Theorem 2.1.13, we can obtain the follow theorem

Theorem 2.6.4. Under the conditions (2.6.17)-(2.6.19), if $f \in L^2_{loc}((0,\infty), L^2(\Omega, R^m))$, $\phi \in W^{2,2}(\Omega, R^m) \cap H^1_0(\Omega, R^m)$, then the system (2.6.16) has an unique strong solution

$$u \in L^2_{loc}((0,\infty), H^1_0(\Omega, R^m)) \cap L^s_{loc}((0,\infty), W^{2,s}(\Omega, R^m))$$
$$u_t \in L^\infty_{loc}((0,\infty), L^2(\Omega, R^m))$$

where $s = \min_{1\le k\le m}\{2, \frac{2(n+2)}{n} p_1^{-1}\}$.

The proof is the same as that of Theorem 2.6.3, we omit the details.

Finally, we consider the regularity of the following quasilinear systems possessing some variational structure

(2.6.20) $$\begin{cases} \frac{\partial u_k}{\partial t} - D_i A^k_i(x,u,\nabla u) + A^k_0(x,u,\nabla u) = B^k(x,u,\nabla u) \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi \end{cases}$$

Suppose that there exists a c^2 function $F(x,z,\xi)$ on $x \in \Omega \subset R^n$, $z = \{z_1, \cdots, z_m\}$ and $\xi = \{\xi_{kj} | 1 \le k \le m, 1 \le j \le n\}$, such that

(2.6.21) $$\begin{cases} A^k_i(x,z,\xi) = \frac{\partial F(x,z,\xi)}{\partial \xi_{ki}} \\ A^k_0(x,z,\xi) = \frac{\partial F(x,z,\xi)}{\partial z_k} \end{cases}$$

and

(2.6.22) $$F(x,z,\xi) \ge \lambda |\xi|^p - c, \quad \lambda, c > 0, p \ge 2$$

(2.6.23) $$[A^k_i(x,z,\xi) - A^k_i(x,z,\eta)][\xi_{ki} - \eta_{ki}] \ge c|\xi - \eta|^2; \quad c > 0.$$

(2.6.24) $$\begin{cases} |A^k_i(x,z,\xi)| \le c|\xi|^{p-1} + \mu_1(z) \\ |A^k_0(x,z,\xi)| \le c|\xi|^{p_1} + \mu_2(z) \\ |B^k(x,z,\xi)| \le c[|\xi|^{\frac{p}{2}} + |z|^{\frac{p}{2}} + 1] \end{cases}$$

where

$$p_1 = \begin{cases} < p - \frac{n-p}{n}, & p < n \\ < p, & p = n \\ = p, & p > n \end{cases}$$

$$\mu_1(z) = \begin{cases} \alpha|z|^{q_1} + \alpha \begin{cases} q_1 < \frac{n(p-1)}{n-p}, & p < n \\ q_1 < \infty, & p = n \end{cases} \\ \mu_1 \in c(R^m) \ a \ positive \ function, \ as \ p > n \end{cases}$$

$$\mu_2(z) = \begin{cases} \alpha|z|^{q_2} + \alpha \begin{cases} q_2 < \frac{np}{n-p} - 1, & p < n \\ q_2 < \infty, & p = n \end{cases} \\ \mu_2 \in c(R^m) \ a \ positive \ function, \ as \ p > n. \end{cases}$$

In addition, we also assume that $B^k(x, z, \xi)$ are c^1, and

$$\frac{\partial^2 F}{\partial \xi_{ki} \partial \xi_{lj}} \eta_{ki} \eta_{lj} + \frac{\partial^2 F}{\partial z_k \partial z_l} y_k y_l + 2 \frac{\partial^2 F}{\partial \xi_{ki} \partial z_l} \eta_{ki} y_l$$

$$-\frac{\partial B^k}{\partial \xi_{lj}} \eta_{lj} y_k - \frac{\partial B^k}{\partial z_l} y_k y_k \geq c_1 |\eta|^2 - c_2 |y|^2 \tag{2.6.25}$$

$\forall (x, z, \xi) \in \Omega \times R^m \times R^{mn}$ and $(y, \eta) \in R^m \times R^{mn}$, where $c_1 > 0, c_2 \geq 0$.

Theorem 2.6.5. Under the conditions (2.6.21)-(2.6.24), for any $\phi \in W_0^{1,p}(\Omega, R^m)$, the problem (2.6.20) has a global weak solution

$$u \in L^\infty_{loc}((0, \infty), W_0^{1,p}(\Omega, R^m))$$

$$u_t \in L^2_{loc}((0, \infty), L^2(\Omega, R^m))$$

Moreover, if the condition (2.6.25) is also satisfied, then for $\phi \in W^{2,p}(\Omega, R^m) \cap H_0^1(\Omega, R^m)$, the solution of (2.6.20) is unique, which has the regularity

$$u \in L^2_{loc}((0, \infty), H^2_{loc}(\Omega, R^m))$$

$$u_t \in L^2_{loc}((0, \infty), H_0^1(\Omega, R^m)) \cap L^\infty_{loc}((0, \infty), L^2(\Omega, R^m))$$

This theorem can be proven by using Theorem 2.2.5 and Theorem 2.2.7 in the same methods as used in Theorem 2.4.7 and Theorem 2.4.10, and we omit the proof here.

