Chapter I
Existence and Regularity of Nonlinear Stationary Equations

The existence of nonlinear stationary equations is very important for the understanding of dynamical properties of nonlinear evolution equations, and only under the existence conditions of nonlinear stationary equations, can the stability and bifurcation problems of nonlinear evolution equations be effectively researched. On the other hand, the stationary equations (i.e. the equations independent of time) themselves are also of important significance in mathematical and the other fields.

Because the majority of infinite dimensional evolution equations describing the motion in nature is of the dissipative structure, and the dissipative structure are always connected with the elliptic differential operators. Naturally, the theory of elliptic equations and elliptic systems is a core subject in the field of partial differential equations.

In this chapter, we present a basic theory for the abstract operators, which can generally treat the existence problem of not only a large class of elliptic equations and elliptic systems which can not be treated by the traditional methods, i.e. the monotone operator theory, variational principle and the Green expression method (fixed point theory+priori estimates), but also a large class of the fully nonlinear elliptic boundary value problems, the degenerate equations, and the equations with nonnegative characteristic form in general domains. Combining with the acute-angle principle and L^P-estimates, we can obtain some regularity results on elliptic equations and degenerated elliptic equations in general domains.

1.1. Preliminary Material

1.1.1. Sobolev spaces

Let $\Omega \subset R^n$ be an open set. We denote by $C^k(\Omega)$(resp. $C^k(\overline{\Omega})$), $k =$integer$\geq$

0, the space of all k times continuously differentiable functions on Ω(resp. on $\overline{\Omega}$), and

$$\begin{cases} C_0^k(\Omega) = \{u \in C^k(\Omega)| \ supp\ u \ \subset \Omega\} \\ supp\ u = the\ closure\ of\ \{x \in \Omega | u(x) \neq 0\} \end{cases}$$

The spaces $C^k(\overline{\Omega})$ are Banach spaces endowed with the norm

$$\|u\|_{C^k} = \sum_{|\alpha| \leq k} \sup_{x \in \overline{\Omega}} |D^\alpha u|$$

where $\alpha = \{\alpha_1, \cdots, \alpha_n\}, \alpha_i$=integer$\geq 0, 1 \leq i \leq n, |\alpha| = \sum_{i=1}^n \alpha_i$. We always denote

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

$$D_i u = \frac{\partial u}{\partial x_i};\ D_{ij} u = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

For $1 \leq p \leq \infty, L^p(\Omega)$ is the space of all measurable functions on Ω which are of L^p integrability. It is a Banach space with the norm

$$\|u\|_{L^p} = [\int_\Omega |u|^p dx]^{\frac{1}{p}}.$$

For $1 \leq p \leq \infty$ and nonnegative integer k, we introduce the Sobolev space which is denoted by

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq k\}$$

endowed with the norm

$$\|u\|_{W^{k,p}} = [\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx]^{\frac{1}{p}}$$

When $p = 2$, we write $W^{k,2}(\Omega) = H^k(\Omega)$ which is a Hilbert space with the scalar product

$$< u, v >_{H^k} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u \cdot D^\alpha v dx$$

The space $W_0^{k,p}(\Omega)$ is defined by

$$\begin{cases} W_0^{k,p}(\Omega) = \ the\ closure\ of\ C_0^\infty(\Omega)\ in\ W^{k,p}(\Omega), \\ \\ H_0^k(\Omega) = W_0^{k,2}(\Omega). \end{cases}$$

A equivalent norm of $W_0^{k,p}(\Omega)$ is

$$\|u\|_{W_0^{k,p}} = [\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^p dx]^{\frac{1}{p}}$$

The space $C^{k,\alpha}(\overline{\Omega}), k$=integer$\geq 0, 0 < \alpha < 1$, is a Banach space, which is defined by (without confusion, we always denote $C^{0,\alpha}(\overline{\Omega})$ by $C^\alpha(\overline{\Omega})$).

$$C^{k,\alpha}(\overline{\Omega}) = \{u \in C^k(\overline{\Omega}) | [D^\beta u]_\alpha < \infty, |\beta| = k\}$$

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + \sum_{|\beta|=k} [D^\beta u]_\alpha$$

$$[v]_\alpha = \sup_{x,y\in\Omega, x\neq y.} \frac{|v(x) - v(y)|}{|x-y|^\alpha}$$

For the Sobolev spaces, there are the following important properties, and the reader can be referred to [Ad] and [Mj] for the details.

Density Theorem.

Theorem 1.1.1. For $k \geq 1$ and $1 \leq p < \infty, W_0^{k,p}(\Omega)$ is dense in $W_0^{k-1,p}(\Omega)$; and when $\Omega \subset R^n$ is of class Lipschitz, then $C^k(\overline{\Omega})$ and $W^{k,p}(\Omega)$ are dense respectively in $W^{k,p}(\Omega)$ and $W^{k-1,p}(\Omega)$.

Embedding Theorems.

Theorem 1.1.2. Let $\Omega \subset R^n$ be a bounded domain, and $1 \leq p < \infty$, then

$$\begin{cases} W_0^{k,p}(\Omega) \subset L^q(\Omega), \forall 1 \leq q \leq \frac{np}{n-kp}, n > kp \\ W_0^{k,p}(\Omega) \subset L^q(\Omega), \forall 1 \leq q < \infty, n = kp \\ W_0^{k,p}(\Omega) \subset C^{m,\alpha}(\overline{\Omega}), \forall kp > n, m + \alpha = k - \frac{n}{p} \end{cases}$$

moreover, the inclusions are continuous, namely

$$\|u\|_{L^q} \leq C\|u\|_{W_0^{k,p}}, q \leq \frac{np}{n-kp}. \tag{1.1.1}$$

$$\|u\|_{C^{m,\alpha}} \leq C\|u\|_{W_0^{k,p}}, m + \alpha = k - \frac{n}{p}, kp > n. \tag{1.1.2}$$

where $c = c(n, \Omega, q)$.

Theorem 1.1.3. Let $\Omega \subset R^n$ be a Lipschitz domain, not necessarily bounded, then

$$\begin{cases} W^{k,p}(\Omega) \subset L^q(\Omega), q = \frac{np}{n-kp}, n > kp, \\ W^{k,p}(\Omega) \subset C^{m,\alpha}(\overline{\Omega}), m + \alpha = k - \frac{n}{p}, kp > n. \end{cases}$$

and the inclusions are continuous.

Trace Theorem (see [Te 1]).

Theorem 1.1.4. Let Ω be a domain of class C^{m+1}. For any $u \in W_0^{m,p}(\Omega), p \geq 1$, we have

$$D^\alpha u|_{\partial\Omega} = 0, \ a.e., \ \forall |\alpha| \leq m - 1.$$

Compactness Theorems.

Theorem 1.1.5. Let Ω be a bounded domain. Then the embeddings are compact

$$\begin{cases} W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega), q < \frac{np}{n-p}, n > p \\ W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega), q < \infty, n = p \\ W_0^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \alpha < 1 - \frac{n}{p}, n < p \end{cases}$$

Theorem 1.1.6. Let Ω be a bounded domain. Then the embedding is compact

$$W^{1,2}(\Omega) \cap L^p(\Omega) \hookrightarrow L^q(\Omega), 1 \leq q < p.$$

The theorem follows from Theorem 8.22 in [Ad. 1].

Interpolation Theorems.

Theorem 1.1.7. For $p < q < r$, we have the L^p interpolation inequality

$$\begin{cases} \|u\|_{L^q} \leq \epsilon \|u\|_{L^r} + \epsilon^{-\mu} \|u\|_{L^p} \\ \\ \mu = (\frac{1}{p} - \frac{1}{q})/(\frac{1}{q} - \frac{1}{r}) \end{cases} \tag{1.1.2}$$

where $\epsilon > 0$ is an arbitrary real number.

The interpolation theorem can be found in [GT]

Theorem 1.1.8. There is a constant $C = C(m,p,n)$, for any $\Omega \subset R^n, \epsilon > 0$ and $u \in W_0^{m,p}(\Omega)$, we have

$$
(1.1.3) \qquad \begin{cases} |u|_{j,p} \le C[\epsilon |u|_{m,p} + \epsilon^{-\mu}\|u\|_{L^p}] \\ |u|_{k,p} = [\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^p dx]^{\frac{1}{p}} \end{cases}
$$

where $0 \le j \le m-1, \mu = j/(m-j)$.

Theorem 1.1.9. Suppose that $\Omega \subset R^n$ is of Lipschitz and $\epsilon_0 > 0$ given. Then there is a constant $C = C(\epsilon_0, m, p, \Omega)$, for any $\epsilon, 0 < \epsilon \le \epsilon_0$ and $u \in W^{m,p}(\Omega)$

$$
|u|_{j,p} \le C[\epsilon |u|_{m,p} + \epsilon^{-\mu}\|u\|_{L^p}]
$$

where $0 \le j \le m-1, \mu = j/(m-j)$.

From Theorem 1.1.9, we can get an equivalent norm of $W^{k,p}(\Omega)$ as follows

$$
\|u\|_{W^{k,p}} = [\int_\Omega |u|^p dx]^{\frac{1}{p}} + [\int_\Omega \sum_{|\alpha|=k} |D^\alpha u|^p dx]^{\frac{1}{p}}.
$$

1.1.2 Regularity estimates

We consider the linear elliptic equations

$$
(1.1.4) \qquad \begin{cases} -\sum_{i,j=1}^n a_{ij} D_{ij} u + \sum_{i=1}^n b_i D_i u + cu = f(x), x \in \Omega \\ u|_{\partial\Omega} = \phi \end{cases}
$$

where $\Omega \subset R^n$ is a bounded domain, and

$$
a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2, \forall x \in \overline{\Omega}, \xi \in R^n, \lambda > 0.
$$

For the equations (1.1.4), we introduce the Schauder global estimate theorem and L^p-estimate theorem, which are very important for the theory of nonlinear elliptic equations.

Schauder Global Estimate Theorem (see [GT]).

Theorem 1.1.10. Let $\Omega \subset R^n$ be of class $C^{2,\alpha}$, and $a_{ij}, b_i, c, f \in C^{0,\alpha}(\overline{\Omega}), \phi \in C^{2,\alpha}(\overline{\Omega})$. If $u \in C^{2,\alpha}(\overline{\Omega})$ is a solution of (1.1.4), then

$$
(1.1.5) \qquad \|u\|_{C^{2,\alpha}} \le C[\|u\|_{C^0} + \|\phi\|_{C^{2,\alpha}} + \|f\|_{C^{0,\alpha}}]
$$

where C is a constant only depending on $n, \alpha, \Omega, \lambda$ and the Holder modular of $a_{ij}, b_i, c(x)$.

L^p-Estimates of Strong Solutions (see [ADN]).

Theorem 1.1.11. Suppose that Ω is of class $C^2, a_{ij} \in C^0(\overline{\Omega}), b_i, c \in L^\infty(\Omega), \phi \in W^{2,p}(\Omega), f \in L^p(\Omega), 1 < p < \infty$. If $u \in W^{2,p}(\Omega)$ is a strong solution of (1.1.4), then

$$\|u\|_{W^{2,p}} \leq C[\|f\|_{L^p} + \|\phi\|_{W^{2,p}} + \|u\|_{L^p}] \tag{1.1.6}$$

where C is a constant depending on n, p, Ω, λ and the modulars $\|a_{ij}\|_{L^\infty}, \|b_i\|_{L^\infty}, \|C\|_{L^\infty}$.

Now, we consider the linear elliptic equations of order $2m (m \geq 1)$,

$$Au = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D_\alpha (a_{\alpha,\beta} D_\beta u) = f(x), x \in \Omega. \tag{1.1.7}$$

where $\Omega \subset R^n$ is bounded, and

$$\lambda |\xi|^{2m} \leq \sum_{|\alpha|=|\beta|=m} a_{\alpha,\beta}(x) \xi^\alpha \xi^\beta, \forall x \in \overline{\Omega}$$

$\xi = (\xi_1, \cdots, \xi_n) \in R^n, \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. The adjoint operator of A is defined by

$$A^* u = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\beta|} D_\beta (a_{\alpha,\beta}(x) D_\alpha u).$$

Agmon's Theorem (see [Ag] and [Au]).

Theorem 1.1.12. Let Ω be C^{2m} and $a_{\alpha,\beta} \in C^m(\overline{\Omega})$. Let $u \in L^q(\Omega)$ and $f \in L^p(\Omega), q, p > 1$. If for any $v \in C^{2m}(\overline{\Omega}) \cap W_0^{m,p}(\Omega)$,

$$\int_\Omega u \cdot A^*(v) dx = \int_\Omega f \cdot v dx$$

then $u \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ is a strong solution of (1.1.7), and

$$u\|_{W^{2m,p}} \leq C[\|f\|_{L^p} + \|u\|_{L^p}]$$

where $C = C(n, p, \Omega, A)$.

1.1.3. Maximum principle

Hopf Maximum Principle (see [GT]).

Theorem 1.1.13. We denote

$$Lu = -\sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{i=1}^{n} b_i D_i u + cu,\ x \in \Omega.$$

where $a_{ij}, b_i, c \in C(\overline{\Omega}), c(x) \le 0$, and Ω is of class C^2. Suppose that L is uniformly elliptic in Ω, and $u \in C^2(\overline{\Omega})$ satisfies that $Lu \ge 0$. If $x_0 \in \partial\Omega, u(x_0) \le 0$ and $u(x) > u(x_0) \forall x \in \Omega$, then

$$\frac{\partial u(x_0)}{\partial n} < 0 \tag{1.1.8}$$

where n is the outward normal at $x_0 \in \partial\Omega$.

In the following, we give the maximum principle of quasilinear elliptic equations in divergence form, referred to [GT].

$$-\sum_{i=1}^{n} D_i A_i(x, u, Du) + B(x, u, Du) = 0, x \in \Omega \tag{1.1.9}$$

where Ω is bounded, and

$$\sum_{i=1}^{n} A_i(x, z, p) p_i \ge \lambda |p| - b_0 \tag{1.1.10}$$

$$-B(x, z, p) \cdot signz \le b_1 |p|^{k-1} + b_2 \tag{1.1.11}$$

where $k > 1, \lambda, b_0, b_1, b_2 > 0$ are constants.

Theorem 1.1.14. Let $u \in W^{1,2}(\Omega)$ be a weak solution of (1.1.9), i.e. u satisfies

$$\int_\Omega [\sum_{i=1}^{n} A_i(x, u, Du) D_i v + B(x, u, Du) v] dx = 0$$

$\forall v \in c_0^1(\overline{\Omega})$. Then, under the conditions (1.1.10) and (1.1.11), we have the estimate

$$\sup_\Omega |u| \le \sup_{\partial\Omega} |u| + c(b_0 + b_2)$$

where $c = c(n, k, \lambda, \Omega, b_1)$.

1.2. Basic Theories and Methods

1.2.1. Green expression method

In this subsection, we illustrate the ideal of Green expression method by discussing the existence of the following equation

$$\begin{cases} -\Delta u + |u|^p u = f(x), x \in \Omega \subset R^n \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.2.1}$$

where $f \in C^\alpha(\overline{\Omega})(0 < \alpha < 1), \Omega$ is bounded and $C^{2,\alpha}$.

First, we transfer from the existence problem of (1.2.1) to the existence of fixed points of an abstract operator as follows. By the theory of linear elliptic equations, for any $u \in C^\alpha(\overline{\Omega})$, the linear equation

$$\begin{cases} -\Delta v = f(x) - |u|^p u \\ v|_{\partial\Omega} = 0 \end{cases}$$

has an unique solution $v \in C^{2,\alpha}(\overline{\Omega})$. Then we define a mapping $T : C^\alpha(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ by

$$Tu = v, \tag{1.2.2}$$

Obviously, the existence of equation (1.2.1) is equivalent to the fixed point existence of the operator equation (1.2.2) in $C^\alpha(\overline{\Omega})$.

It is well known that the embedding is compact

$$C^{2,\alpha}(\overline{\Omega}) \hookrightarrow C^\alpha(\overline{\Omega}).$$

From the Schauder estimate theorem (Theorem 1.1.10) one can deduce

$$\|v\|_{C^{2,\alpha}} = \|Tu\|_{C^{2,\alpha}} \leq C[\|f\|_{C^\alpha} + \|u\|_{C^\alpha}^{p+1}]$$

which means that the operator T is compact.

Next, we consider the homotopical completely continuous field

$$id - T_\lambda : C^\alpha(\overline{\Omega}) \to C^\alpha(\overline{\Omega}), 0 \leq \lambda \leq 1$$

where T_λ is defined by that $T_\lambda u = v$ is the unique solution of the equation

$$\begin{cases} -\Delta v = f - \lambda |u|^p u, \\ v|_{\partial\Omega} = 0 \end{cases}$$

It is easy to see that $T_1 = T$ and $T_0 : C^\alpha(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ is a constant value mapping. Therefore, by the topological degree theory, we have

$$deg(id - T_0, U, 0) = 1$$

where $U \subset C^\alpha(\overline{\Omega})$ is an arbitrary open set including the T_0 value.

Let $u_\lambda(x)$ be the solution of the equation

$$\left\{ \begin{array}{l} -\Delta u = f - \lambda |u|^p u, \quad 0 \leq \lambda \leq 1 \\ u|_{\partial\Omega} = 0 \end{array} \right. \tag{1.2.3}$$

For the solution u_λ of (1.2.3) if we can get the uniform Holder estimate

$$\|u_\lambda\|_{C^\alpha} \leq C \tag{1.2.4}$$

where C is a constant independent of λ, then for all $0 \leq \lambda \leq 1$ the operator equations

$$u - T_\lambda u = 0, \quad u \in C^\alpha(\Omega) \tag{1.2.5}$$

have no solutions on the boundary ∂U for $U \subset C^\alpha(\overline{\Omega})$ great enough. By the homotopy invariability of the topological degree, we have

$$deg(id - T_1, U, 0) = deg(id - T_0, U, 0) = 1.$$

Therefore the equations (1.2.5) have solutions in $C^\alpha(\overline{\Omega})$ for all $0 \leq \lambda \leq 1$, which implies that (1.2.2) has fixed point.

Finally, we show the uniform Holder estimate (1.2.4). By the Sobolev embedding theorem (Theorem 1.1.2)

$$\|u\|_{C^\alpha} \leq C\|u\|_{W^{2,q}}, \quad q > \frac{n}{2} \tag{1.2.6}$$

And by the L^p-estimate theorem (Theorem 1.1.11), the solution of (1.2.3) satisfies

$$\begin{aligned} \|u_\lambda\|_{W^{2,q}} &\leq C[\|u_\lambda\|_{L^q} + \|f - \lambda u_\lambda^{p+1}\|_{L^q}] \\ &\leq C[\|f\|_{L^q} + \|u_\lambda\|_{C^0}^{p+1}] \end{aligned} \tag{1.2.7}$$

where C is independent of $\lambda (0 \leq \lambda \leq 1)$.

Obviously, the nonlinear term $B(x, u, Du) = \lambda |u|^p u - f(x)$ in (1.2.3) satisfies (1.1.10) with $b_0 = b_1 = 0, b_2 = \sup_\Omega |f|$. Therefore, from Theorem 1.1.14 one deduce

$$\|u_\lambda\|_{C^0} \leq \sup_{\partial\Omega} |u| + cb_2 = c\|f\|_{C^0} \tag{1.2.8}$$

where C is independent of λ.

From the inequalities (1.2.6)-(1.2.8), we get

$$\|u_\lambda\|_{C^\alpha} \leq C[\|f\|_{L^q} + \|f\|_{C^0} + \|f\|_{C^0}^{p+1}]$$

which is the estimate (1.2.4).

Remark 1.2.1. By using the Green expression method, one can also discuss the quasilinear elliptic equations, see [GT].

$$\begin{cases} -\sum_{i,j=1}^n a_{ij}(x,u,Du)D_{ij}u + b(x,u,Du) = 0 \\ u|_{\partial\Omega} = \phi \end{cases} \tag{1.2.9}$$

In this case, we transform the problem (1.2.9) into the fixed point problem of operator

$$T : C^{1,\alpha}(\overline{\Omega}) \to C^{1,\alpha}(\overline{\Omega}), \tag{1.2.10}$$

where, for $u \in C^{1,\alpha}(\overline{\Omega}), Tu = v$ is the solution of the equation

$$\begin{cases} -\sum_{i,j=1}^n a_{ij}(x,u,Du)D_{ij}v + b(x,u,Du) = 0 \\ v|_{\partial\Omega} = \phi \end{cases}$$

And the homotopical operator $T_\lambda (0 \le \lambda \le 1)$ is defined by that $T_\lambda u = v$ is the solution of the equation

$$\begin{cases} -\sum_{i,j=1}^n a_{ij}(x,\lambda u,\lambda Du)D_{ij}v + \lambda b(x,u,Du) = 0 \\ v|_{\partial\Omega} = \phi \end{cases}$$

Thus, the existence of (1.2.9) in $C^{2,\alpha}(\overline{\Omega})$ is reduced to the proof of the uniformly estimate of solutions of (1.2.9)

$$\|u\|_{C^{1,\alpha}} \le C. \tag{1.2.11}$$

In the proof of (1.2.11), the De Giorgi estimates and the maximum principle for quasilinear equations are crucial.

The operator T of (1.2.10) can be expressed by the Green function, e.g. for $\phi = 0, u \in C^{1,\alpha}(\overline{\Omega})$

$$Tu = \int_\Omega G_u(x,y)b(y,u(y),Du(y))dy$$

where $G_u(x,y)$ is the Green function of the elliptic operator $-\sum_{i,j=1}^n a_{ij}(x,u,Du)D_{ij}$. It is why we call the method above the Green expression method.

Virtue of Green Expression Method.

i). The existence of solutions is in the $C^{2,\alpha}$ classical sense.

ii). Many of the existence results of quasilinear elliptic equations of second order obtained by this method can not be covered by the other methods and theories.

For example, for the quasilinear elliptic equations with divergence form as follows

$$(1.2.12)\qquad \begin{cases} -\sum_{i=1}^n D_i A_i(x,u,Du) + B(x,u,Du) = 0 \\ u|_{\partial\Omega} = \phi \end{cases}$$

by using the Green expression method, we have the following results, which can not be obtained by the other methods. Suppose that

$$(1.2.13)\qquad \begin{cases} \sum_{i=1}^n A_i(x,z,p)p_i \geq b_1|p|^k - b_2 \\ -B(x,z,p)\cdot signz \leq b_3|p|^{k-1} + b_4 \\ \lambda(x,z,p)|p|^2 \leq \sum_{i,j=1}^n D_{p_i}A_j(x,z,p)p_ip_j, \\ |p|^\tau \leq 0(\lambda(x,z,p)) \\ D_p\overline{A}(x,z,p) = 0(|p|^\tau) \\ |p|D_z\overline{A}, D_x\overline{A}, B = 0(|p|^k) \end{cases}$$

as $p \to \infty$ uniformly for $x \in \Omega$ and $z < \infty, 0 < \lambda(x,z,p) \forall (x,z,p) \in \overline{\Omega} \times R \times R^n, k = 2+\tau, \tau > -1, \overline{A} = \{A_1, \cdots, A_n\}$.

Theorem 1.2.2. Let $\Omega \subset R^n$ be $C^{2,\alpha}$ and bounded, $A_i \in C^{1,r}(\overline{\Omega} \times R \times R^n), B \in C^r(\overline{\Omega} \times R \times R^n), 0 < r < 1, \phi \in C^{2,\alpha}(\overline{\Omega})$. Then under the condition (1.2.13), the problem (1.2.12) has a solution $u \in C^{2,r}(\overline{\Omega})$.

Imperfection of Green Expression Method.

i). It is too tedious for the proof of uniformly boundedness of $C^{1,\alpha}$-modular.

ii). It is unavailable for the cases of unbounded domains; higher order equations; elliptic systems and degenerate elliptic equations.

1.2.2 Monotone operator theory

Let X be a Banach space. Mapping $G: X \to X^*$ is a monotone operator, if

$$(1.2.14)\qquad < Gu - Gv, u - v > \geq 0, \forall u, v \in X.$$

Theorem 1.2.3. Let X be a reflective Banach space. Suppose that $G : X \to X^*$ is continuous and monotone, and

$$\frac{< Gu, u >}{\|u\|} \to \infty, \; as \; \|u\| \to \infty \tag{1.2.15}$$

then, for any $f \in X^*, Gu = f$ has a solution in X.

We shall show the use of Theorem 1.2.2 by discussing the quasilinear elliptic equations

$$\begin{cases} -\sum_{i=1}^{n} D_i(|D_i u|^p D_i u) + |u|^q u = f(x), x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.2.16}$$

where $\Omega \subset R^n$ is an arbitrary domain, $p, q \geq 0$, and $f \in L^{q+2}(\Omega)$. Let

$$X = W_0^{1,p+2}(\Omega) \cap L^{q+2}(\Omega).$$

Definition 1.2.4. $u \in X$ is called a weak solution of (1.2.16), if for any $v \in X$, we have

$$\int_\Omega [\sum_{i=1}^{n} |D_i u|^p D_i u D_i v + |u|^q uv - f \cdot v] dx = 0$$

We define a mapping $G : X \to X^*$ by

$$< Gu, v >= \int_\Omega [\sum_{i=1}^{n} |D_i u|^p D_i u D_i v + |u|^q u \cdot v] dx$$

Obviously, the existence of solutions of the equation

$$Gu = f, f \in X^*$$

is equivalent to the existence of weak solution of (1.2.16). It is easy to check that $G : X \to X^*$ is continuous, and

$$< Gu, u >= \int_\Omega [\sum_{i=1}^{n} |D_i u|^{p+2} + |u|^{q+2}] dx$$

which means (1.2.15) holds.

We know that, for a monotone increase function $g(x), g'(x) \geq 0$. Therefore, by the mean value theorem,

$$(g(x_1) - g(x_2))(x_1 - x_2) = g'(\widetilde{x})(x_1 - x_2)^2 \geq 0$$

Since $g(x) = |x|^p x (p \geq 0)$ is a monotone increase function, hence we have

$$< Gu - Gv, u - v >= \int_\Omega [\sum_{i=1}^{n} (|D_i u|^p D_i u - |D_i v|^p D_i v) \times$$

$$\times (D_i u - D_i v)] dx + \int_\Omega (|u|^q u - |v|^q v)(u - v) dx \geq 0$$

Thus, from Theorem 1.2.3, we obtain the existence of weak solution of (1.2.16).

Virtue of Monotone Operator Theory

i). It is simple and easy to understand.

ii). It is a generally method, which can treat a large class of quasilinear elliptic equations of order $2m (m \geq 1)$ in general domains.

Imperfection of Monotone Operator Theory.

i). The monotone condition is sharper for the general differential equations, especially, for the nonlinear elliptic systems.

ii). The solutions obtained are in weak sense.

1.2.3. Variational principle.

Let $\Omega \subset R^n$ be a bounded domain, and $F \in C^1(\Omega \times R \times R^n)$ be a given function. We consider the equation

$$\begin{cases} -\sum_{i=1}^{n} D_i A_i(x, u, Du) + B(x, u, Du) = 0 \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.2.17}$$

where

$$\begin{cases} A_i(x, z, \xi) = \frac{\partial F(x,z,\xi)}{\partial \xi_i} \\ \\ B(x, z, \xi) = \frac{\partial F(x,z,\xi)}{\partial z} \end{cases} \tag{1.2.18}$$

The equation (1.2.17) with condition (1.2.18) is called to be of variational structure.

The existence problem of weak solutions of (1.2.17) can be reduced to the existence problem of minimum points of the below functional in $W_0^{1,p}(\Omega), p > 1$,

$$I(u) = \int_\Omega F(x, u, Du) dx, \ u \in W_0^{1,p}(\Omega) \tag{1.2.19}$$

where

$$F(x, z, \xi) = O(|\xi)^p), \ for \ x \in \Omega \ and \ |z| < \infty.$$

Suppose that $u_0 \in W_0^{1,p}(\Omega)$ is a minimum point of (1.2.19), namely there is a neighborhood $O \subset W_0^{1,p}(\Omega)$ of u_0 such that

$$I(u_0) \leq I(u), \forall u \in O \subset W_0^{1,p}(\Omega)$$

then we have

$$\frac{I(u_0 \pm \epsilon v) - I(u_0)}{\epsilon} \geq 0$$

$\forall v \in W_0^{1,p}(\Omega)$ and $\epsilon > 0$ small enough.

From (1.2.19) and (1.2.18), one deduces that

$$\lim_{\epsilon \to 0^+} \frac{I(u_0 \pm \epsilon v) - I(u_0)}{\epsilon} = \pm \int_\Omega [\sum_{i=1}^n A_i(x, u_0, Du_0) D_i v$$

$$+ B(x, u_0, Du_0) v] dx \geq 0, \forall v \in W_0^{1,p}(\Omega)$$

Thus, we obtain

$$\int_\Omega [\sum_{i=1}^n A_i(x, u_0, Du_0) D_i v + B(x, u_0, Du_0) v] dx = 0$$

$\forall v \in W_0^{1,p}(\Omega)$, which says that u_0 is a weak solution of (1.2.17)

Now, we give an existence theorem of minimum points of abstract functional on a Banach space.

Definition 1.2.5. Let X be a Banach space, and $I : X \to R^1$ be a C^1 functional. We say that I is weakly inferior semi-continuous, if as $x_n, x_0 \in X$, and $x_n \rightharpoonup x_0$($\rightharpoonup$ is weakly convergent), we have

$$\underline{\lim}_{n\to\infty} I(x_n) \geq I(x_0)$$

Theorem 1.2.6. Let X be a reflective Banach space, and $I : X \to R^1$ be a C^1 functional. Suppose that I is weakly inferior semi-continuous, and $I(x) \to \infty$ as $\|x\| \to \infty, I$ has at least a minimum point in X.

In the following, we shall show how to apply Theorem 1.2.6 to the partial differential equations by the given example

$$(1.2.20) \qquad \begin{cases} -\sum_{i=1}^n D_i(|D_i u_1|^{p_1} D_i u_1) + u_1 u_2^2 = f_1(x) \\ -\sum_{i=1}^n D_i(|D_i u_2|^{p_2} D_i u_2) + u_1^2 u_2 = f_2(x), \quad x \in \Omega \\ u_1|_{\partial\Omega} = 0, u_2|_{\partial\Omega} = 0 \end{cases}$$

where $p_1, p_2 \geq 0$ and $\Omega \subset R^3$ is bounded.

The elliptic system (1.2.20) is of the variational structure, in fact, if we write the system (1.2.20) in this form

$$\begin{cases} -\sum_{i=1}^n D_i A_i^1(x,u,Du) + B^1(x,u,Du) = 0 \\ -\sum_{i=1}^n D_i A_i^2(x,u,Du) + B^2(x,u,Du) = 0 \end{cases}$$

$u = (u_1, u_2) \in W_0^{1,p_1+2}(\Omega) \times W_0^{1,p_2+2}(\Omega)$, then we have

$$\begin{cases} F(x,z_1,z_2,\xi,\eta) = \sum_{i=1}^n (\frac{1}{p_1+2}|\xi_i|^{p_1+2} + \frac{1}{p_2+2}|\eta_i|^{p_2+2}) + \frac{1}{2}z_1^2 z_2^2 - f_1 z_1 - f_2 z_2 \\ A_i^1(x,z_1,z_2,\xi,\eta) = \frac{\partial F}{\partial \xi_i} = |\xi_i|^{p_1}\xi_i, \\ A_i^2(x,z_1,z_2,\xi,\eta) = \frac{\partial F}{\partial \eta_i} = |\eta_i|^{p_2}\eta_i \\ B^1(x,z_1,z_2,\xi,\eta) = \frac{\partial F}{\partial z_1} = z_1 z_2^2 - f_1 \\ B^2(x,z_1,z_2,\xi,\eta) = \frac{\partial F}{\partial z_2} = z_1^2 z_2 - f_2 \end{cases}$$

Let $X = W_0^{1,p_1+2}(\Omega) \times W_0^{1,p_2+2}(\Omega)$. The functional corresponding to (1.2.20) is as follows

$$I(u) = \int_\Omega [\sum_{i=1}^n (\frac{1}{p_1+2}|D_i u_1|^{p_1+2} + \frac{1}{p_2+2}|D_i u_2|^{p_2+2}) + \frac{1}{2}u_1^2 u_2^2 - f_1 u_1 - f_2 u_2] dx$$

$u = (u_1, u_2) \in X$. Obviously, X is reflective, $I : X \to R^1$ is C^1, and

$$I(u) \to \infty, \ as \ \|u\|_X \to \infty \ (by \ Holder \ inequality).$$

Next we check the weak inferior semi-continuousness of the functional I. Let $u_n \in X$ and $u_n \rightharpoonup v$ in X. We notice that

$$I(u) = \|u_1\|^{p_1+2}_{W_0^{1,p_1+2}} + \|u_2\|^{p_2+2}_{W_0^{1,p_2+2}} + J(u)$$

$$J(u) = \int_\Omega [\frac{1}{2}u_1^2 u_2^2 - f_1 u_1 - f_2 u_2] dx$$

By the compact embedding theorem (Theorem 1.1.5), we have

$$\lim_{n\to\infty} J(u_n) = J(v)$$

Hence

$$\underline{\lim}_{n\to\infty} I(u_n) = \underline{\lim}_{n\to\infty} [\|u_1^n\|^{p_1+2}_{W_0^{1,p_1+2}} + \|u_2^n\|^{p_2+2}_{W_0^{1,p_1+2}}] + J(v)$$

By the Mazur theorem, it is known that the functionals $\|u_k\|^{p_k+2}_{W_0^{1,p_k+2}}(k=1,2)$ are weakly inferior semi-continuous. Therefore the functional

$$\|u_1\|^{p_1+2}_{W_0^{1,p_1+2}} + \|u_2\|^{p_2+2}_{W_0^{1,p_2+2}}$$

is also weakly inferior semi-continuous. Thus, $I(u)$ is weakly inferior semi-continuous.

Theorem 1.2.6 tells us that $I(u)$ has a minimum point in X, which implies that the system (1.2.20) has a weak solution.

In the applications of variational principle, the variational structure is of great restriction. But, in some spacial problems, the variational is a strong means.

We notice that the two conditions below are the same in essential:
i). function $f(x,z,\xi)$ is convex on variable ξ;
ii). functional $I(u)=\int_\Omega f(x,u,Du)dx$ is weakly inferior semi-continuous.

1.3. Abstract Theory of the Inner Product Operators

1.3.1. Upward weakly continuous operators

Let X be a linear space, X_1, X_2 be the completion of X respectively with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Suppose that X_1 is a reflexive Banach space, and X_2 is a separable Banach space.

Definition 1.3.1. A mapping $G: X_1 \to X_2^*$ is called to be weakly continuous, if for any $x_n, x_0 \in X_1, x_n \rightharpoonup x_0$ in X_1, we have

$$\lim_{n\to\infty} < Gx_n, y >=< Gx_0, y >, \ \ \forall y \in X_2$$

Theorem 1.3.2. Suppose that $G: X_1 \to X_2^*$ is weakly continuous, if there exists a bounded open set $\Omega \subset X_1$, such that

$$< Gu, u >\geq 0, \ \ \forall u \in \partial\Omega \cap X$$

then the equation $Gu = 0$ has a solution in X_1.

This theorem is a corollary of the following theorem (Theorem 1.3.3), here we specially state it in order to emphasize the difference between the upward operators and the downward operators in the later applications.

In some problems of partial differential equations, the space X_2 should be taken to be embedded in X_1, i.e. $X_2 \hookrightarrow X_1$, which means that the regularity of $u \in X_2$ is higher than that of $u \in X_1$. Therefore the mapping $G : X_1 \to X_2^*$ maps the lower differentiability elements into the dual space of a higher differentiability space. In general, the degenerate elliptic equations and the equations with nonnegative characteristic form correspond to the upward mappings, hence, Theorem 1.3.2 is a basic tool to treat these equations.

1.3.2. Downward weakly continuous operators.

Let X be a linear space, X_2 be a reflexive Banach space and X_1 be a separable Banach space. Let $X \subset X_2$ and there exist an one to one linear mapping

$$L : X \to X_1$$

which is dense.

Theorem 1.3.3. Let $G : X_2 \to X_1^*$ be weakly continuous and $\Omega \subset X_2$ be a bounded open set. If

$$< Gu, Lu > \geq 0, \ \ \forall u \in \partial\Omega \cap X \tag{1.3.1}$$

then $Gu = 0$ has a solution in X_2.

Proof. Because X_1 is separable, and $LX \subset X_1$ is dense, there exists a sequence $\{e_n\} \subset X_1 \cap L(X)$ such that span$\{e_n\}$ is dense in X_1. Denote by

$$Z_n = span\{\widetilde{e}_1, \cdots, \widetilde{e}_n\},$$

$$Y_n = span\{e_1, \cdots, e_n\}, \quad e_i = L\widetilde{e}_i \tag{1}$$

We define a mapping $A_n : Z_n \to Y_n$ by

$$< A_n u, v >=< Gu, v >, \ \ \forall u \in Z_n, v \in Y_n$$

Since $LZ_n = Y_n$, by (1.3.1) we have

$$< A_n u, Lu >=< Gu, Lu > \geq 0, \ \ \forall u \in \partial\Omega \cap Z_n$$

Thanks to the Poincare-Bohl theorem of the Brouwer degree (cf. [Ze]), we get

$$deg(A_n, \Omega_n, 0) = deg(L, \Omega_n, 0), \quad \Omega_n = \Omega \cap Z_n$$

provided $A_n u \neq 0, \forall u \in \partial\Omega_n$. Because $L : Z_n \to Y_n$ is a linear homeomorphism, hence

$$deg(L, \Omega_n, 0) \neq 0$$

Thus, it implies that $A_n u = 0$ has a solution $u_n \in \overline{\Omega}_n$, i.e.

$$< Gu_n, v >= 0, \quad \forall v \in Y_n \tag{1.3.2}$$

Because $\{u_n\} \subset \overline{\Omega} \subset X_2$ is bounded, and X_2 is reflexive, there is a $u_0 \in X_2$ such that $u_n \rightharpoonup u_0$ in X_2. Then from (1.3.2) it follows

$$\lim_{n\to\infty} < Gu_n, v >=< Gu_0, v >, \quad \forall v \in Y_n$$

Due to the denseness of $U_{n=1}^{\infty} Y_n \subset X_1$, it follows that $Gu_0 = 0$. The proof is complete.

Remark 1.3.4. When $X \subset X_1$, and $L = id : X \to X_1$ is a inclusion mapping, then Theorem 1.3.2 is a corollary of Theorem 1.3.3. When $L \neq id$ and $X_2 \hookrightarrow X_1$, the mapping $G : X_2 \to X_1^*$ is downward, which can treat a class of fully nonlinear elliptic boundary value problems.

1.3.3. Downward operators with monotone structure.

Let X_1, X_2 be separable and reflexive Banach spaces, and $L : X_2 \to X_1$ is an one to one and dense linear bounded operator. In the following, we shall state and prove the existence theorems for the operators $G : X_2 \to X_1^*$, which are of some monotone structure.

Definition 1.3.5. A bounded mapping $G : X_2 \to X_1^*$ is called to be coerceively continuous, if for any $u_n \rightharpoonup u_0$ in X_2, and

$$\lim_{n\to\infty} < Gu_n - Gu_0, Lu_n - Lu_0 >= 0$$

then we have

$$\lim_{n\to\infty} < Gu_n, v >=< Gu_0, v >, \quad \forall v \in X_1.$$

Theorem 1.3.6. Let $\Omega \subset X_2$ be a bounded open set. Suppose that $G : X_2 \to X_1^*$ is coerceively continuous, and

$$< Gu, Lu >\geq 0, \quad \forall u \in \partial\Omega \tag{1.3.3}$$

then $Gu = 0$ has a solution in X_2.

Theorem 1.3.7. If $G : X_2 \to X_1^*$ is continuous, which satisfies (1.3.3) and

$$< Gu - Gv, Lu - Lv >\geq 0, \quad \forall u, v \in X_2 \tag{1.3.4}$$

then $Gu = 0$ has a solution in X_2.