2.6.3. Uniform boundedness in $W^{2,p}$-norms

We conclude this section by considering the uniform bundedness of global strong solutions in the $W^{2,p}$-norms for the following semilinear parabolic systems

$$\left\{\begin{array}{l} \frac{\partial u_k}{\partial t} - \triangle u_k + f^k(x,u) = g^k(x), \quad 1 \leq k \leq m \\ u|_{\partial\Omega} = 0 \\ u(x,0) = \phi \end{array}\right. \tag{2.6.26}$$

Suppose that there is a c^2 function $F_1(x, z_1, \cdots, z_m)$ such that

$$f^k(x,z) = \frac{\partial F_1(x,z)}{\partial z_k} \tag{2.6.27}$$

$$F_1(x,z) \geq c^k|z_k|^{p_k} - c, \quad p_k > 1 \ or \ = 0 \tag{2.6.28}$$

$$\frac{\partial^2 F_1(x,z)}{\partial z_k \partial z_l} y_k y_l \geq -c|y|^2 \tag{2.6.29}$$

$$\left\{\begin{array}{l} |f(x,z)| \leq \beta^k |z_k|^{q_k} + \beta \\ q_k = \max\{p_k - 1, \frac{n+2}{n-2}\} \end{array}\right. \tag{2.6.30}$$

We have the following result.

Theorem 2.6.6. Under the conditions (2.6.27)-(2.6.30) if $g \in L^2(\Omega, R^m)$, then for any $\phi \in X_2, X_2 = H^2(\Omega, R^m) \cap H_0^1(\Omega, R^m) \cap L^{2q}(\Omega, R^m), q = \max_k\{q_k\}$, the system (2.6.26) has an unique global strong solution

$$u \in L^\infty((0,\infty), W^{2,p}(\Omega, R^m) \cap X_1)$$

$$u_t \in L^\infty((0,\infty), L^2(\Omega, R^m)) \cap L^2((0,\infty), H_0^1(\Omega, R^m))$$

where $p = 1 + q^{-1}, X_1 = H_0^1(\Omega, R^m) \cap L^{q+1}(\Omega, R^m)$. Moreover, the solutions of (2.6.26) are uniformly bounded for (u_t, u) in $L^2(\Omega, R^m) \times [W^{2,p}(\Omega, R^m) \cap X_1]$, i.e. for any bounded subset $\sum \subset X_2$, there is a constant $c > 0$ such that

$$\|u_t\|_{l^2} + \|u\|_{W^{2,p}} + \|u\|_{X_1} \leq c, \quad \forall \phi \in \sum \ and \ t \geq 0$$

Proof. We use Theorem 2.2.1 and Theorem 2.2.2 to prove this theorem. Let $G : X_1 \to X_1^*$ be defined by

$$< Gu, v >= \int_\Omega [\nabla u \cdot \nabla v + f^k(x,u) v_k - g^k \cdot v_k] dx$$

and the functional $F : X_1 \to R$ defined by

$$F(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 + F_1(x,u) - g^k(x) \cdot u_k]dx$$

Obviously, $G = DF : X_1 \to X_1^*$ is weakly continuous, and by (2.6.28) we have

$$F(u) \geq \int_\Omega [\frac{1}{2}|\nabla u|^2 + c^k|u_k|^{p_k} - \frac{\epsilon}{2}|u|^2]dx$$
$$-c|\Omega| - \frac{1}{2\epsilon}\int_\Omega |g|^2 dx$$

where $\epsilon > 0$ is an arbitrary number. Hence the condition (2.2.3) in Theorem 2.2.1 is satisfied. On the other hand, by the hypothese (2.6.29) we have

$$\begin{aligned} < DG(u)v, v > &= \int_\Omega [|\nabla v|^2 + \frac{\partial^2 F_1(x,u)}{\partial z_k \partial z_l} v_k v_l]dx \\ &\geq \int_\Omega |\nabla v|^2 dx - c\int_\Omega |v|^2 dx \end{aligned}$$

and

$$\begin{aligned} | < Gu, v > | &\leq \int_\Omega [|\triangle u| + |f(x,u)| + |g|]|v|dx \\ &\leq \frac{1}{2}\int_\Omega |v|^2 dx + c\int_\Omega [|\nabla^2 u|^2 + |u|^{2q} + 1]dx \end{aligned}$$

Hence, by Theorem 2.2.1 and Theorem 2.2.2, there exists an unique solution of (2.6.26)

$$u \in L^\infty((0,\infty), X_1)$$
$$u_t \in L^\infty((0,\infty), L^2(\Omega, R^m)) \cap L^2((0,\infty), H_0^1(\Omega, R^m))$$

and the solutions (u_t, u) of (2.6.26) are uniformly bounded in $L^2(\Omega, R^m) \times X_1$:

(2.6.31) $$\|u_t\|_{L^2} + \|u\|_{X_1} \leq c, \quad \forall t \geq 0, \phi \in \sum \subset X_2$$

By the L^p-estemates of elliptic equations, from (2.6.30) and (2.6.31) one can deduce that the solutions of (2.6.26) have the regularity $u \in W^{2,p}(\Omega, R^m)$, and satisfy

$$\begin{aligned} \|u\|_{W^{2,p}} &\leq c[\|u_t\|_{L^p} + \|f(x,u)\|_{L^p} + \|g\|_{L^p} + \|u\|_{L^p}] \\ &\leq (\ by\ 1 < p < 2\ and\ p \cdot q = q + 1) \\ &\leq c[\|u_t\|_{L^2} + \|g\|_{L^2} + \|u\|_{L^{q+1}} + 1] \end{aligned}$$

This theorem follows from (2.6.31). The proof is complete.