Theorem 1.3.8. Let $G : [0,1]\times X_2 \to X_1$ be bounded continuous. Suppose that there exists a constant $R > 0$ such that

i).$< G_0 u, Lu >\geq 0, \forall u \in X_2$ with $\|u\|_{X_2} > R$.

ii).$G_\lambda u = 0 \Rightarrow \|u\|_{X_2} < R, \forall 0 \leq \lambda \leq 1$, and

iii).for any $u_n \rightharpoonup u_0$ in $X_2, \lambda_n \to \lambda_0$

$$\lim_{n\to\infty,\lambda_n\to\lambda_0} < G_{\lambda_n} u_n - G_{\lambda_0} u_0, Lu_n - Lu_0 >= 0$$

$$\Rightarrow u_n \to u_0 \ in \ X_2$$

then $G_1 u = 0$ has a solution in X_2.

Proof of Theorem 1.3.6. In the same manner as the proof of Theorem 1.3.3. there exists a sequence $\{u_n\} \subset \overline{\Omega}$ such that $u_n \rightharpoonup u_0$ in X_2, and

$$< Gu_n, v >= 0, \quad \forall v \in Y_n \tag{1.3.5}$$

which implies that

$$< Gu_n, Lu_n >= 0, \ and \tag{1.3.6}$$

$$\lim_{n\to\infty} < Gu_n, v >= 0, \quad \forall v \in \bigcup_{n=1}^{\infty} Y_n. \tag{1.3.7}$$

Because $\bigcup_{n=1}^{\infty} Y_n$ is dense in X_1, the equality (1.3.7) holds true for all $v \in X_1$. From (1.3.6) and (1.3.7) one obtains

$$\lim_{n\to\infty} < Gu_n - Gu_0, Lu_n - Lu_0 >= 0$$

Due to the coerceive continuity of G, from (1.3.7) it follows

$$< Gu_0, v >= 0, \quad \forall v \in X_1$$

The proof of Theorem 1.3.6 is complete.

Proof of Theorem 1.3.7. It is known that there exists $u_n \in \overline{\Omega}$ which satisfies (1.3.5). By the condition (1.3.4) is follows that for any $v \in Z_n$ and $k \geq n$

$$\begin{aligned} 0 &\leq\ < Gv - Gu_k, Lv - Lu_k > \\ &=\ < Gv, Lv - Lu_k > \end{aligned}$$

Letting $u_k \rightharpoonup u_0$ in X_2, then

$$\lim_{k\to\infty} < Gv, Lv - Lu_k > = < Gv, Lv - Lu_0 >$$

Hence

$$< Gv, Lv - Lu_0 > \geq 0, \ \ \forall v \in \bigcup_{n=1}^{\infty} Z_n \tag{1.3.8}$$

Because $L : X_2 \to X_1$ is one to one and dense, and $\bigcup_{n=1}^{\infty} Y_n = L(\bigcup_{n=1}^{\infty} Z_n)$ is dense in $X_1, \bigcup_{n=1}^{\infty} Z_n$ is dense in X_2. Hence (1.3.8) holds for all $v \in X_2$. Replacing v by $u_0 + \lambda v$ in (1.3.8). We can obtain

$$< G(u_0 + \lambda v), Lv > \geq 0, \ \ \forall v \in X_2, \lambda > 0$$

Passing to $\lambda \to 0^+$ we get

$$< Gu_0, Lv > \geq 0, \ \ \forall v \in X_2$$

which implies that $Gu_0 = 0$. The proof is complete.

Proof of Theorem 1.3.8. For any $0 \leq \lambda \leq 1$, we define a mapping $A_\lambda^n : Z_n \to Y_n^*$ by

$$< A_\lambda^n u, v > = < G_\lambda u, v >, \ \ \forall u \in Z_n, v \in Y_n$$

Letting $\Omega = \{u \in X_2 | \|u\|_{X_2} < R\}$, from the condition i) we get

$$deg(A_0^n, \Omega_n, 0) = deg(L, \Omega_n, 0) \neq 0 \tag{1.3.9}$$

Now, we need to verify that

$$A_\lambda^n u \neq 0, \ \ \forall 0 \leq \lambda \leq 1, u \in \partial\Omega. \tag{1.3.10}$$

If (1.3.10) is not true, then there exist sequences $\{u_n\} \subset \partial\Omega$ and $\{\lambda_n\} \subset [0, 1]$, such that $A_{\lambda_n}^n u_n = 0$. Letting $u_n \rightharpoonup u_0$ in $X_2, \lambda_n \to \lambda_0$, in the same fashion as the proof of Theorem 1.3.6, one can deduce that

$$\lim_{n\to\infty, \lambda_n \to \lambda_0} < G_{\lambda_n} u_n - G_{\lambda_0} u_0, Lu_n - Lu_0 > = 0$$

From the condition iii), it follows that $u_n \to u_0 \in \partial\Omega$, which means that $G_{\lambda_0} u_0 = 0$, and a contradiction to the condition ii). Hence (1.3.10) holds true. By the homotopy invariance of the Brouwer degree, from (1.3.9) we get

$$deg(A_1^n, \Omega_n, 0) = deg(A_0^n, \Omega_n, 0) \neq 0$$

It implies that there is a $u_n \in \Omega_n$ such that

$$< G_1 u_n, v >= 0, \ \ \forall v \in Y_n.$$

Letting $u_n \rightharpoonup u_0$, as the proof of Theorem 1.3.6, from the condition iii), one can derive that

$$< G_1 u_0, v >= 0, \ \ \forall v \in X_1$$

The proof is complete.

Remark 1.3.9. As $X_1 = X_2$ and $L = id : X_2 \to X_1$ is an identity mapping, Theorem 1.3.7 is the well known monotone operator theorem.

1.3.4. Remarks and examples.

It is known that the monotone operator theory and variational principle have a common character that the operators involved are the mappings which map a Banach space to its own dual space, namely

$$G : X \to X^*$$

which are termed to be horizontal operators. In applications to nonlinear partial differential equations this condition makes the both theories to have the limitations that they only can efficiently deal with the elliptic quasilinear problems, and the solutions obtained are in the weak sense.

In fact, a complete theory of the inner product operators should include the upward and downward operators, which can treat a lot of nonlinear problems which can not be solved by the horizontal operator theory. Example, the theory of upward operators may efficiently treat the degenerate elliptic nonlinear problems and the equations with nonnegative characteristic form, and the theory for downward operators can deal with the existence of strong solutions of a large class of fully nonlinear elliptic boundary value problems.

In the following, we shall illustrate how to apply the basic theorems to the problems of partial differential equations by some simple examples.

We first introduce a lemma which is useful for the later discussion,. (Cf.[Te]).

Lemma 1.3.10. Assume that the operator $\sum_{|\alpha|=2m} a_\alpha(x)D^\alpha$ is elliptic, i.e.

$$\lambda|\xi|^{2m} \leq \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha, \ \ \forall \xi \in R^n$$

where $\lambda > 0$ is a constant, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}, \alpha = \{\alpha_1, \cdots, \alpha_n\}, m \geq 1, a_\alpha \in C^0(\overline{\Omega})$, then

$$[\int_\Omega (|\sum_{|\alpha|=2m} a_\alpha(x)D^\alpha u|^p)dx]^{\frac{1}{p}} \tag{1.3.11}$$

is a norm on the space

$$D^{2m} = \{u \in W^{2m,p}(\Omega) | u|_{\partial\Omega} = 0, \cdots, D^{m-1}u|_{\partial\Omega} = 0\}$$

$1 < p < \infty$, which is equivalent too the $W^{2m,p}$-norm.

In fact, if the L^p-estimate theorems hold for the boundary value problem, which has unique solution

$$\begin{cases} \sum_{|\alpha|=2m} a_\alpha(x)D^\alpha u = f(x), x \in \Omega \\ Bu|_{\partial\Omega} = 0 \end{cases}$$

then the norm (1.3.11) on the space

$$B^{2m} = \{u \in W^{2m,p}(\Omega) | Bu|_{\partial\Omega} = 0\}$$

is equivalent to the $W^{2m,p}$-norm.

Example 1.3.11. Consider the degenerate elliptic equations

$$\begin{cases} \frac{\partial}{\partial x}(e^{-\frac{1}{x+y}}\frac{\partial u}{\partial x}) + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - u = f(x), \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.3.12}$$

where $\Omega = \{(x, y) \in R^2 | 0 < x < 1, 0 < y < 1\}$ is a square, and $(x, y) = (0, 0)$ is a degenerate point of (1.3.12) on $\partial\Omega$.

As usual, the weak solution u of (1.3.12) is defined as to satisfy the integration

$$< Gu, v >= \int_\Omega [e^{-\frac{1}{x+y}}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial v}{\partial x} + uv$$
$$+fv]dx = 0, \ \ \forall v \in W_0^{1,2}(\Omega).$$

For the problem (1.3.12), if we choose $X_1 = X_2 = W_0^{1,2}(\Omega)$, then although the inner product $< Gu, v >$ defines a bounded linear mapping $G : X_1 \to X_2^*$, but G doesn't satisfy the acute angle condition

$$< Gu, u > \geq 0, \ \ \forall u \in \partial\Omega$$

for some bounded open set $\Omega \subset W_0^{1,2}(\Omega)$.

Let $X = C_0^\infty(\Omega)$, and X_1 be the completion of X with the norm

$$\|u\|_{X_1} = [\int_\Omega (e^{-\frac{1}{x+y}}|\frac{\partial u}{\partial x}|^2 + |\frac{\partial u}{\partial y}|^2 + u^2)dx]^{\frac{1}{2}}$$

If we take $X_2 = X_1$, then the term $\int_\Omega u\frac{\partial v}{\partial x}dx$ in $< Gu, v >$ maybe has no sense for some $u, v \in X_1$. Hence it is a basic requirement that $X_2 \neq X_1$ for the degenerate elliptic equations.

We take $X_2 = W_0^{1,2}(\Omega)$, thus the inner product $< Gu, v >$ define a linear bounded mapping $G : X_1 \to X_2^*$, and

$$\begin{aligned} < Gu, u > &= \int_\Omega [e^{-\frac{1}{x+y}}|\frac{\partial u}{\partial x}|^2 + |\frac{\partial u}{\partial y}|^2 + \frac{1}{2}u^2 - \frac{1}{2}|f|^2]dx \\ &\geq 0, \ \ \forall u \in X = C_0^\infty(\Omega) \ and \ \|u\|_{X_1} = R. \end{aligned}$$

where $R = \|f\|_{L^2}$. From Theorem 1.3.2 it follows that (1.3.12) has a weak solution $u \in X_1$.

The next example shows the usage of downward operators.

Example 1.3.12. We consider the fully nonlinear elliptic equation as follows

$$\begin{cases} -|\triangle u|^{p-2}\triangle u + u = f, x \in \Omega \subset R^n, p > 2 \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.3.13}$$

where Ω is C^∞ and bounded, $f \in L^{p'}(\Omega)(\frac{1}{p'} + \frac{1}{p} = 1)$.

Let $X_2 = W^{2,p}(\Omega) \cap W_0^{1,2}(\Omega), X_1 = L^p(\Omega)$ and $L : X_2 \to X_1$ be defined by

$$Lu = \triangle u.$$

It is known that L is a homeomorphism. The mapping $G : X_2 \to X_1^*$ is defined by

$$< Gu, v >= \int_\Omega [|\triangle u|^{p-2}\triangle u - u + f]vdx.$$

It is easy to see that $G : X_2 \to X_1^*$ is continuous, and satisfies the acute condition (1.3.3) and the monotone condition (1.3.4). Then by Theorem 1.3.7, the problem (1.3.13) has a strong solution $u \in W^{2,p}(\Omega) \cap W_0^{1,2}(\Omega)$.

1.4. Strong Solutions of A Class of Fully Nonlinear Elliptic Equations

1.4.1. Some lemmas

Before our discussion, it is necessary to introduce three lemmas which are helpful to the later contents.

Lemma 1.4.1. For any $\alpha > 0$, the functional

$$[\int_\Omega (|\nabla \triangle u|^2 + \alpha |\triangle u|^2) dx]^{\frac{1}{2}}$$

is a norm on the space

$$\{u \in H^3(\Omega) | u|_{\partial\Omega} = 0\}$$

which is equivalent to the H^3-norm.

Proof. We consider the equations

$$\begin{cases} \triangle u = f, \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

By the L^2-regularity estimates (Cf.[GT]), for any $f \in H^1$, the solution u obeys the estimate

$$\begin{aligned} \|u\|_{H^3} &\leq C[\|f\|_{H^1} + \|u\|_{L^2}] \\ &= C[\int_\Omega |\nabla \triangle u|^2 + |\triangle u|^2 dx]^{\frac{1}{2}} + \|u\|_{L^2} \end{aligned}$$

From Lemma 1.3.10, this lemma follows.

The following lemmas are basic in the nonlinear functional analysis (Cf.[Ch]),[ZFC] and [CH]).

Definition 1.4.2. A function $f : \Omega \times R^N \to R$ is called to satisfy the Caratheodory condition, if i) for any $\xi \in R^N, f(x, \xi)$ is a measurable function with respect to $x \in \Omega$;and ii) for almost all $x \in \Omega, f(x, \xi)$ is a continuous function with respect to $\xi \in R^n$.

Lemma 1.4.3. Let $f : \Omega \times R^N \to R$ satisfy the Caratheodory condition and $\Omega \subset R^n$ be bounded. If

$$|f(x, \xi)| \leq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i/p} + b(x) \tag{1.4.1}$$

where $\alpha > 0, p_i, p \geq 1, b \in L^p(\Omega)$, then the mapping

$$F : L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega) \to L^p(\Omega)$$

defined by $F(u_1, \cdots, u_N) = f(x, u_1, \cdots, u_N)$ is continuous.

Lemma 1.4.4. Let $\Omega \subset R^n$ be an open set (not necessarily bounded), and $f : \Omega \times R^N \to R$ satisfy the Caratheodory condition and the condition (1.4.1) with $p_i, p > 1$. If $\{u_{ik}\} \subset L^{p_i}(\Omega)(1 \leq i \leq N)$ is bounded, and for any bounded subdomain $\Omega_0 \subset \Omega, u_{ik}$ converges to u_i in measure on Ω_0, then for any $v \in L^{p'}(\Omega)$, we have

$$\lim_{k \to \infty} \int_\Omega f(x, u_{1k}, \cdots, u_{Nk}) v dx = \int_\Omega f(x, u_1, \cdots, u_N) v dx.$$

1.4.2. $W^{2,p}$-strong solutions

Let us consider the fully nonlinear elliptic Dirichlet boundary value problem given by

$$\begin{cases} F(x, u, \nabla u, \triangle u) = g(x, u, \nabla u, D^2 u), x \in \Omega \\ \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.4.2}$$

where $\Omega \subset R^n$ is C^∞ and bounded.

We shall apply Theorem 1.3.6 to investigate the existence of $W^{2,p}$-strong solutions of (1.4.2).

According to Lemma 1.3.10, we denote by $K > 0$ the best constant which satisfies

$$K^2 \|u\|_{H^2}^2 \leq \int_\Omega |\triangle u|^2 dx. \tag{1.4.3}$$

The following assumptions are imposed. Let $F(x,z,\xi,y)$ and $g(x,z,\xi,\eta)$ satisfy the Caratheodory condition, and

$$F(x,z,\xi,y)y \geq C_1|y|^p - C_2, \quad p \geq 2, \tag{1.4.4}$$

$$(F(x,z,\xi,y_1) - F(x,z,\xi,y_2))(y_1 - y_2) \geq k|y_1 - y_2|^2 \tag{1.4.5}$$

$$\begin{cases} |g(x,z,\xi,\eta_1) - g(x,z,\xi,\eta_2)| \leq K_1|\eta_1 - \eta_2| \\ K_1^2 < K^2 \cdot k^2, K \ is \ as \ in \ (1.4.3), k > 0 \ as \ in \ (1.4.5). \end{cases} \tag{1.4.6}$$

$$|F(x,z,\xi,y)| \leq \begin{cases} a(x,z,\xi)(|y|^{p-1}+1), \ as \ n < p \\ b(x,z)(|y|^{p-1}+|\xi|^{q_1}+1), \ as \ p \leq n < 2p \\ C(|y|^{p-1}+|\xi|^{q_1}+|z|^{q_2}+1), \ as \ 2p \leq n \end{cases} \tag{1.,4.7}$$

$$|g(x,z,\xi,\eta)| \leq C[|\eta|^{p_1} + |\xi|^{p_1} + |z|^{p_1} + 1] \tag{1.4.8}$$

where $q_1 < \frac{n(p-1)}{n-p}, q_2 < \frac{n(p-1)}{n-2p}, p_1 < p-1$, and $a \in C^0(\overline{\Omega} \times R \times R^n), b \in C^0(\overline{\Omega} \times R), C > 0$ is a constant.

Theorem 1.4.5. Under the conditions (1.4.4)-(1.4.8), the problem (1.4.2) has a strong solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Proof. We shall use Theorem 1.3.10 to prove this theorem. Let

$$X_2 = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

$$X_1 = L^p(\Omega). \tag{2}$$

and the linear mapping $L : X_2 \to X_1$ be as to read

$$Lu = \triangle u.$$

We define the mapping $G : X_2 \to X_1^*$ by

$$< Gu, v >= \int_\Omega (F(x,u,\nabla u, \triangle u) - g(x,u,\nabla u, D^2u))v dx$$

$u \in X_2, v \in X_1$. By the conditions (1.4.7)(1.4.8) it is clear that $G : X_2 \to X_1^*$ is a bounded mapping.

According to (1.4.4) and (1.4.8), we see that

$$\begin{aligned} < Gu, Lu > &= \int_\Omega [F(x,u,\nabla u, \triangle u)\triangle u - g(x,u,\nabla u, D^2u)\triangle u]dx \\ &\geq \int_\Omega [C_1|\triangle u|^p - C_2 - |g(x,u,\nabla u, D^2 u)||\triangle u|]dx \\ &\geq \int_\Omega [\frac{C_1}{2}|\triangle u|^p - C|g(x,u,\nabla u, D^2u)|^{p'} - C]dx \\ &\geq \frac{C_1}{2}\int_\Omega |\triangle u|^P dx - C\int_\Omega \sum_{|\alpha|\leq 2} |D^\alpha u|^{p' p_1} dx - C \end{aligned}$$

Because $p' p_1 < p$, by Lemma 1.3.10 it follows that

$$< Gu, Lu > \geq \alpha_1 \int_\Omega |\triangle u|^p dx - \alpha_2, \quad \alpha_1, \alpha_2 > 0 \ are\ constnts$$

which implies that the condition (1.3.4) is fulfilled.

We are now in a position to check the coerceive continuity of $G : X_2 \to X_1^*$. Let $u_n \rightharpoonup u_0$ in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, and

$$\lim_{n\to\infty} < Gu_n - Gu_0, Lu_n - Lu_0 >= 0 \tag{1.4.9}$$

We notice that

$$\begin{aligned} < Gu_n - Gu_0, Lu_n - Lu_0 > &= \int_\Omega [(F(x,u_n,\nabla u_n, \triangle u_n) \\ &-F(x,u_n,\nabla u_n,\triangle u_0))(\triangle u_n - \triangle u_0) - \\ &-(g(x,u_n,\nabla u_n, D^2u_n) - g(x,u_n,\nabla u_n, D^2u_0))(\triangle u_n - \triangle u_0)]dx \\ &+\int_\Omega [(F(x,u_n,\nabla u_n,\triangle u_0) - F(x,u_0,\nabla u_0,\triangle u_0))(\triangle u_n - \triangle u_0) \\ &-(g(x,u_n,\nabla u_n, D^2u_0) - g(x,u_0,\nabla u_0, D^2u_0))(\triangle u_n - \triangle u_0)]dx \end{aligned} \tag{1.4.10}$$

Because

$$\{u_n, \nabla u_n\} \to \{u_0, \nabla u_0\} \ in \begin{cases} C(\overline{\Omega}) \times C(\overline{\Omega}, R^n), \ as \ n < p \\ C(\overline{\Omega}) \times L^{q_1}(\Omega, R^n), \ as \ p \le n < 2p \\ L^{q_2}(\Omega) \times L^{q_1}(\Omega, R^n), \ as \ 2p \le n. \end{cases}$$

where $q_1 < \frac{np}{n-p}, q_2 < \frac{np}{n-2p}$, by Lemma 1.4.3 and Lemma 1.4.4, from (1.4.7) and (1.4.8) one obtain

$$\lim_{n\to\infty} \int_\Omega [F(x, u_n, \nabla u_n, \triangle u_0) - F(x, u_0, \nabla u_0, \triangle u_0)][\triangle u_n - \triangle u_0] = 0$$

$$\lim_{n\to\infty} \int_\Omega [g(x, u_n, \nabla u_n, D^2 u_0) - g(x, u_0, \nabla u_0, D^2 u_0)][\triangle u_n - \triangle u_0]dx = 0$$

On the other hand, by (1.4.5) and (1.4.6).

$$\begin{aligned} & \int_\Omega [(F(x, u_n, \nabla u_n, \triangle u_n) - F(x, u_n, \nabla u_n, \triangle u_0))(\triangle u_n - \triangle u_0) \\ & -(g(x, u_n, \nabla u_n, D^2 u_n) - g(x, u_n, \nabla u_n, D^2 u_0))(\triangle u_n - \triangle u_0)]dx \\ \ge \ & \int_\Omega [k|\triangle u_n - \triangle u_0|^2 - K_1 |D^2 u_n - D^2 u_0||\triangle u_n - \triangle u_0|]dx \\ \ge \ & \int_\Omega [\frac{k}{2}|\triangle u_n - \triangle u_0|^2 - \frac{K_1^2}{2k}|D^2 u_n - D^2 u_0|^2]dx \\ \ge \ & \frac{K^2 k^2 - K_1^2}{2k} \int_\Omega |D^2 u_n - D^2 u_0|^2 dx \end{aligned}$$

Hence, from (1.4.9)(1.4.10) it follows that

$$\lim_{n\to\infty} \int_\Omega |D^2 u_n - D^2 u_0|^2 dx = 0$$

namely, $D^2 u_n$ converges to $D^2 u_0$ in measure on Ω. Thus, by Lemma 1.4.4, from (1.4.7) and (1.4.9) we get

$$\begin{aligned} & \lim_{n\to\infty} \int_\Omega [F(x, u_n, \nabla u_n, \triangle u_n) - g(x, u_n, \nabla u_n, D^2 u_n)]v dx \\ & = \int_\Omega [F(x, u_0, \nabla u_0, \triangle u_0) - g(x, u_0, \nabla u_0, D^2 u_0)]v dx \end{aligned}$$

$\forall v \in X_1 = L^p(\Omega)$, which shows that $G : X_2 \to X_1^*$ is coerceively continuous. The proof is complete.

Example 1.4.6. According to Theorem 1.4.5, it is easy to see that the below equation

$$\begin{cases} (1+|\triangle u|^2)\triangle u = ke^{-|D^2u|} + g(x), g \in L^{\frac{4}{3}}(\Omega) \\ u|_{\partial\Omega} = 0 \end{cases}$$

has a strong solution $u \in W^{2,4}(\Omega) \cap W_0^{1,4}(\Omega)$ provided $0 < |k| < K$, where K is the constant as in (1.4.3), and

$$|D^2u| = \sqrt{\sum_{|\alpha|=2} |D^\alpha u|^2}.$$

Next, we investigate the below equations

$$\begin{cases} F(x, u, \nabla u, D^2u, \triangle u) = g(x), x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.4.11}$$

Assume that

$$F(x, z, \xi, \eta, y)y \geq c_1|y|^p - c_2, p > 1, \tag{1.4.12}$$

$$\int_\Omega [F(x, u, \nabla u, D^2u, \triangle u) - F(x, v, \nabla v, D^2v, \triangle v)][\triangle u - \triangle v]dx \tag{1.4.13}$$

$$\geq 0, \quad \forall u, v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

$$\begin{cases} |F(x, z, \xi, \eta, y)| \leq c[|y|^{p-1} + |z|^{p_1} + |\xi|^{p_1} + |\eta|^{p-1} + 1] \\ p_1 \leq \frac{n(p-1)}{n-p}, \end{cases} \tag{1.4.14}$$

Applying Theorem 1.3.7, we can deduce the following theorem.

Theorem 1.4.7. Under the conditions (1.4.12)-(1.4.14), if $g \in L^{p'}(\Omega)$, then the problem (1.4.11) has a strong solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

The proof of Theorem 1.4.7 is similar to that of Theorem 1.4.5, here we omit the details.

Example 1.4.8. We consider the following example

$$(1.4.15) \qquad \begin{cases} \triangle u + e^{-|\triangle u|^2} = f(x), \quad x \in \Omega, \quad f \in L^2(\Omega) \\ \\ u|_{\partial\Omega} = 0 \end{cases}$$

Let $F(x, y) = y + e^{-y^2}$. It is easy to verify that F satisfies conditions (1.4.12) and (1.4.14). We notice

$$F'_y(x, y) = 1 - 2ye^{-y^2} \geq 1 - \sqrt{2}e^{-\frac{1}{2}} > 0, \quad \forall y \in R$$

which implies that

$$[F(x, y_1) - F(x, y_2)][y_1 - y_2] \geq 0, \ \ \forall y_1, y_2 \in R.$$

Hence the condition (1.4.13) is satisfied. By Theorem 1.4.7, the problem (1.4.15) has a strong solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Finally, we shall apply Theorem 1.3.8 to discuss the equation given by

$$(1.4.16) \qquad \begin{cases} F(x, u, \nabla u, \triangle u) = B(x, u, \nabla u), x \in \Omega \\ \\ u|_{\partial\Omega} = 0 \end{cases}$$

We assume that

$$(1.4.17) \qquad F(x, z, \xi, y)y \geq ky^2, \quad k > 0$$

$$(1.4.18) \qquad (F(x, , \xi, y_1) - F(x, z, \xi, y_2))(y_1 - y_2) \geq \alpha|y_1 - y_2|^2, \quad \alpha > 0$$

$$(1.4.19) \qquad |F(x, z, \xi, y)| \leq \begin{cases} a(x, z)(|y| + |\xi|^{q_1} + 1), & n < 4 \\ \\ c(|y| + |\xi|^{q_1} + |z|^{q_2} + 1), & 4 \leq n \end{cases}$$

where $a \in c(\overline{\Omega} \times R), q_1 < \frac{n}{n-2}, q_2 < \frac{n}{n-4}$, and

$$(1.4.20) \qquad -B(x, z, \xi) signz \leq c(|\xi| + 1)$$

$$(1.4.21) \qquad |B(x, z, \xi)| \leq \begin{cases} b(x, z)(|\xi|^{p_1} + 1), & n < 4 \\ \\ c[|\xi|^{p_1} + |z|^{p_2} + 1], & 4 \leq n \end{cases}$$

where $b \in c(\overline{\Omega} \times R), p_1 < \frac{n+2}{n}, p_2 < \frac{n}{n-4}$.

Theorem 1.4.9. Under the assumptions (1.4.17)-(1.4.21), the problem (1.4.16) has a strong solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. Let the spaces be taken by

$$X_2 = H^2(\Omega) \cap H_0^1(\Omega),$$

$$X_1 = L^2(\Omega). \tag{3}$$

and the linear operator $L : X_2 \to X_1$ be defined as in Theorem 1.4.5. Define the mapping $G : [0,1] \times X_2 \to X_1^*$ by

$$< G_\lambda u, v >= \int_\Omega [F(x, u, \nabla u, \triangle u) - \lambda B(x, u, \nabla u)] v dx$$

$\lambda \in [0,1], u \in X_2$, and $v \in X_1$.

In the same fashion as the proof of Theorem 1.4.5, the conditions i) and iii) in Theorem 1.3.8 are readily checked, and we only need to check the condition ii). To this end, it suffices to verify that there exists a constant $C > 0$ such that for all solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of (1.4.16), we have the estimate

$$\|u\|_{H^2} \leq C. \tag{1.4.22}$$

Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ be a solution of (1.4.16). Then u_0 satisfies the equation

$$\begin{cases} \triangle u - b(x, u, \nabla u) = 0 \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.4.23}$$

where

$$b(x, u, \nabla u) = B(x, u, \nabla u) G^{-1}(x)$$
$$G(x) = F(x, u_0, \nabla u_0, \triangle u_0) / \triangle u_0$$

Due to (1.4.17),$\forall (x, z, \xi, y) \in \overline{\Omega} \times R \times R^n \times R$,

$$F(x, z, \xi, y)/y \geq k > 0 \tag{1.4.24}$$

From (1.4.20) and (1.4.24) one gets

$$-b(x, z, \xi) sign z \leq C k^{-1} (|\xi| + 1) \tag{1.4.25}$$

Thanks to the maximum principle of elliptic equations (Cf.[GT]), the solution u_0 of (1.4.23) satisfies

$$sup|u_0| \leq c k^{-1} \tag{1.4.26}$$

where $C > 0$ is a constant only dependent on Ω, n and k. Besides, u_0 satisfies

$$\begin{aligned} 0 &= \int_\Omega [|\nabla u_0|^2 + b(x, u_0, \nabla u_0)u_0]dx \\ &\geq \int_\Omega [|\nabla u_0|^2 - Ck^{-1}|\nabla u_0||u_0| - Ck^{-1}]dx \quad (by\ (1.4.25)) \\ &\geq \int_\Omega [\frac{1}{2}|\nabla u_0|^2 - \frac{1}{2}Ck^{-1}|u_0|^2 - Ck^{-1}]dx \end{aligned}$$

Hence we have

$$\begin{aligned} (1.4.27) \qquad \int_\Omega |\nabla u_0|^2 dx &\leq 2Ck^{-1}|\Omega| + Ck^{-1}\int_\Omega |u_0|^2 dx \\ &\leq 2Ck^{-1}|\Omega| + C^3k^{-3}|\Omega|; \quad (by\ (1.4.26)) \end{aligned}$$

By (1.4.21)(1.4.26) and (1.4.27), one can see that there is a constant $C_1 > 0$ such that for all solutions $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of (1.4.16), we have

$$g \in L^q(\Omega), \ \ q = \frac{2}{p_1}, \ and$$

$$\|g\|_{L^q} \leq C_1 \tag{4}$$

where

$$(1.4.28) \qquad g(x) = B(x, u, \nabla u)\triangle u \cdot F^{-1}(x, u, \nabla u, \triangle u).$$

By the L^p-estimates of elliptic equations, it implies

$$\|u\|_{W^{2,q}} \leq C\|q\|_{L^q} \leq CC_1$$

Because $p_1 < \frac{n+2}{n}$, we have $2 < nq/n - q$, and by the Sobolev embedding theorems

$$\|\nabla u\|_{L^{q^*}} \leq C\|u\|_{W^{2,q}} \leq C, \quad q^* = \frac{nq}{n-q}$$

Consequently, for $r = nq/(n-q)p_1 > q = 2/p_1$,

$$\begin{cases} g \in L^r(\Omega) \ and \\ \|g\|_{L^r} \leq C, \quad C > 0 \ independent\ of\ u \end{cases}$$

By iteration, one can deduce that (1.4.22) holds true. The proof is complete.

Remark 1.4.10. Under the conditions (1.4.17)-(1.4.21), for any $p < \infty$ the solutions $u \in W^{2,p}(\Omega)$ of (1.4.16). In fact the solutions u of (1.4.16) satisfy

$$\begin{cases} \triangle u = g(x), \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $g(x)$ is defined by (1.4.28), and by using L^p-estimates and the iteration, one can obtain $u \in W^{2,p}(\Omega)$ for any $p < \infty$.

1.4.3. H^3-strong solutions

In this subsection, we shall use Theorem 1.3.3 to discuss the existence of H^3-strong solutions for a class of fully nonlinear elliptic Dirichlet and Neumann boundary value problems.

Let us first consider the elliptic Dirichlet boundary value problem given by

$$\begin{cases} -f(x, \triangle x) = g(x, u, \nabla u, D^2 u), \quad x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.4.29}$$

According to Lemma 1.4.1, we denote by $k > 0$ the best constant which satisfies

$$k^2 \|u\|_{H^3}^2 \leq \int_\Omega [|\nabla \triangle u|^2 + |\triangle u|^2] dx \tag{1.4.30}$$

$\forall u \in H^3(\Omega) \cap H_0^1(\Omega)$. For $f(x, y)$ and $g(x, z, \xi, \eta)$ we assume that $f \in C^1(\overline{\Omega} \times R), g \in C^1(\overline{\Omega} \times R \times R^n \times R^{n^2})$, and

$$\begin{cases} f'_y(x, y) \geq \alpha > 0 \\ f(x, y)y \geq c_1 |y|^p - c_2, \quad p \geq 2 \quad (c_1 \geq \alpha \ as \ p = 2) \end{cases} \tag{1.4.31}$$

$$\begin{cases} |f(x, y)| \leq c[|y|^{p-1} + 1] \\ |f'_x(x, y)| \leq c[|y|^{p_1} + 1], \quad p_1 < p/2 \end{cases} \tag{1.4.32}$$

$$\begin{cases} |g| + |D_x g| + |D_z g||\xi| + |D_\xi g||\eta| \leq c[|z|^{p_1} + |\xi|^{p_1} + |\eta|^{p_1} + 1] \\ |D_\eta g|^2 \leq k_1^2 < \alpha^2 k^2, \quad k \ as \ in \ (1.4.30), \ p_1 < p/2 \end{cases} \tag{1.4.33}$$

The spaces are taken as to write

$$X = \{u \in C^\infty(\overline{\Omega}) | u|_{\partial\Omega} = 0, \frac{\partial \triangle u}{\partial n}|_{\partial\Omega} = 0\}$$

$$X_1 = L^p(\Omega), \ and$$

$$X_2 = H^3(\Omega) \cap W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \qquad (5)$$

and the linear mapping $L : X \to X_1$ is defined by

$$Lu = \triangle^2 u - \triangle u \qquad (1.4.34)$$

By the theory of linear elliptic equations, the problem

$$\begin{cases} \triangle^2 u - \triangle u = f(x), x \in \Omega \\ u|_{\partial\Omega} = 0, \frac{\partial \triangle u}{\partial n}|_{\partial\Omega} = 0 \end{cases} \qquad (1.4.35)$$

has an unique solution $u \in c^\infty(\overline{\Omega})$ provided $f \in c^\infty(\overline{\Omega})$. In fact, (1.4.35) is equivalent to the system

$$\begin{cases} \triangle u = v \\ \triangle v - v = f \\ u|_{\partial\Omega} = 0, \frac{\partial v}{\partial n}|_{\partial\Omega} = 0 \end{cases} \qquad (1.4.36)$$

and it is well known that (1.4.36) has an unique solution $u \in c^\infty(\overline{\Omega}), v \in c^\infty(\overline{\Omega}), \forall f \in c^\infty(\overline{\Omega})$. Hence the linear mapping $L : X \to X_1$ is one to one and dense.

Now we shall state and prove the existence theorem of H^3-strong solution for the problem (1.4.29).

Theorem 1.4.11. Under the condition (1.4.31)-(1.4.33), the problem (1.4.29) has a strong solution $u \in H^3(\Omega) \cap W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Proof. We define the mapping $G : X_2 \to X_1^*$ by

$$< Gu, v >= \int_\Omega [-f(x, \triangle u) - g(x, u, \nabla u, D^2 u)] v dx$$

$\forall v \in X_1 = L^p(\Omega)$. By the compact embedding theorems and conditions (1.4.32)(1.4.33), from Lemma 1.4.4 it is easy to see that $G : X_2 \to X_1^*$ is weakly continuous. It remains to check the acute angle condition (1.3.1). We see that $\forall u \in X$,

$$< Gu, Lu >= \int_\Omega [\nabla f(x, \triangle u) \cdot \nabla \triangle u + f(x, \triangle u)\triangle u +$$

$$+\nabla g(x, u, \nabla u, D^2 u) \cdot \nabla \triangle u + g(x, u, \nabla u, D^2 u)\triangle u]dx$$

$$\geq \int_\Omega [\alpha|\nabla \triangle u|^2 + c_1|\triangle u|^p - c_2 - |D_x f(x, \triangle u)||\nabla \triangle u|$$

$$-|\nabla g(x, u, \nabla u, D^2 u)||\nabla \triangle u| - |g(x, u, \nabla u, D^2 u)||\triangle u|]dx$$

$$(by\ (1.4.31))$$

$$\geq \int_\Omega [\frac{\alpha}{2}|\nabla \triangle u|^2 + \frac{c_1}{2}|\triangle u|^p - |D_x f(x, \triangle u)||\nabla \triangle u| - c_2$$

$$-\frac{1}{2\alpha}|\nabla g(x, u, \nabla u, D^2 u)|^2 - (\frac{2}{c_1})^{\frac{1}{p-1}}|g(x, u, \nabla u, D^2 u)|^{p'}]dx$$

We notice that

$$\nabla g(x, u, \nabla u, D^2 u) = \nabla_x g + D_z g \cdot \nabla u + D_z g \cdot \nabla D u + D_\eta g \cdot \nabla D^2 u$$

From (1.4.30)(1.4.32) and (1.4.33) it follows that

$$< Gu, Lu > \geq \frac{1}{2\alpha}(\alpha^2 k^2 - k_1^2)\|u\|_{H^3}^2 + \int_\Omega [\frac{c_1}{2}|\triangle u|^p - \frac{\alpha}{2}|\triangle u|^2$$

$$-c \sum_{|\beta| \leq 2} |D^\beta u|^{p_1 p'} - |D_x f(x, \triangle u)||\nabla \triangle u| - c]dx$$

$$\geq k\|u\|_{H_3}^2 + \int_\Omega [\frac{c_1}{2}|\triangle u|^p - \frac{\alpha}{2}|\triangle u|^2 - c|\triangle u|^{2p_1} - c \sum_{|\beta| \leq 2} |D^\beta u|^{p_1 p'} - c]dx$$

where $0 < k < \frac{1}{2\alpha}(\alpha^2 k^2 - k_1^2)$. Due to $2p_1 < p$, $p_1 p' < p$ and $\alpha \leq c_1$ as $p = 2$, hence we obtain

$$< Gu, Lu > \geq 0\ for\ \|u\|_{X_2} \geq\ some\ constant.$$

This proof is complete.

In the following, we shall discuss the fully nonlinear elliptic Neumann boundary value problem

$$(1.4.37) \qquad \begin{cases} -f(x, \triangle u) = g(x, u, \nabla u),\ mod\ constant \\ \frac{\partial u}{\partial n}|_{\partial \Omega} = 0, \int_\Omega u dx = 0 \end{cases}$$

Suppose that $f \in C^1(\overline{\Omega} \times R), g \in C^1(\overline{\Omega} \times R \times R^n)$, and

$$(1.4.38) \qquad \begin{cases} |g| + |D_x g| + |D_z g||\xi| \leq [|z|^{p_1} + |\xi|^{p_1} + 1] \\ |D_\xi g| \leq \begin{cases} k_1 < kc_1 \ as \ p = 2, \\ c[|z|^{p_2} + |\xi|^{p_2} + 1], \quad p_2 < \frac{p-2}{2}, \quad p > 2, \end{cases} \end{cases}$$

where $p_1 < \frac{p}{2}$, and p, c_1 are as in (1.4.31), and $k > 0$ is the best constant which satisfies

$$k^2 \int_\Omega |D^2 u|^2 dx \leq \int_\Omega |\triangle u|^2 dx, \quad \forall u \in H^2_N(\Omega)/R$$

$$H^2_N(\Omega) = \{u \in H^2(\Omega)| \ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}.$$

We have the below existence result.

Theorem 1.4.12. Let the conditions (1.4.31)(1.4.32) and (1.4.38) be fulfilled. Then the problem (1.4.37) has a strong solution

$$u \in X_2 = \{v \in H^3(\Omega) \cap W^{2,p}(\Omega)| \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \int_\Omega u dx = 0\}.$$

Proof. Let the spaces be taken as follows

$$X = \{u \in c^\infty(\overline{\Omega}) \Big| \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \frac{\partial \triangle u}{\partial n}|_{\partial\Omega} = 0, \int_\Omega u dx = 0\}.$$

$$X_1 = L^p(\Omega)/R = \{u \in L^p(\Omega)| \int_\Omega u dx = 0\}$$

X_2=the completion of the space Y under the norm

$$\|u\|_{X_2} = [\int_\Omega |\nabla \triangle u|^2 dx]^{\frac{1}{2}} + \|u\|_{W^{2,p}}.$$

$$Y = \{u \in H^3(\Omega) \cap W^{2,p}(\Omega)| \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \int_\Omega u dx = 0\}$$

The linear mapping $L : X \to X_1$ is defined by (1.4.34). It is known that the operator L is one to one and dense. We define the mapping $G : X_2 \to X_1^*$ by

$$< Gu, v >= \int_\Omega [-f(x, \triangle u) - g(x, u, \nabla u)] v dx$$

$\forall u \in X_2, v \in X_1$. We know that

$$\begin{cases} L^p(\Omega) = X_1 \bigoplus R \\ \\ L^{p'}(\Omega) = X_1^* \bigoplus R \end{cases}$$

namely $X^* = L^{p'}(\Omega)/R = \{f \in L^{p'}(\Omega) | \int_\Omega f(x)dx = 0\}$. Hence $Gu = 0$ in X_1^* implies that (1.4.37) holds for all $x \in \Omega$.

Let $u_n \rightharpoonup u_0$ in X_2. Then $\triangle u_n \rightharpoonup \triangle u_0$ in H^1, by the compact embedding theorems, we have

$$\triangle u_n \to \triangle u_0 \ in \ L^2(\Omega)$$

which implies, by (1.4.32) and (1.4.38), that $G : X_2 \to X_1^*$ is weakly continuous. The remaining proof is parallel to that of Theorem 1.4.11. The proof is complete.

Remark 1.4.13. When $g(x, z, \xi) = \beta z + g_1(x, \xi), \beta \neq 0$, then under the conditions (1.4.31)(1.4.32) and (1.4.38) the problem

$$\begin{cases} -f(x, \triangle u) = \beta u + g_1(x, \nabla u), x \in \Omega \\ \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0 \end{cases}$$

has a solution $u = u_0 + c$, where u_0 satisfies (1.4.37), and

$$c = \frac{1}{\beta}\int_\Omega [-f(x, \triangle u_0) - g_1(x, \nabla u_0)]dx.$$

1.5. Nonlinear Elliptic Systems of Second Order

The nonlinear elliptic systems of second order are very different from the elliptic equations. First, we know that the maximum principle and the de Giovgi estimates generally don't hold true anymore for the nonlinear elliptic systems, and next, a few elliptic systems have the monotonicity structure. Hence the many traditional theories and methods, such as the Green expression method, the method of super-lower solutions and the monotonicity theory, are unavailable. Now, the variational principle is the most widely used method in the nonlinear elliptic systems. In this section we shall use the acute angle principle to discuss the existence problem of nonlinear elliptic systems.

1.5.1. Quasilinear elliptic systems

First, we consider the elliptic systems

$$(1.5.1)\qquad \begin{cases} -D_i(a_{ij}^{kl}(x,u)D_ju_l + b_i^k(x,u)) + h_i^{kl}(x,u)D_iu_l \\ +c^k(x,u) = f^k(x), x\in\Omega, k=1,\cdots,m. \\ u_i|_{\partial\Omega} = 0, 1\le i\le m. \end{cases}$$

where $u=\{u,\cdots,u_n\}, \Omega\subset R^n$ is bounded. We use the summation convention, i.e. $a^ku_k=\sum_{k=1}^m a^ku_k$ and $a_iD_iu=\sum_{j=1}^n a_jD_ju$.

Let the system be elliptic, namely

$$(1.5.2)\qquad \begin{cases} \lambda|\xi|^2\le a_{ij}^{kl}\xi_{ki}\xi_{lj}, \ \ \forall(x,z)\in\overline{\Omega}\times R^m, \xi\in R^{mn} \\ \lambda>0 \ a\ constant, \ a_{ij}(x,z)=a_{ji}(x,z) \end{cases}$$

Suppose that the coefficients satisfy the Caratheodory condition, and

$$(1.5.3)\qquad \begin{cases} \int_\Omega[b_i^k(x,u)D_iu_k + h_i^{kl}(x,u)u_kD_iu_l + c^k(x,u)u_k]dx \\ \ge \int_\Omega \alpha^k|u_k|^{p_k}dx - \alpha, \quad \forall u\in[c_0^\infty(\Omega)]^m \\ p_k\ge 0, \quad \alpha^k, \alpha>0\ be\ constants \end{cases}$$

$$(1.5.4)\qquad \begin{cases} |a_{ij}^{rl}(x,z)|, |h_i^{rl}(x,z)| \le \beta^k|z_k|^{q_k}+\beta \\ 0\le q_k< max\{p_k/2, \frac{n}{n-2}\}, \quad \beta^k,\beta>0 \end{cases}$$

$$(1.5.5)\qquad \begin{cases} |c^l(x,z)|, |b_i^l(x,z)| \le \gamma^k|z_k|^{\widetilde{q}_k}+\gamma \\ 0\le \widetilde{q}_k< max\{p_k, \frac{2n}{n-2}\}, \quad \gamma,\gamma^k>0 \end{cases}$$

Let $X=C_0^\infty(\Omega,R^m)$, and X_1 be the completion of X under the norm

$$\begin{cases} \|u\|_{X_1} = [\int_\Omega|\nabla u|^2dx]^{\frac{1}{2}} + \sum_{k=1}^m \delta_k[\int_\Omega|u_k|^{p_k}dx]^{\frac{1}{p_k}} \\ |\nabla u|^2 = \sum_{k=1}^m|\nabla u_k|^2, \delta_k = \begin{cases} 1, \ as\ p_k>1 \\ 0, \ as\ p_k\le 1 \end{cases} \end{cases}$$

Let $X_2 = C_0^1(\Omega, R^m)$.

$u = (u_1, \cdots, u_m) \in X_1$ is said a weak solution of (1.5.1) if $\forall v = (v_1, \cdots, v_m) \in X_2$ we have

$$\int_\Omega [a_{ij}^{kl}(x,u) D_j u_l D_i v_k + b_i^k(x,u) D_i v_k + \tag{1.5.6}$$

$$+ h_i^{kl}(x,u) D_i u_l \cdot v_k + c^k(x,u) v_k - f^k v_k] dx = 0$$

Theorem 1.5.1. Under the conditions (1.5.2)-(1.5.5), if $f \in L^2(\Omega, R^m)$, then the system (1.5.1) has a weak solution $u \in X_1$.

Proof. Denote by $< Gu, v >$ the left part of (1.5.6). It is not difficult to verify that the inner product $< Gu, v >$ defines a bounded mapping $G : X_1 \to X_2^*$. In fact, $\forall u \in X_1$ and $v \in X_2$ we have

$$\begin{aligned} & |\int_\Omega a_{ij}^{kl}(x,u) D_j u_l D_i v_k dx| \\ \le\ & [\int_\Omega |a_{ij}^{kl}(x,u) D_j u_l| dx] \cdot \|v\|_{X_2} \\ \le\ & [\int_\Omega (\beta^k |u_k|^{q_k} + \beta) |Du| dx] \cdot \|v\|_{X_2} \qquad (by\ (1.4.4)) \\ \le\ & c\|u\|_{X_1}^q \cdot \|v\|_{X_2}; \quad q = \max_{1 \le k \le m} \{p_k, \frac{2n}{n-2}\}\ (by\ Holder\ inequality) \end{aligned}$$

For the other terms in $< Gu, v >$, we can also get the similar inequality as above in the same fashion.

Now we show the weak continuousness of $G : X_1 \to X_2^*$. Let $u_n \rightharpoonup u_0$ in X_1. For $v \in X_2$ given, we only need to check that

$$\lim_{n \to \infty} \int_\Omega a_{ij}^{kl}(x, u_n) D_j u_{ln} D_i v_k dx = \int_\Omega a_{ij}^{kl}(x, u_0) D_j u_{l0} D_i v_k dx \tag{1.5.7}$$

$$\lim_{n \to \infty} \int_\Omega h_i^{kl}(x, u_n) D_i u_{ln} v_k dx = \int_\Omega h_i^{kl}(x, u_0) D_j u_{l0} v_k dx \tag{1.5.8}$$

$$\lim_{n \to \infty} \int_\Omega b_i^k(x, u_n) D_i v_k dx = \int_\Omega b_i^k(x, u_0) D_i v_k dx \tag{1.5.9}$$

$$\lim_{n\to\infty}\int_\Omega c^k(x,u_n)v_k dx = \int_\Omega c^k(x,u_0)v_k dx \tag{1.5.10}$$

Thanks to Theorem 1.1.6, we have

$$u_{kn} \to u_{k0} \ in \ L^{q_k}(\Omega), \quad \forall q_k < \max\{p_k, \frac{2n}{n-2}\}$$

From Lemma 1.4.3, Lemma 1.4.4 and (1.5.5), one can deduce that (1.5.9) and (1.5.10) hold true.

For the proof of (1.5.7), we take the form

$$\begin{aligned}
& \int_\Omega [a_{ij}^{kl}(x,u_n)D_j u_{ln} - a_{ij}^{kl}(x,u_0)D_j u_{l0}]D_i v_k dx \\
= \ & \int_\Omega [a_{ij}^{kl}(x,u_n) - a_{ij}^{kl}(x,u_0)]D_j u_{ln} D_i v_k dx \\
& + \int_\Omega a_{ij}^{kl}(x,u_0)(D_j u_{ln} - D_j u_{l0})D_i v_k dx
\end{aligned}$$

By the condition (1.5.4),

$$\{a_{ij}^{kl}(x,u_n)\} \subset L^p(\Omega) \ bounded \ for \ some \ p > 2$$

and $a_{ij}^{kl}(x,u_n)$ is convergent to $a_{ij}^{kl}(x,u_0)$ in measure. Hence we have

$$a_{ij}^{kl}(x,u_n) \to a_{ij}^{kl}(x,u_0) \ in \ L^2(\Omega)$$

which means that

$$\lim_{n\to\infty}\int_\Omega [a_{ij}^{kl}(x,u_n) - a_{ij}^{kl}(x,u_0)]D_j u_{ln} D_i v_k dx = 0$$

From the definition of weak convergence, it is evidently

$$\lim_{n\to\infty}\int_\Omega [a_{ij}^{kl}(x,u_0)D_i v_k(D_j u_{ln} - D_j u_{l0})]dx = 0$$

Thus we get

$$\lim_{n\to\infty}\int_\Omega [a_{ij}^{kl}(x,u_n)D_j u_{ln} - a_{ij}^{kl}(x,u_0)D_j u_{l0}]D_i v_k dx = 0$$

i.e. (1.5.7) holds true. By the same fashion, one can also get the equality (1.5.8).

It remains to check the acute angle condition. Taking $u \in X = C_0^\infty(\Omega, R^m)$, we have

$$
\begin{aligned}
< Gu, u > \quad &= \int_\Omega [a_{ij}^{kl}(x,u) D_j u_l D_i u_k + b_i^k(x,u) D_i u_k \\
&\quad + h_i^{kl}(x,u) D_i u_l u_k + c^k(x,u) u_k - f^k \cdot u_k] dx \\
&\geq \lambda \int_\Omega |\nabla u|^2 dx + \int_\Omega \alpha^k |u_k|^{p_k} dx - \int_\Omega f^k \cdot u_k dx - \alpha \\
&\quad (\textit{by } (1.5.2) \textit{ and } (1.5.3)) \\
&\geq \frac{1}{2}\lambda \int_\Omega |\nabla u|^2 dx + \int_\Omega \alpha^k |u_k|^{p_k} dx - c\int_\Omega |f|^2 dx - \alpha
\end{aligned}
$$

The last inequality is obtained by the Holder inequality and Young inequality as follows

$$
\begin{aligned}
\int_\Omega f^k u_k dx \quad &\leq \quad [\int_\Omega |f|^{p'} dx]^{\frac{1}{p'}} [\int_\Omega |u|^p dx]^{\frac{1}{p}} \\
&\leq \quad \epsilon^{-\frac{p'}{p}} \int_\Omega |f|^{p'} dx + \epsilon \int_\Omega |u|^p dx, \quad \frac{1}{p} + \frac{1}{p'} = 1
\end{aligned}
$$

as well as the Poincare inequality

$$
\int_\Omega |u|^p dx \leq c \int_\Omega |\nabla u|^p dx, \ \ \forall u \in W_0^{1,p}(\Omega).
$$

where $\epsilon > 0$ is an arbitrary number, and $c > 0$ a constant.

From the inequality above, one follows that

$$
< Gu, u > \geq 0, \quad \forall u \in X_1 \textit{ and } \|u\|_{X_1} \textit{ great enough}
$$

By Theorem 1.3.2, the system (1.5.1) has a weak solution in X_1. The proof is complete.

Next, we consider the quasilinear systems

$$
(1.5.11) \qquad \begin{cases} -D_i A_i^k(x,u,Du) + B^k(x,u,Du) = f^k(x), \\ u_k|_{\partial\Omega} = 0, k = 1, \cdots, m \end{cases}
$$

Suppose that

$$(1.5.12)\qquad \begin{cases} \lambda|\xi|^p \leq A_i^k(x,z,\xi)\xi_{ki}, \quad \forall (x,z)\in \overline{\Omega}\times R^m, \xi\in R^{nm} \\ \\ \lambda>0, p>1 \ be\ constants \end{cases}$$

$$(1.5.13)\qquad \int_\Omega B^k(x,u,Du)u_k dx \geq \int_\Omega \alpha^k |u_k|^{p_k} dx - \alpha$$

$p_k \geq 0, \alpha, \alpha^k > 0.$

$$(1.5.14)\qquad [A_i^k(x,z,\xi) - A_i^k(x,z,\eta)][\xi_{ki} - \eta_{ki}]$$

$$\geq \lambda_1 |\xi-\eta|^q; \quad for\ some\ q\geq 1, \lambda_1 > 0$$

$$(1.5.15)\qquad \begin{cases} |A_i^l(x,z,\xi)| \leq a^k|z_k|^{q_k} + a|\xi|^{p-1} + a \\ \\ 0\leq q_k < q^*, \quad a, a^k > 0 \end{cases}$$

where

$$\begin{cases} q_k^* = max\{\frac{(p-1)}{p}p_k, \beta\} \\ \\ \beta = \begin{cases} \frac{n(p-1)}{n-p},\ as\ n>p \\ \\ \infty,\ as\ p\geq n \end{cases} \end{cases}$$

$$(1.5.16)\qquad \begin{cases} |B^l(x,z,\xi)| \leq b^k|z_k|^{q_{lk}} + b|\xi|^{s_l} + b \\ \\ 0\leq q_{lk} < q_{lk}^*, 0\leq s_l < s_l^*, b, b^k > 0 \end{cases}$$

where

$$\begin{cases} q_{lk}^* = max\{\frac{(p_l-1)}{p_l}p_k, \frac{(np+p-n)}{np}p_k, \beta_1\} \\ \\ \beta_1 = \begin{cases} \frac{np+p-n}{n-p},\ as\ n>p \\ \\ \infty,\ as\ p\geq n \end{cases} \end{cases}$$

$$\begin{cases} s_l^* = max\{\frac{(p_l-1)}{p_l}p, \beta_2\} \\ \\ \beta_2 = \begin{cases} p - \frac{n-p}{n},\ as\ n>p \\ \\ p,\ as\ p\geq n \end{cases} \end{cases}$$

Let $X = C_0^\infty(\Omega, R^m)$, and X_1 be the completion of X with the norm

$$\|u\|_{X_1} = [\int_\Omega |\nabla u|^p dx]^{\frac{1}{p}} + \sum_{k=1}^m \delta_k [\int_\Omega |u_k|^{p_k} dx]^{\frac{1}{p_k}}$$

For $u, v \in X_1$, we define an inner product

$$< Gu, v >= \int_\Omega [A_i^k(x, u, Du) D_i v_k + B^k(x, u, Du) v_k - f^k v_k] \tag{1.5.17}$$

By the Sobolev embedding theorems and the definition of norm $\| \cdot \|_{X_1}$, from conditions (1.5.15) and (1.5.16), one can deduce that the inner product (1.5.17) defines a bounded continuous mapping $G : X_1 \to X_1^*$.

$u \in X_1$ is said a weak solution of (1.5.11), if

$$< Gu, v >= 0, \ \ \forall v \in X_1$$

Theorem 1.5.2 Under the assumptions (1.5.12)-(1.5.16), if $f \in L^{p'}(\Omega, R^m), p' = \frac{p}{p-1}$, then the system (1.5.11) has a weak solution in X_1.

Proof. We apply Theorem 1.3.4 to prove the theorem. To this end, first of all we need to verify the acute angle condition.

For $u \in X_1$, we have

$$\begin{aligned}
< Gu, u > &= \int_\Omega [A_i^k(x, u, Du) D_i u_k + B^k(x, u, Du) u_k - f^k u_k] dx \\
&\geq \int_\Omega [\lambda |\nabla u|^p + \alpha^k |u_k|^{p_k} - f^k u_k] dx - c \\
&\quad (by\ conditions\ (1.5.12)\ and\ (1.5.13)) \\
&\geq \int_\Omega [\frac{\lambda}{2} |\nabla u|^p + \alpha^k |u_k|^{p_k}] dx - c \int_\Omega |f|^{p'} dx - c \\
&\quad (by\ Holder-Young\ inequality\ and\ Poincare\ inequality) \\
&\geq 0, \quad \forall u \in X_1\ and\ \|u\|_{X_1}\ great\ enough
\end{aligned}$$

We are now in a position to verify that $G : X_1 \to X_1^*$ satisfies the continuity condition ii) in Theorem 1.3.4.

Let $u_n \in X_1, u_n \rightharpoonup u_0$ in X_1, and

$$\lim_{n\to\infty} < Gu_n - Gu_0, u_n - u_0 >= 0$$

which is

$$\lim_{n\to\infty} \int_\Omega [(A_i^k(x, u_n, Du_n) - A_i^k(x, u_0, Du_0))(D_i u_{kn} - D_i u_{k0}) \tag{1.5.18}$$

$$+(B^k(x, u_n, Du_n) - B^k(x, u_0, Du_0))(u_{kn} - u_{k0})]dx = 0$$

In following, we need to show that

$$\lim_{n\to\infty} \int_\Omega [B^k(x, u_n, Du_n) - B^k(x, u_0, Du_0)][u_{kn} - u_{k0}]dx = 0 \tag{1.5.19}$$

$$\lim_{n\to\infty} \int_\Omega [A_i^k(x, u_n, Du_0) - A_i^k(x, u_0, Du_0)][D_i u_{kn} - D_i u_{k0}]dx = 0 \tag{1.5.20}$$

Obviously, by $u_n \rightharpoonup u_0$ in X_1, we have

$$\lim_{n\to\infty} \int_\Omega B^k(x, u_0, Du_0)(u_{kn} - u_{k0})dx = 0$$

By Holder inequality

$$|\int_\Omega B^l(x, u_n, Du_n)(u_{ln} - u_{l0})dx| \tag{1.5.21}$$

$$\leq [\int_\Omega |B^l(x, u_n, Du_n)|^{r_l'} dx]^{\frac{1}{r_l'}} [\int_\Omega |u_{ln} - u_{l0}]^{r_r} dx]^{\frac{1}{r_l}}$$

here $r_l' = r_l/(r_l - 1)$.

By Theorem 1.1.2, Theorem 1.1.6 and the definition of the norm $\|\cdot\|_{X_1}, u_{ln}, u_{l0} \in L^{r_l^*}(\Omega), r_l^* = max\{p_l, r\}$, where

$$r = \begin{cases} np/n - p, n > p \\ \\ arbitrary\ number > 0, n = p \\ \\ \infty,\ n < p \end{cases}$$

moreover,

$$u_{ln} \to u_{l0}\ in\ L^{r_l}(\Omega),\ \ \forall r_l < r_l^* \tag{1.5.22}$$

From (1.5.16), we get that for any bounded sequence $\{u_n\} \subset X_1, \{B^l(x, u_n, Du_n)\} \subset L^{r_l'}(\Omega)$ is bounded for some $r_l' > r_l^{*\prime} = r_l^*/(r_l^* - 1)$. Hence from (1.5.21) and (1.5.22) we derive

$$\lim_{n\to\infty} \int_\Omega B^l(x, u_n, Du_n)(u_{ln} - u_{l0})dx = 0$$

Therefore (1.5.19) holds true.

Let the Caratheodory mappings

$$A_i^l : L^{N_1}(\Omega) \times \cdots \times L^{N_m}(\Omega) \to L^{p'}(\Omega), \quad p' = \frac{p}{p-1}$$

be defined by

$$A_i^l(u) = A_i^l(x, u, Du_0), \ for \ u \in L^{N_1}(\Omega) \times \cdots \times L^{N_m}(\Omega)$$

where $N_k = q_k \cdot p', 1 \le k \le m$. The condition (1.5.15) means that $N_k < p_k^*, p_k^* = max\{p_k, p^*\}$, here

$$p^* = \begin{cases} \frac{np}{n-p}, & n > p \\ \\ \infty, & p \ge n \end{cases}$$

By Lemma 1.4.3, we know that the mappings A_i^l are continuous. On the other hand, the compact theorems (Theorem 1.1.5 and Theorem 1.1.6) say that

$$u_n \rightharpoonup u_0 \ in \ X_1 \Rightarrow u_n \to u_0 \ in \ L^{N_1}(\Omega) \times \cdots \times L^{N_m}(\Omega)$$

Thus we get

$$A_i^l(x, u_n, Du_0) \to A_i^l(x, u_0, Du_0) \ in \ L^{p'}(\Omega)$$

which implies the equality (1.5.20) holds true.

From (1.5.18)-(1.5.20), ones derive

$$\lim_{n\to\infty} \int_\Omega [A_i^k(x, u_n, Du_n) - A_i^k(x, u_n, Du_0)][D_i u_{kn} - D_i u_{k0}] = 0$$

and by condition (1.5.14), it implies

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u_0|^q dx = 0, \ for \ some \ q \ge 1.$$

Hence we deduce that

$$\begin{cases} \nabla u_n \to \nabla u_0 \ in \ L^N(\Omega), \forall 1 \le N < p \\ \\ u_n \to u_0 \ in \ L^{N_1}(\Omega) \times \cdots \times L^{N_m}(\Omega), N_k < p_k^* \end{cases}$$

From the structural conditions (1.5.15) and (1.5.16) we get

$$\lim_{n\to\infty} < Gu_n, v >=< Gu_0, v >, \forall v \in X_1$$

Then the theorem follows from Theorem 1.3.4. The proof is complete.

1.5.2. $W^{2,p}$-strong solutions of nonlinear elliptic systems.

In this subsection, we consider the existence of the $W^{2,p}$-strong solutions of nonlinear elliptic system. Let $\Omega \subset R^n$ be bounded and C^∞.

Given the semilinear elliptic systems

$$\begin{cases} -D_i(a_{ij}^{kl}(x)D_j u_l) + b_i^{kl}(x,u)D_i u_l + c^k(x,u) = f^k(x) \\ \\ u_1|_{\partial\Omega} = 0, \cdots, u_m|_{\partial\Omega} = 0 \end{cases} \tag{1.5.23}$$

where $a_{ij}^{kl} \in c^1(\overline{\Omega})$, and

$$\lambda|\xi|^2 \le a_{ij}^{kl}(x)\xi_{ki}\xi_{lj}, \forall x \in \overline{\Omega}, \xi \in R^{mn} \tag{1.5.24}$$

Suppose that

$$\begin{aligned} &\int_\Omega [b_i^{kl}(x,u)D_i u_l \cdot u_k + c^k(x,u)u_k]dx \\ \ge &\int_\Omega \alpha^k |u_k|^{p_k} dx - \alpha, \quad \forall u \in [c_0^\infty(\overline{\Omega})]^m \end{aligned} \tag{1.5.25}$$

$$\begin{cases} |b_i^{kl}(x,z)| \le \beta^k |z_k|^{q_k} + \beta; \\ \\ 0 \le q_k < max\{p_k/2, n/n-2\} \end{cases} \tag{1.5.26}$$

$$\begin{cases} |c^k(x,z)| \le \gamma^k |z_k|^{\widetilde{q_k}} + \gamma, \\ \\ 0 \le \widetilde{q}_k < max\{p_k, 2n/n-2\}. \end{cases} \tag{1.5.27}$$

We denote

$$s_k = min\{t_k/\widetilde{q}_k, 2t_k/t_k + 2q_k\}$$
$$s = \min_{1\le k\le m}\{s_k\}; (1 < s \le 2)$$
$$t_k = \max\{p_k, 2n/n-2\}$$

Theorem 1.5.3. Let the conditions (1.5.24)-(1.5.27) be satisfied. If $f \in L^2(\Omega, R^m)$, then the system (1.5.23) has a strong solution $u \in W^{2,s}(\Omega, R^m) \cap X_1, X_1$ as that in Theorem 1.5.1. Moreover, if $a_{ij}^{kl}, b_i^{kl}, c^k, f^k \in c^\infty(\overline{\Omega} \times R^m)$ and either $n = 2$ or $s > nt_k/(n+2t_k)$ for $n \geq 3, (1 \leq k \leq m)$, then (1.5.23) has a classical solution $u \in C^\infty(\Omega, R^m)$.

Proof. By Theorem 1.5.1, the equation (1.5.23) has a weak solution $u \in X_1$. Let

$$g^k(x) = f^k - c^k(x, u) - b_i^{kl}(x, u) D_i u_l$$

Then $g^k \in L^{s_k}(\Omega)$(by (1.5.26) and (1.5.27)). Using Theorem 10.5 of [ADN], the weak solution in $W_0^{1,2}(\Omega, R^m)$ of equation

$$\text{(1.5.28)} \qquad \begin{cases} -D_i(a_{ij}^{kl}(x) D_j v_l) = g^k(x), \\ \\ v_k|_{\partial\Omega} = 0, k = 1, \cdots, m \end{cases}$$

is unique and $v \in W^{2,s}(\Omega, R^m)$. Because the solution u of (1.5.23) is also a weak solution of (1.5.28), $u = v \in W^{2,s}(\Omega, R^m)$.

When $n = 2, s < \infty$ is an arbitrary number. Take $s > n$, the $g^k \in c^{0,\alpha}(\overline{\Omega})$(by Theorem 1.1.3). Applying Theorem 9.3 in [ADN], the solution $u \in c^{2,\alpha}(\overline{\Omega})$,which means $g^k \in c^{1,\alpha}(\overline{\Omega})$, and $u \in c^{3,\alpha}(\overline{\Omega})$. We derive $u \in c^\infty(\Omega)$ by iteration.

When $n = 3$, and $s > nt_k/(n+2t_k), W^{2,s}(\Omega) \hookrightarrow L^{sn/n-2s}(\Omega), sn/(n-2s) > t_k$, which implies, by (1.5.26) and (1.5.27), that $g^k \in L^{\widetilde{s}}(\Omega)$ for some $\widetilde{s} > s$, then ones obtain $u \in c^\infty(\Omega)$ by iteration.

The proof of Theorem 1.5.3 is complete.

Remark 1.5.4. For the equation in diagonal form

$$\begin{cases} -\triangle u_k + b_i^{kl}(x, u) D_i u_l + c^k(x, u) = f^k(x) \\ \\ u_k|_{\partial\Omega} = 0, k = 1, \cdots, m \end{cases}$$

if b_i^{kl}, c^k satisfy (1.5.25)-(1.5.27), then the solution $u = (u_1, \cdots, u_m) \in [W^{2,s_1}(\Omega) \times \cdots \times W^{2,s_m}(\Omega)] \cap X_1$ for $f^k \in L^2(\Omega)(1 \leq k \leq m)$.

We consider the semilinear elliptic system

$$\text{(1.5.29)} \qquad \begin{cases} -D_i(a_{ij}^{kl}(x) D_j u_l) + b^k(x, u, Du) = f^k(x) \\ \\ u_k|_{\partial\Omega} = 0, 1 \leq k \leq m \end{cases}$$

$a_{ij}^{kl} \in c^1(\overline{\Omega})$ satisfy (1.5.24), and

$$\int_\Omega b^k(x, u, Du)u_k dx \geq \int_\Omega \alpha^k |u_k|^{p_k} dx - \alpha \tag{1.5.30}$$

$$|b^l(x, z, \xi)| \leq b^k |z_k|^{q_{lk}} + b|\xi|^{r_l} + b \tag{1.5.31}$$

where

$$q_{lk} < \max\{\frac{(p_l - 1)}{p_l} p_k, \frac{n}{n-2}\}$$

$$r_l < \max\{\frac{2(p_l - 1)}{p_l}, \frac{n+2}{2}\}$$

We denote

$$p = \min_{1 \leq k, l \leq m} \{t_k / q_{lk}, 2/r_l\} > 1$$

$$t_k = \max\{p_k, 2n/n - 2\}.$$

Theorem 1.5.5. Let the conditions (1.5.30)-(1.5.31) be satisfied. If $f \in L^2(\Omega, R^m)$, then (1.5.29) has a strong solution $u \in W^{2,p}(\Omega, R^m) \cap X_1$. Moreover, if $a_{ij}^{kl}, b^k, f^k \in c^\infty(\overline{\Omega} \times R^m \times R^{mn})$, and either $N = 2$ or $p > nt_k/(n + 2t_k)$ for $n \geq 3 (1 \leq k \leq m)$, then (1.5.29) has a classical solution $u \in C^\infty(\Omega, R^m)$.

The proof of Theorem 1.5.5, by applying Theorem 1.5.2, is similar to that of Theorem 1.5.3, here we omit the details.

By using the de Giorgi estimates, we can obtain the $c^{0,\alpha}$ regularity of weak solutions of the quasilinear elliptic system in diagonal form

$$\begin{cases} -D_i(a_{ij}^1(x, u)D_j u_1 + b_i^1(x, u)) + c^1(x, u, Du) = f^1 \\ \vdots \\ -D_i(a^m(x, u)D_j u_m + b_i^m(x, u)) + c^m(x, u, Du) = f^m \\ u_1|_{\partial\Omega} = 0, \cdots, u_m|_{\partial\Omega} = 0 \end{cases} \tag{1.5.32}$$

Before discussing the problem (1.5.32), it is necessary to introduce the de Giorgi estimate theorem. Give the elliptic equations with divergence form

$$\begin{cases} -D_i(a_{ij}(x)D_j u) = g(x) + D_i g_i(x) \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.5.33}$$

where $a_{ij} \in L^{\infty}(\Omega)$, and

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j$$

Theorem 1.5.6(see [GT]). Let $\Omega \subset R^n$ be bounded and c^{∞}. For some $q > n, g_i \in L^q(\Omega), g \in L^{q/2}(\Omega)$, if $u \in W_0^{1,2}(\Omega)$ is a weak solution of (1.5.33), then $u \in c^{\alpha}(\overline{\Omega})(0 < \alpha < 1)$, and

$$\|c\|_{c^{\alpha}} \leq c[\|u\|_{L^2} + \|g\|_{L^{q/2}} + \sum_{i=1}^{n} \|g_i\|_{L^q}]$$

Now we return to consider the existence of regularity solution of (1.5.32). Suppose that

$$\lambda|\xi|^2 \leq a_{ij}^k(x,z)\xi_i\xi_j, \quad \forall 1 \leq k \leq m, \lambda > 0 \tag{1.5.34}$$

$$\int_{\Omega} [b_i^k(x,u)D_i u_k + c^k(x,u)u_k]dx \geq -c \tag{1.5.35}$$

$$\begin{cases} |a_{ij}^k(x,z)| < c \\ |b_i^k(x,z)| < c|z|^{p_1} + c \\ |c^k(x,z,\xi)| < c|z|^{p_2} + c|\xi|^{p_3} + c \end{cases} \tag{1.5.36}$$

where $0 \leq p_1, \ \ p_2 < \infty$ are arbitrarily for $n = 2$, and $0 \leq p_1 < \frac{n}{n-2}, \ \ 0 \leq p_2 < \frac{4}{n-2}$ for $n \geq 3, 0 \leq p_3 < \frac{4}{n}$ for $n \geq 2$.

Theorem 1.5.7. Let the conditions (1.5.34)-(1.5.36) hold. If $f \in L^{\frac{q}{2}}(\Omega, R^m)(q > n)$, then (1.5.29) has a weak solution $u \in W_0^{1,2}(\Omega, R^m) \cap C^{\alpha}(\overline{\Omega}, R^m)$ for some $0 < \alpha < 1$.

Proof. The existence of weak solution $u \in W_0^{1,2}(\Omega, R^m)$ follows from Theorem 1.5.2, and by (1.5.36)

$$g^k(x) = f^k(x) - c^k(x,u,Du) \in L^{q/2}(\Omega), \ \ for\ some\ q > n.$$

$$g_i^k(x) = b_i^k(x,u) \in L^q(\Omega).$$

Thus we get from Theorem 1.5.6 that the weak solution u of (1.5.32) belongs to $C^{\alpha}(\overline{\Omega}, R^m)$. The proof is complete.

Remark 1.5.8. By the H^2-estimates of quasilinear elliptic systems (see [LU]), if $u \in W_0^{1,2}(\Omega, R^m) \cap C^0(\overline{\Omega}, R^m)$ is a weak solution of (1.5.29), then $u \in H^2_{loc}(\Omega, R^m)$.

Remark 1.5.9. For the quasilinear elliptic systems with non-diagonal form, M.Giaquinta [Gi] obtain some inner $c^{0,\alpha}$-regularity estimates, which amounts to saying that a weak solution $u \in W_0^{1,2}(\Omega, R^m)$ of the system below belongs to $c^\alpha_{loc}(\Omega)(0 < \alpha < 1)$

$$\begin{cases} -D_i(a^{kl}_{ij}(x,u)D_j u_l + b^k_i(x,u)) + c^k(x,u,Du) = 0 \\ \\ u_k|_{\partial\Omega} = 0, 1 \le k \le m \end{cases}$$

where a^{kl}_{ij}, b^k_i, c^k satisfy the conditions of (1.5.34) and (1.5.35)

1.6. Keldys-Fichera Boundary Value Problem for Degenerate Elliptic Equations

1.6.1. Background

An important example relating to degenerate elliptic equations is the following well known Tricomi equation, which is of especially interest in the aerodynamics

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x,y) \in R^2 \tag{1.6.1}$$

The Tricomi equation is a mixed equation of elliptic-hyperbolic type. As $y > 0$, (1.6.1) is elliptic and when $y < 0$ it is hyperbolic. The equation (1.6.1) can be divided into two equations to be considered respectively as follows

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad for\ (x,y) \in R^2_+ \tag{1.6.2}$$

where $R^2_+ = \{(x,y) \in R^2 | y > 0\}$, and

$$\frac{\partial^2 u}{\partial y^2} - y\frac{\partial^2 u}{\partial x^2} = 0, \quad for\ (x,y) \in R^2_+ \tag{1.6.3}$$

It is easy to see that the equation (1.5.2) is a degenerate elliptic equation and (1.6.3) is a degenerate hyperbolic equation in R^2_+. If $u_i(x,y)(i = 1,2)$ are respectively the solutions of (1.5.2) and (1.5.3), with

$$u_1(x,0) = u_2(x,0), \forall x \in R^1$$

then the function

$$u(x,y) = \begin{cases} u_1(x,y), & as\ y \geq 0 \\ u_2(x,-y), & as\ y \leq 0 \end{cases}$$

is a weak solution of Tricomi equation (1.6.1).

In general, most of the mixed equations of elliptic-hyperbolic type can be divided into the degenerate elliptic and hyperbolic equations to be discussed respectively.

For the degenerate elliptic equations, generally to say, the set of degenerate points on boundary $\partial\Omega$ is of nonzero measure on $\partial\Omega$. It implies that the Dirichlet boundary value problem for degenerate elliptic is not well posed anymore, and instead of it the Keldys-Fichera boundary value problem works. On the well posedness of Keldys-Fichera boundary value problem for degenerate elliptic equations, the readers are referred to next section or O.A.Oleinik and E.V.Radkevich [OR].

1.6.2 Existence of the quasilinear equation

In this subsection, we consider the existence of the Keldys-Fichera boundary value problem for the following degenerate quasilinear elliptic equations

$$(1.6.4) \quad \begin{cases} Lu = D_i[a_{ij}(x,u)D_j u + b(x)u] - C(x,u) = f(x), & x \in \Omega \\ u(x) = 0, & x \in \sum_2 \cup \sum_3 \end{cases}$$

where $\Omega \subset R^n$ is an open set, and $\sum_i (i = 1,2,3)$ are defined by

$\sum_3 = \{x \in \partial\Omega\} | a_{ij}(x,0)N_iN_j > 0\}$

$\sum_2 = \{x \in \partial\Omega\} \setminus \sum_3 | b_i(x) \cdot N_i > 0\}$

$\sum_1 = \partial\Omega \setminus (\sum_2 \cup \sum_3)$.

$\vec{N} = (N_1, \cdots, N_n)$ is the unit outward normal vector on $\partial\Omega$.

Suppose that the coefficients satisfy Caratheodory condition, and

(L_1) Symmetry: $a_{ij}(x,z) = a_{ji}(x,z)$,

(L_2) There exist a constant $\beta > 0$ and a nonnegative continuous function $\lambda(x)$ on $\overline{\Omega}$ such that

$$(1.6.5) \qquad \beta^{-1} a_{ij}(x,0)\xi_i\xi_j \leq a_{ij}(x,z)\xi_i\xi_j \leq \beta a_{ij}(x,0)\xi_i\xi_j$$

$$(1.6.6) \qquad \lambda(x)|\xi|^2 \leq a_{ij}(x,0)\xi_i\xi_j.$$

$(L_3)\Omega' = \{x \in \Omega | \lambda(x) = 0\}$ is a measure zero set in R^n, and there exist bounded subdomains with the cone property $\Omega_n \subset\subset \Omega \backslash \Omega'$, such that $\Omega_n \subset \Omega_{n+1}$ and $\cup_{n=1}^{\infty} \Omega_n = \Omega \backslash \Omega'$.

$(L_4) b_i \in C^1(\overline{\Omega})(1 \le i \le n)$, and

$$|a_{ij}(x,z)| \le C \tag{1.6.7}$$

$$C[|z|^k + |z|^2] - g_1(x) \le C(x,z)z - \frac{1}{2} D_i b_i(x) z^2 \tag{1.6.8}$$

$$|C(x,z)| \le C|z|^{k-1} + g_2(x) \tag{1.6.9}$$

where $k > 1, C > 0$ are constants, $g_1 \in L^1(\Omega), g_2 \in L^{k'}(\Omega), 1/k + 1/k' = 1$.

Remark 1.6.1. The condition (1.6.5) implies that the degenerate points of (1.6.4) have nothing to do with z, hence $\sum_3$ can be written as $\sum_3 = \{x \in \partial\Omega | a_{ij}(x,z) N_i N_j > 0, \quad \forall z \in R^1\}$.

Remark 1.6.2. If Ω is a bounded domain, the condition (1.6.8) may be weakened to read

$$C|z|^k - g_1(x) \le C(x,z)z - \frac{1}{2} D_i b_i(x) z^2$$

Denote by

$$X = \{v \in C^1(\overline{\Omega}) \quad |v|_{\sum_3} = 0, \qquad and\ \|v\|_2 < \infty\}$$

endowed with the norm

$$\|v\|_2 = [\int_\Omega (|\nabla v|^2 + |v|^2) dx + \int_{\partial\Omega} |v|^2 ds]^{\frac{1}{2}} + [\int_\Omega |v|^k dx]^{\frac{1}{k}}$$

Let X_1 be the completion of X under the norm

$$\|v\|_1 = [\int_\Omega (a_{ij}(x,0) D_i v D_j v + |v|^2) dx]^{\frac{1}{2}} +$$

$$+ [\int_{\sum_1 \cup \sum_2} |\vec{b} \cdot \vec{N}| v^2 ds]^{\frac{1}{2}} + [\int_\Omega |v|^k dx]^{\frac{1}{k}}$$

where $\vec{b} = \{b_1(x), \cdots, b_n(x)\}$.

Let X_2 be the completion of X under the norm $\|\cdot\|_2$. Obviously, X_1 is a reflexive Banach space, and X is separable endowed norm space. A weak solution of (1.6.4) is defined to be a element $u \in X_1$ such that

$$\int_\Omega [a_{ij}(x,u)D_j u D_i v + b_i \cdot u D_i v + C(x,u)v + f \cdot v]dx$$

$$(1.6.10) \qquad -\int_{\sum_1} \vec{b} \cdot \vec{N} uvds = 0, \quad \forall v \in X_2$$

Theorem 1.6.3. Under the conditions $(L_1) - (L_4)$, if $f \in L^{k'}(\Omega)$, then problem (1.6.4) has a weak solution in X_1.

Proof. Let $< Lu, v >$ be the left-hand side of (1.6.10). The first thing to be checked is that the inner product $< Lu, v >$ defines a bounded mapping $L : X_1 \to X_2^*$.

Given $u \in X_1$, let $\widehat{a}_{ij}(x) = a_{ij}(x,u)$, then $(\tilde{a}_{ij}(x))$ is a symmetric semi-positive definite matrix. Hence for any $v \in X_2$ we have

$$\begin{aligned} & \int_\Omega a_{ij}(x,u) D_j u D_i v dx = \int_\Omega \tilde{a}_{ij}(x) D_j u D_i v dx \\ \leq \ & [\int_\Omega \tilde{a}_{ij}(x) D_i u D_j u]^{\frac{1}{2}} \times [\int_\Omega \tilde{a}_{ij}(x) D_j v D_i v dx]^{\frac{1}{2}} \\ \leq \ & \beta[\int_\Omega a_{ij}(x,0) D_i u D_j u dx]^{\frac{1}{2}} [\int_\Omega a_{ij}(x,0) D_i v D_j v dx]^{\frac{1}{2}} \\ \leq \ & C\|u\|_1 \cdot \|v\|_2 \end{aligned}$$

where $C > 0$ is a constant.

By (1.6.9) and $b_i \in C^1(\Omega)$, from the definitions of $\|\cdot\|_1$ and $\|\cdot\|_2$ we can deduce

$$\begin{aligned} & |\int_\Omega [b_i u D_i v + C(x,u)v + f \cdot v]dx| \\ \leq \ & C\|u\|_{L^2} \cdot \|Dv\}_{L^2} + \|f\|_{L^{k'}} \|v\|_{L^k} + C\|u\|_{L^k}^{k-1} \|v\|_{L^k} + \|g_2\|_{L^{k'}} \|v\|_{L^k} \\ \leq \ & [C\|u\|_1^{k-1} + C] \cdot \|v\|_2 \end{aligned}$$

Furthermore, we have

$$\begin{aligned} |\int_{\sum_1} \vec{b}\cdot\vec{N}u\cdot vds| &\leq [\int_{\sum_1} |\vec{b}\cdot\vec{N}|u^2 ds]^{\frac{1}{2}}[\int_{\sum_1} |\vec{b}\cdot\vec{N}|v^2 ds]^{\frac{1}{2}} \\ &\leq C\|u\|_1\cdot\|v\|_2. \end{aligned}$$

It follows from the three inequalities above that $L : X_1 \to X_2^*$ is a bounded mapping.

Now we check the acute angle condition. Let $u \in X$, then

$$< Lu, u >= \int_\Omega [a_{ij}(x,u)D_iuD_ju + C(x,u)u - \frac{1}{2}D_ib_iu^2 + fu]dx$$

$$+\frac{1}{2}\int_{\sum_2} \vec{b}\cdot\vec{N}\cdot u^2 ds - \frac{1}{2}\int_{\sum_1}\vec{b}\cdot\vec{N}u^2 ds \tag{1.6.11}$$

Since $\vec{b}\cdot\vec{N} > 0$ on $\sum_2$ and $\vec{b}\cdot\vec{n} \leq 0$ on $\sum_1$, from (1.6.5), (1.6.8) and (1.6.11) we derive

$$\begin{aligned} < Lu, u > &\geq \int_\Omega [\beta^{-1}a_{ij}(x,0)D_iuD_ju + C|u|^k + C|u|^2 - fu - g_1]dx \\ &\quad +\frac{1}{2}\int_{\sum_1\cup\sum_2} |\vec{b}\cdot\vec{N}|u^2 ds \\ &\geq \int_\Omega [\beta^{-1}a_{ij}(x,0)D_iuD_ju + \frac{1}{2}C|u|^k + Cu^2]dx \\ &\quad +\frac{1}{2}\int_{\sum_1\cup\sum_2} |\vec{b}\cdot\vec{N}|u^2 ds - \int_\Omega [\frac{2}{Ckk'}|f|^{k'} + |g_1|]dx \end{aligned}$$

which means that ther is a constant $R > 0$ such that

$$< Lu, u >\geq 0, \quad \forall u \in X \ and \ \|u\|_1 = R.$$

It remains to show that $L : X_1 \to X_2^*$ is weakly continuous. Suppose that $u_n \rightharpoonup u_0$ in X_1. We shall prove that u_n converges to u_0 in measure on any bounded subdomain $\Omega_0 \subset \Omega$. According to (L_3), there are bounded subdomains with cone property $\Omega_n \subset\subset \Omega\backslash\Omega', \Omega_n \subset \Omega_{n+1}, \cup_{n=1}^\infty \Omega_n = \Omega\backslash\Omega'$. Since Ω' is a set of measure zero in R^m, we only have to show that u_n converges to u_0 in measure on Ω_k for any k.

For any integer $k_0 > 0$, it follows from the continuity of $\lambda(x)$ on $\overline{\Omega}$ and $\Omega_{k_0} \subset\subset \Omega\backslash\Omega'$ that there exists $\epsilon_{k_0} > 0$ such that $\lambda(x) \geq \epsilon_{k_0}$ in Ω_{k_0}. We denote by $X_1(\Omega_{k_0})$ the Banach space which is the completion of functions in X_1 restricted to Ω_{k_0} under the norm

$$\|u\| = [\int_{\Omega_0} |\nabla u|^2 dx]^{\frac{1}{2}} + [\int_{\Omega_{k_0}} |u|^k dx]^{\frac{1}{k}}$$

Clearly, X_1 can be embedded into $X_1(\Omega_{k_0})$. We denote by I the embedding operator, evidently $Iu_n \rightharpoonup Iu_0$ in $X(\Omega_{k_0})$. By means of the compact embedding theorems, $u_n \to u_0$ in $L^2(\Omega_{k_0})$, which means that u_n converges to u_0 in measure on Ω_0.

We only have to prove that for any $v \in X_2$,

$$\lim_{n\to\infty} \int_\Omega a_{ij}(x, u_n) D_j u_n D_i v dx = \int_\Omega a_{ij}(x, u_0) D_j u_0 D_i v dx \tag{1.6.12}$$

$$\lim_{n\to\infty} \int_\Omega C(x, u_n) v dx = \int_\Omega C(x, u_0) v dx. \tag{1.6.13}$$

By the condition (1.6.9), $\{C(x, u_n)\} \subset L^{k'}(\Omega)$ is bounded. Because u_n converges to u_0 in measure on any bounded subdomain $\Omega_0 \subset \Omega$, we get that

$$C(x, u_n) \to C(x, u_0) \ in \ L^p(\Omega_0), \quad \forall 1 \le p < k'$$

Hence (1.6.13) holds true for any $v \in C_0^\infty(\Omega)$. On the other hand, $C_0^\infty(\Omega)$ is dense in $L^k(\Omega)$, so by (1.6.9) we conclude that (1.6.13) is satisfied for any $v \in L^k(\Omega)$.

Now we consider (1.6.12). It is possible that Du does not belong to any $L^p(\Omega)(p \ge 1)$, hence the proof of (1.6.12) cannot be as simple as that of (1.6.13).

Make the following decomposition

$$\begin{aligned} & \int_\Omega [a_{ij}(x, u_n) D_j u_n D_i v - a_{ij}(x, u_0) D_j u_0 D_i v] dx \\ = & \int_\Omega a_{ij}(x, u_0) D_i v (D_j u_n - D_j u_0) dx + \\ & + \int_\Omega [a_{ij}(x, u_n) - a_{ij}(x, u_0)] D_j u_n D_i v dx \end{aligned}$$

We know from (1.6.5) that the following two semi-norms are equivalent

$$|u|_0 = [\int_\Omega a_{ij}(x, 0) D_i u D_j u dx]^{\frac{1}{2}},$$

$$|u|_{u_0} = [\int_\Omega a_{ij}(x, u_0) D_i u D_j u dx]^{\frac{1}{2}}.$$

By $u_n \rightharpoonup u_0$ in X_1, we get that

$$\lim_{n\to\infty} \int_\Omega a_{ij}(x, u_0) D_i v (D_j u_n - D_j u_0) dx = 0$$

$\forall v \in X_2$. It is easy to verify that the following inner product defines a bounded mapping $A : X_1 \to L^2(\Omega, R^n)$

$$< Au, w >= \int_\Omega [a_{ij}(x, u) - a_{ij}(x, u_0)] D_j u w_i dx$$

where $w = (w_1, \cdots, w_n) \in L^2(\Omega, R^n)$. If we can prove

$$\lim_{n\to\infty} < Au_n, w >= 0, \quad \forall w \in L^2(\Omega, R^n) \tag{1.6.14}$$

then we will have finished the proof of (1.6.12).

Let $\widetilde{\Omega} = \Omega \backslash \Omega'$. Because Ω' is of measure zero in $R^n, C_0^\infty(\widetilde{\Omega}, R^n)$ is dense in $L^2(\Omega, R^n)$. Hence it suffices to prove (1.6.14) for $w \in C_0^\infty(\widetilde{\Omega}, R^n)$.

Given a $w \in C_0^\infty(\widetilde{\Omega}, R^n)$, then there is a bounded subdomain $\Omega_0 \subset\subset \widetilde{\Omega}$ such that supp $w \subset \Omega_0$, while u_n converges to u_0 in measure in Ω_0. From (1.6.7) we obtain

$$a_{ij}(x, u_n) \to a_{ij}(x, u_0) \; in \; L^2(\Omega_0)$$

On the other hand, there is a $\epsilon_0 > 0$ such that $\lambda(x) > \epsilon_0, \forall x \in \Omega_0$. Hence $\{Du_n\}$ is bounded in $L^2(\Omega_0, R^n)$, which implies (1.6.14) holds true for $w \in C_0^\infty(\widetilde{\Omega}, R^n)$. Thus the weak continuousness of L is proved. The theorem follows from Theorem 1.3.2. The proof is complete.

Example 1.6.4. We take the following quasilinear Keldys equation as an example to illustrate the application of Theorem 1.6.3.

$$\begin{cases} \frac{\partial}{\partial x}(x^p f_1(u)) \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(y^q f_2(u) \frac{\partial u}{\partial y}) - \frac{\partial u}{\partial x} - u^3 = f \\ u(x, y) = 0, \quad (x, y) \in \sum_2 \cup \sum_3 \end{cases} \tag{1.6.15}$$

where $0 < p, q, (x, y) \in \Omega \subset R^2$ as shown in Fig.1.1 below and $f_1, f_2 \in C(R)$ satisfy the condition

$$0 < C_1 \leq f_1(z), f_2(z) \leq C_2 < \infty$$

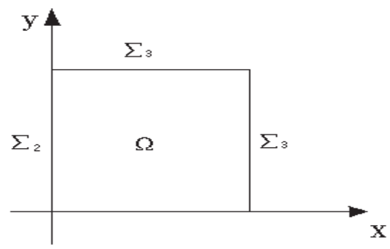

Fig. 1.1

It is easy to see that $\sum_2 = \{(x,y)|x = 0, 0 < y \leq 1\}, \sum_3 = \{(x,y)|x = 1, 0 < y \leq 1$ and $y = 1, 0 < x \leq 1\}$. Applying Theorem 1.6.3 we claim that if $f \in L^{4/3}$, then problem (1.6.15) has a weak solution u satisfying

$$\int_\Omega [x^p|\frac{\partial u}{\partial x}|^2 + y^q|\frac{\partial u}{\partial y}|^2 + u^4]dxdy \leq C$$

where $C > 0$ is a constant dependent on f.

1.6.3 Maximum principle and L^∞-modular estimates

In this subsection, we mainly discuss the maximum principle, L^∞-modular estimates and the comparison principle for weak solutions of degenerate elliptic Keldys-Fichera boundary value problem.

Let $\Omega \subset R^n$ be a bounded domain, we first consider the linear case. Give the following operator

$$L_1 u = D_i(a_{ij}(x)D_j u + b_i(x)u) - C(x)u$$

$b_i \in C^1(\overline{\Omega})$ and $a_{ij}(x) = a_{ji}(x)$, furthermore

$$0 \leq a_{ij}(x)\xi_i\xi_j, \quad \forall x \in \overline{\Omega}, \xi \in R^n.$$

Let $\widetilde{X}_1$ be the completion of $C^1(\overline{\Omega})$ with the norm

$$\|u\|_{\widetilde{X}_1} = [\int_\Omega (a_{ij}(x)D_i u D_j u + u^2)dx + \int_{\sum_1 \cup \sum_2} |\bar{b} \cdot \bar{N}|u^2 ds]^{\frac{1}{2}}$$

We say $u \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^q(\Omega)(1 < p \leq 2, \frac{1}{p} + \frac{1}{q} = 1)$ satisfies $L_1 u \geq 0$(or ≤ 0) in weak sense, if $\forall v \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^q(\Omega)$ with $v|_{\sum_2 \cup \sum_3} = 0, v \geq 0$

in Ω, the following inequality holds

$$\int_\Omega [a_{ij}(x)D_i u D_j v + b_i(x)u \cdot D_i v + Cu \cdot v]dx$$

$$-\int_{\sum_1} \vec{b} \cdot \vec{N} u \cdot v ds \leq 0 \ (or \ \geq 0) \tag{1.6.16}$$

Theorem 1.6.5. Let $\sum_2 \cup \sum_3 \neq \phi$, and

$$b^*(x) < C(x), \quad \forall x \in \Omega \tag{1.6.17}$$

where $b^*(x) = max\{D_i b_i(x), \frac{1}{2} D_i b_i(x)\}$. If $u \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^q(\Omega) (\frac{1}{p} + \frac{1}{q} = 1)$ satisfies $L_1 u \geq 0$(or ≤ 0) in weak sense, then the nonnegative maximum (nonpositive minimum) of u must be achieved in $\overline{\sum_2 \cup \sum_3}$.

Proof. Let

$$k = \sup_{\sum_2 \cup \sum_3} u \ (or \ k = \inf_{\sum_2 \cup \sum_3} u)$$

By the claims of the theorem, we may assume $k \geq 0 (k \leq 0)$. Let $v_0 = [u-k]^+$(or $v_0 = [k-u]^+$), where for a function $f, [f]^+$ is defined by

$$[f]^+ = \begin{cases} f(x) & if \ f(x) > 0 \\ 0 & if \ f(x) \leq 0 \end{cases}$$

From the chain rule for weak derivative (see [GT]) and the assumptions on u, we obtain that $v_0 \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^q(\Omega), v_0|_{\sum_2 \cup \sum_3} = 0, v_0 \geq 0$ on Ω, and

$$D_i v_0 = \begin{cases} D_i u, & u(x) > k \\ 0, & u(x) \leq k \end{cases} \tag{1.6.18}$$

Now we only need to deal with the case of $k \geq 0$, as for the case of $k \leq 0$, the proof is the same as the case of $k \geq 0$.

Putting $v = v_0$ in (1.6.16), by (1.6.18) and the integration by parts, we obtain the inequality

$$\int_\Omega [a_{ij}(x)D_j u D_i v_0 - b_i(x)D_i u v_0 + c(x)u v_0 - D_i b_i(x) u v_0]dx$$

$$
\begin{aligned}
&= \int_\Omega [a_{ij}(x)D_jv_0D_iv_0 - b_i(x)D_iv_0\cdot v_0 + c(x)uv_0 - D_ib_iuv_0]dx \\
&= \int_\Omega [a_{ij}(x)D_iv_0D_jv_0 + (c - D_ib_i)uv_0 + \frac{1}{2}D_ib_iv_0^2]dx \\
&\leq \frac{1}{2}\int_{\sum_1} \vec{b}\cdot\vec{N}v_0^2ds
\end{aligned}
$$

Let $\Omega^+ = \{x \in \Omega | u(x) > k\}$. Because $u = v_0 + k$ on Ω^+ and $\vec{b}\cdot\vec{n} \leq 0$ on $\sum_1$, from the above inequality, we have

$$
\int_{\Omega^+} [a_{ij}(x)D_iv_0D_jv_0 + (c(x) - \frac{1}{2}D_ib_i(x))v_0^2 + (c - D_ib_i)v_0k]dx \leq 0
$$

which implies by (1.6.17) that Ω^+ is a set of measure zero. The proof is complete.

Now we consider the modular estimate theorem for weak solutions of equation (1.6.4). The condition (L_2) is changed to read

$$
0 \leq a_{ij}(x,0)\xi_i\xi_j, \quad \forall x \in \Omega, \xi \in R^n. \tag{1.6.19}
$$

Theorem 1.6.6. Assume that $\sum_2 \cup \sum_3 \neq \phi$ and L satisfies $(L_1), (L_3)$, (1.6.19) and

$$
b^*(x) < c(x,z)z^{-1}, \quad for\ (x,z) \in \Omega \times R \tag{1.6.20}
$$

If $u \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^k(\Omega)(\frac{p}{p-1} \leq k)$ satisfies (1.6.10) $\forall v \in \widetilde{X}_1 \cap W^{1,p}(\Omega) \cap L^k(\Omega)$ with $v|_{\sum_2\cup\sum_3} = 0$, then

$$
|u| \leq max\{\sup_\Omega |\frac{f}{c^*}|, \sup_{\sum_2\cup\sum_3} |u|\} = M
$$

whe $c^*(x) = \inf_{z\in R^1}[c(x,z)z^{-1} - D_ib_i(x)]$.

Proof. First, we assume that L is a linear operator. Let $w = M \pm u$, then $w|_{\sum_2\cup\sum_3} \geq 0$. Taking $v \in W^{1,p}(\Omega) \cap \widetilde{X}_1 \cap L^k(\Omega)$ with $v|_{\sum_2\cup\sum_3} = 0, v \geq 0$ on Ω, we have

$$
\int_\Omega [a_{ij}(x)D_jwD_iv + b_i(x)D_ivw + cwv]dx - \int_{\sum_1} b_iN_iwvds
$$

$$= \int_\Omega \pm[a_{ij}(x)D_j u D_i v + b_i(x)D_i v u + cuv]dx$$

$$- \int_{\sum_1} \pm[\bar{b}\cdot \bar{N}uv]ds + \int_\Omega [c(x)M - D_i b_i(x)M]vdx$$

$$= \int_\Omega [(c(x) - D_i b_i(x))M \pm f]vdx \geq 0$$

By Theorem 1.6.5, $w \geq 0$ on Ω. Then we have $|u| \leq M$ on Ω a.e.

For the case that L is a nonlinear operator, we take $\tilde{a}_{ij}(x) = a_{ij}(x,u), \tilde{b}_i(x) = b_i(x)$, and $\tilde{c}(x) = c(x,u)/u(x)$, then u satisfies (1.6.10) with respect to the linear operator $Lu = D_i[\tilde{a}_{ij}(x)D_j u + \tilde{b}(x)u] - \tilde{c}(x)u$. From the above result the theorem follows. The proof is complete.

Remark 1.6.7. If $k = \sup_{\sum_2 \cup \sum_3} u = 0$(or $k = \inf_{\sum_2 \cup \sum_3} u = 0$), and the set $\Omega' = \{x \in \Omega | a_{ij}(x)\xi_i\xi_j = 0$, for some $\xi \in R^n, |\xi| \neq 0\}$ has measure zero in R^n, the conditions (1.6.17) and (1.6.20) can be weakened respectively as

$$b^*(x) \leq c(x) \ \ \forall x \in \Omega, \quad \textit{and } b^*(x) \leq c(x,z)z^{-1}, \ \ \forall (x,z) \in \Omega \times R^1.$$

Applying Theorem 1.6.6, ones can obtain the L^∞-modular estimates of weak solutions of (1.6.4) as follows.

Corollary 1.6.8. Under the hypotheses of Theorem 1.6.3, let $f \in L^\infty(\Omega)$ and $b^*(x) \leq c(x,z)z^{-1}, \forall (x,z) \in \Omega \times R^1$, if the weak solution $u \in X_1$ of (1.6.4) belongs to $W^{1,p}(\Omega)$ for some $1 < p$ and $p/(p-1) \leq k$, then we have

$$\sup_\Omega |u| \leq \sup_\Omega |\frac{f(x)}{c^*(x)}|$$

Finally, we consider the comparision principle.

Definition 1.6.9. We say that $Lu_1 \geq Lu_2$ in weak sense, if

$$\int_\Omega [a_{ij}(x,u_1)D_j u_1 D_i v + b_i u_1 D_i v + c(x,u_1)v]dx - \int_{\sum_1} \vec{b}\cdot\vec{N}u_1 v ds$$

$$\leq \int_\Omega [a_{ij}(x,u_2)D_j u_2 D_i v + b_i u_2 D_i v + c(x,u_2)v]dx - \int_{\sum_1} \vec{b}\cdot\vec{N}u_2 v ds$$

$\forall v \in \widetilde{X}_1 \cap W^{1,p}(\Omega)$ with $v|_{\sum_2 \cup \sum_3} = 0$ and $v \geq 0$ on Ω.

Theorem 1.6.10. Assume that $a_{ij}(x,z)=a_{ij}(x), c\in c^1(\Omega\times R^1)$, and

$$b^*(x) < c'_z(x,z), \quad \forall(x,z)\in\Omega\times R^1$$

If $u_1,u_2\in \widetilde{X}_1\cap W^{1,p}(\Omega)\cap L^k(\Omega)(p/p-1\le k)Lu_1\ge Lu_2$ in weak sense, and $u_1|_{\sum_2\cup\sum_3}\le u_2|_{\sum_2\cup\sum_3}$, then $u_1\le u_2$ on Ω.

Proof. Let $w=u_1-u_2$, then $w|_{\sum_2\cup\sum_3}\le 0$. From $Lu_1\ge Lu_2$ we have

$$\int_\Omega [a_{ij}(x)D_jwD_iv+b_iD_ivw+(c(x,u_1)-c(x,u_2))v_2]dx$$
$$-\int_{\sum_1}\vec{b}\cdot\vec{N}wvds\le 0$$

for $v\in\widetilde{X}_1\cap W^{1,p}(\Omega)\cap L^k(\Omega), v|_{\sum_2\cup\sum_3}=0, v\ge 0$ on Ω. Let $c(x,u_1)-c(x,u_2)=c'_z(x,\tilde{u})w$, where $\tilde{u}$ is a suitable mean value function of u_1 and u_2. From Theorem 1.6.5 we deduce $w\le 0$ as required. The proof is complete.

Corollary 1.6.11. Under the assumptions of Theorem 1.6.10, if the problem (1.6.4) has a weak solution in $X_1\cap W^{1,p}(\Omega)$ for some $p>1$ and $p/(p-1)\le k$, then this weak solution must be unique.

In the same fashion as the proof of Theorem 1.6.6, one can obtain the following theorem.

Theorem 1.6.12. Let $b^*(x)\le c(x,z)z^{-1}, \forall(x,z)\in\Omega\times R^1$. If $f(x)\le 0$ and u is a weak solution of (1.6.4) in $X_1\cap W^{1,p}(\Omega)(1<p, p/(p-1)\le k)$, then $u(x)\ge 0$ on Ω.

Remark 1.6.13. In the degenerate elliptic equations, if the terms $D_i(b_i(x)u)\equiv 0, 1\le i\le n$, then in all the theorems of this subsection, the condition $u\in\widetilde{X}_1\cap W^{1,p}(\Omega)\cap L^k(\Omega)(\frac{1}{p}+\frac{1}{k}=1)$ can be relaxed as that $u\in\widetilde{X}_1$(or $u\in X_1$ in Corollary 1.6.8).

1.6.4 $W^{1,p}$-solutions of the quasilinear equations

We start with an abstract regularity result which is useful for the existence problem of $W^{m,p}(\Omega)$-solutions of degenerate quasilinear elliptic equations of order $2m$.

Let X, X_1, X_2 be the spaces defined in Theorem 1.3.2, and Y be a reflective Banach space, $Y\hookrightarrow X_1$.

Lemma 1.6.14. Under the hypotheses of Theorem 1.3.2, there exists a sequence of $\{u_n\} \subset X, u_n \rightharpoonup u_0$ in X_1 such that $< Gu_n, u_n >= 0$, furthermore if we can derive that $\|u_n\|_Y < c, c$ is a constant, then the solution u_0 of $Gu = 0$ belongs to Y.

The proof of Lemma 1.6.13 is obviously.

Now we return to discuss the existence of $W^{1,p}$-solutions of equation (1.6.4). Let $\Omega \subset R^n$ be bounded and C^∞.

Theorem 1.6.15. Under the assumptions of Theorem 1.6.3, if there is a real number $\beta > 1$ such that

$$\int_\Omega |\lambda(x)|^{-\beta} dx < \infty, \quad \lambda(x) \; defined \; as \; in \; (1.6.6)$$

then (1.6.4) has a weak solution $u \in X_1 \cap W^{1,p}(\Omega), p = 2\beta/1+\beta > 1$. Moreover, if $\sum_2 \cup \sum_3 \neq \phi$, and when $b_i \not\equiv 0$, for some $1 \leq i \leq n, k \geq 2\beta/\beta - 1, c(x,z)z^{-1} - D_i b_i \geq \alpha > 0, \forall (x,z) \in \Omega \times R^1$, otherwise $c(x,z)z^{-1} \geq \alpha > 0$, then the solution $u \in L^\infty(\Omega)$ provided $f \in L^\infty(\Omega)$.

Proof. According to Lemma 1.6.14, it suffices to prove that there is a constant $c > 0$ such that for any $u \in X (X$ is as that in Theorem 1.6.3) with $< Lu, u >= 0$, we have

$$\|u\|_{W^{1,p}} \leq c, \quad p = \frac{2\beta}{1+\beta}. \tag{1.6.21}$$

From (1.6.10) we know

$$< Lu, u >= \int_\Omega [a_{ij}(x,u) D_i u D_j u + b_i(x) u D_i u$$
$$+ c(x,u)u + f \cdot u] dx - \int_{\sum_1} \vec{b} \cdot \vec{N} u^2 ds = 0, \quad u \in X$$

Due to (L_2) and (1.6.8) we have

$$< Lu, u >= \int_\Omega [a_{ij}(x,u) D_i u D_j u + c(x,u)u - \frac{1}{2} D_i b_i u^2$$
$$+ f \cdot u] dx + \frac{1}{2} \int_{\sum_2} \vec{b} \cdot \vec{N} u^2 ds - \frac{1}{2} \int_{\sum_1} \vec{b} \cdot \vec{N} u^2 ds$$
$$\geq \int_\Omega [\beta^{-1} \lambda(x) |\nabla u|^2 + c|u|^k - f \cdot u - g_1] dx$$

$$+\frac{1}{2}\int_{\sum_1\cup\sum_2}|\vec{b}\cdot\vec{N}|u^2ds$$

Consequently we have

$$\int_\Omega[g_1+c_1|f|^{k'}]dx\geq\int_\Omega[\beta^{-1}\lambda(x)|\nabla u|^2+\frac{c}{2}|u|^k]dx \tag{1.6.22}$$

By the reversed Holder inequality (see [Ad])

$$\int_\Omega\lambda(x)|\nabla u|^2dx\geq[\int_\Omega|\lambda|^{-\beta}dx]^{-\frac{1}{\beta}}[\int_\Omega|\nabla u|^{\frac{2\beta}{1+\beta}}dx]^{\frac{1+\beta}{\beta}} \tag{1.6.23}$$

From (1.6.22) and (1.6.23), the estimates (1.6.21) follows.

The second conclusion follows from Theorem 1.6.6 and Remark 1.6.13. The proof is complete.

Example 1.6.16. We consider the $W^{1,p}$-solutions of the following Keldys equation.

$$\begin{cases}\frac{\partial}{\partial x}(x^{\alpha_1}f_1(u)\frac{\partial u}{\partial x_1})+\frac{\partial}{\partial y}(y^{\alpha_2}f_2(u)\frac{\partial u}{\partial y})-u=f(x), & x\in\Omega\\ u|_{\sum_3}=0\end{cases} \tag{1.6.24}$$

where $\sum_2=\phi$, and $\Omega=(0,1)\times(0,1),\sum_3,f_1,f_2$ are defined as in Example 1.6.4. It is easy to see that

$$\lambda(x,y)=min\{c_1x^{\alpha_1},c_2y^{\alpha_2}\},$$

here $c_1>0$ is the constant defined as in Example 1.6.4. If $0<\alpha_1,\alpha_2<\frac{1}{2}$, then for $\beta=2$, we have

$$\int_0^1\int_0^1|\lambda|^{-2}dx\leq c_1^2\int_0^1x^{-2\alpha_1}dx\cdot\int_0^1y^{-2\alpha_2}dy<\infty$$

Furthermore we have

$$c(x,z)z^{-1}=1>0$$

Therefore, by Theorem 1.6.15, the equation (1.6.24) has a weak solution $u\in W^{1,\frac{4}{3}}(\Omega)\cap L^\infty(\Omega)$ provided $f\in L^\infty(\Omega)$.

Next we investigate the $W^{1,p}$-solutions of the degenerate quasilinear elliptic equations as follows

$$\begin{cases}-D_i(a_{ij}(x,u)D_ju+b_i(x)u)+c(x,u,\nabla u)=f(x),\\ u|_{\sum_2\cup\sum_3}=0\end{cases} \tag{1.6.25}$$

Suppose that

A_1). The conditions $(L_1)(L_2)$ in Theorem 1.6.3 hold, $\sum_i (i=1,2,3)$ is the same as that in (1.6.4), and the measure of $\sum_2 \cup \sum_3$ is nonzero on $\partial\Omega$.

A_2). For the function $\lambda(x)$ in (L_2), there exist $\beta_0 > 1$, such that

$$\int_\Omega |\lambda(x)|^{-\beta_0} dx < \infty.$$

A_3).$b_i \in c^1(\overline{\Omega})$ and there is a $g \in L^1(\Omega)$ such that

$$g(x) \leq c(x,z,\xi)z - \frac{1}{2} D_i b_i \cdot z^2 \tag{1.6.26}$$

$$\begin{cases} |a_{ij}(x,z)| \leq c \\ |c(x,z,\xi)| \leq c[|z|^{\alpha_1} + |\xi|^{\alpha_2} + 1] \end{cases} \tag{1.6.27}$$

$$0 \leq \alpha_1 < \frac{n(\beta_0 - 1) + 2\beta_0}{n(1+\beta_0) - 2\beta_0}; \quad 0 \leq \alpha_2 < \frac{2\beta_0 + n(\beta_0 - 1)}{n(1+\beta_0)}$$

Let $X = \{u \in c^1(\overline{\Omega}) | u|_{\sum_3} = 0\}$, and Y be the completion of X with the norm

$$\|u\|_Y = [\int_\Omega a_{ij}(x,0) D_i u D_j u dx]^{\frac{1}{2}} + [\int_\Omega |\nabla u|^p dx]^{\frac{1}{p}}$$
$$+ [\int_{\sum_1 \cup \sum_2} |\vec{b} \cdot \vec{N}| u^2 ds]^{\frac{1}{2}}$$

Since mes $\sum_2 \cup \sum_3 \neq 0$, by the generalized Poincare inequalities (see [Te]), we know that $\|u\|_Y \geq c\|u\|_{W^{1,p}}$, i.e. $Y \hookrightarrow W^{1,p}(\Omega)$. For the equation (1.6.25), we always take $p = 2\beta_0/1 + \beta_0$, and

$$\beta_0 \geq \begin{cases} 1, & if\ b_i \equiv 0, \ \forall 1 \leq i \leq n \\ n, & if\ b_i \not\equiv 0, \quad for\ some\ 1 \leq i \leq n \end{cases} \tag{1.6.28}$$

$u \in Y$ is called a weak solution of (1.6.25), if $\forall v \in Y$

(1.6.29)
$$\int_\Omega [a_{ij}(x,u) D_j u D_i v + b_i u D_i v + c(x,u,\nabla u)v - fv] dx - \int_{\sum_1} \vec{b} \cdot \vec{N} uv ds = 0$$

By applying Theorem 1.3.4, Theorem 1.6.5 and Remark 1.6.13, we can obtain the following theorem.

Theorem 1.6.17. Let the conditions $(A_1) - (A_3)$ be satisfied and $f \in L^{p'}(\Omega)(\frac{1}{p'} + \frac{1}{p}) = 1)$. Then (1.6.25) has a weak solution $u \in Y$. Moreover, if $f \in L^{\infty}(\Omega)$, and

$$\inf_{z\in R^1,\xi\in R^n}[c(x,z,\xi)z^{-1} - D_i b_i(x)] \geq \alpha > 0 \tag{1.6.30}$$

then the solution $u \in L^{\infty}(\Omega)$.

Proof. Denote by $< Gu, v >$ the left part of equality (1.6.29). It is easy to show that the inner product $< Gu, v >$ defines a bounded continuous mapping $G : Y \to Y^*$ owing to (1.6.27) and (1.6.28). First we check the acute angle condition. Let $u \in Y$, we have

$$\begin{aligned}
< Gu, u > &= \int_\Omega [a_{ij}(x,u)D_i u D_j u - \frac{1}{2}D_i b_i u^2 + c(x,u,\nabla u)u \\
&\quad -fu]dx + \frac{1}{2}\int_{\sum_2} \vec{b}\cdot\vec{N}u^2 ds - \frac{1}{2}\int_{\sum_1} \vec{b}\cdot\vec{N}u^2 ds \\
&\geq \int_\Omega [\beta^{-1}a_{ij}(x,0)D_i u D_j u + g - fu]dx + \frac{1}{2}\int_{\sum_1\cup\sum_2} |\vec{b}\cdot\vec{N}|u^2 ds \\
&\quad \textit{(due to (1.6.5) and (1.6.26))} \\
&\geq \frac{\beta^{-1}}{2}\int_\Omega \lambda(x)|\nabla u|^2 dx + \frac{\beta^{-1}}{2}\int_\Omega\int_\Omega [a_{ij}(x,0)D_i u D_j u + g - fu]dx \\
&\quad +\frac{1}{2}\int_{\sum_1\cup\sum_2} |\vec{b}\cdot\vec{N}|u^2 ds, \qquad \textit{(by (1.6.6))} \\
&\geq \frac{\beta^{-1}}{2}[\int_\Omega |\lambda|^{-\beta_0}dx]^{-\frac{1}{\beta_0}}[\int_\Omega |\nabla u|^p]^{\frac{2}{p}} + \frac{\beta^{-1}}{2}\int_\Omega a_{ij}(x,0)D_i u D_j u \\
&\quad +\frac{1}{2}\int_{\sum_1\cup\sum_2} |\vec{b}\cdot\vec{N}|u^2 ds - c\int_\Omega |u|^p dx - c\int_\Omega [|g| + |f|^{p'}]dx \\
&\quad \textit{(thanks to the reversed Holder inequality)}
\end{aligned}$$

According to (A_2) and $p < 2$, from the above inequality we can derive

$$< Gu, u > \geq 0, \ \ \forall u \in Y, \ \ \|u\|_Y = R \textit{ great enough.}$$

Next we need to verify the continuous condition ii) in Theorem 1.3.4. Let

$u_n \rightharpoonup u_0$ in Y(as $\beta_0 = 1$, let $u_n \to^* u_0$ in Y), and

$$\lim_{n\to\infty} < Gu_n - Gu_0, u_n - u_0 >= 0$$

From the condition (1.6.27), by using the same manner as the proof of Theorem 1.5.2, ones can show that

$$\lim_{n\to\infty} < Gu_n, v >=< Gu_0, v >, \ \ \forall v \in Y$$

Here we omit the details of proof. Therefore the first conclusion of the theorem follows from Theorem 1.3.4.

Finally, by (1.6.30) we can obtain the second conclusion from Theorem 1.6.5 and Remark 1.6.13 by using the same method as the proof of Theorem 1.6.6. The proof is complete.

1.6.5. Interior regularity

In this subsection, we concern the interior regularity of weak solutions of equations (1.6.25), here a weak solution u of (1.6.25) means that u satisfies (1.6.29) for any $v \in C^1(\overline{\Omega})$ with $v|_{\sum_3} = 0$. We always assume that

$$0 \leq a_{ij}(x,z)\xi_i\xi_j, \ \ \forall (x,z) \in \Omega \times R, \ \ \xi \in R^n$$

and the set $\Omega' = \{x \in \Omega | a_{ij}(x,z)\xi_i\xi_j = 0$, for some $\xi \in R^n$ and $|\xi| \neq 0\}$ is independent of z, mes $\Omega' = 0$ in R^n.

Suppose that $a_{ij}, b_i, c \in c^1(\overline{\Omega} \times R \times R^n)$, and

$$|c(x,z,\xi)| \leq g(x,z), \ \ g \in c(\Omega \times R^1) \tag{1.6.31}$$

Theorem 1.6.18. Let (1.6.31) hold and $f \in c^1(\overline{\Omega})$. If $u \in \widetilde{X} \cap L^\infty(\Omega)$ is a weak solution of (1.6.25), $\widetilde{X}$ defined as that in section 1.6.3, then $u \in c^\alpha(\Omega\backslash\Omega') \cap H^2_{loc}(\Omega\backslash\Omega'), (0 < \alpha < 1)$.

Proof. Because $\Omega\backslash\Omega'$ is open, for any $x_0 \in \Omega\backslash\Omega'$ there is a close ball $\overline{B}_{2\delta}(x_0) = \{x \in \Omega | x - x_0| \leq 2\delta\} \subset \Omega\backslash\Omega'$ for some $\delta > 0$. It suffices to verify that $u \in c^\alpha(B_\delta(x_0)) \cap H^2_{loc}(B_\delta(x_0))$.

Take $\eta \in c_0^\infty(\Omega)$ such that supp $\eta \subset B_{2\delta}(x_0)$, and

$$0 \leq \eta(x) \leq 1, \ \ \eta(x) = 1, \quad as\ x \in B_\delta(x_0)$$

Let $w = \eta \cdot u$, then

$$\int_\Omega a_{ij}(x,u) D_j w D_i v dx = \int_\Omega \eta(x) a_{ij}(x,u) D_j u D_i v dx \tag{1.6.32}$$

$$+\int_\Omega a_{ij}(x,u)uD_j\eta\cdot D_i v dx$$

Puting $\eta\cdot v$ in (1.6.29), we have

$$\int_\Omega \eta a_{ij}(x,u)D_iuD_jvdx = -\int_\Omega[a_{ij}(x,u)D_iuD_j\eta\cdot v+ \tag{1.6.33}$$

$$+b_iuD_i\eta\cdot v + b_iu\eta\cdot D_iv + c(x,u,\nabla u)\eta v - f\eta\cdot v]dx$$

On the other hand

$$-\int_\Omega a_{ij}(x,u)D_iuD_j\eta\cdot vdx = \int_\Omega A_{ij}(x,u)D_j\eta\cdot D_ivdx+$$

$$+\int_\Omega[\frac{\partial A_{ij}}{\partial x_i}D_i\eta + A_{ij}(x,u)D_{ij}\eta]vdx$$

where

$$A_{ij}(x,z)=\int_0^z a_{ij}(x,y)dy.$$

Since supp $\eta\subset B_{2\delta}(x_0)$, from (1.6.32) and (1.6.33), we have

$$\int_{B_{2\delta}} a_{ij}(x,u)D_jwD_ivdx = \int_{B_{2\delta}}[A_{ij}(x,u)D_j\eta + b_iu\cdot\eta$$

$$+a_{ij}(x,u)uD_j\eta]D_ivdx + \int_{B_{2\delta}}[\frac{\partial A_{ij}(x,u)}{\partial x_i}D_j\eta \tag{1.6.34}$$

$$+A_{ij}(x,u)D_{ij}\eta + f\eta - b_iD_i\eta\cdot u]vdx$$

Denote by

$$\begin{cases} g_i(x) = A_{ij}(x,u)D_j\eta + a_{ij}(x,u)D_j\eta u + b_i(x)u\eta, \\ g(x) = \frac{\partial A_{ij}(x,u)}{\partial x_i}D_j\eta + A_{ij}(x,u)D_{ij}\eta + f\cdot\eta - b_iD_i\eta u - c(x,u,\nabla u)\eta \end{cases}$$

Because $\overline{B}_{2\delta}\subset\Omega\backslash\Omega'$, there exists a constant $\epsilon>0$ such that

$$\epsilon|\xi|^2\le a_{ij}(x,z)\xi_i\xi_j,\quad \forall(x,z)\in\overline{B}_{2\delta}(x_0)\times R$$

Hence $w\in W^{1,2}(B_{2\delta})\cap L^\infty(B_{2\delta})$ is a weak solution of the following equation

$$\begin{cases} -D_i(a_{ij}(x,u)D_jw) = g - D_ig_i, \quad x\in B_{2\delta}(x_0) \\ w|_{\partial B_{2\delta}} = 0 \end{cases}$$

Owing to $u \in L^\infty(B_{2\delta})$ and (1.6.31), $g, g_i \in L^\infty(B_{2\delta})$, and thanks to the de Giorgi estimates (Theorem 1.5.6), we get that $w \in c^\alpha(\overline{B}_{2\delta})$, which implies that $u \in c^\alpha(\overline{B}_\delta)$, for some $0 < \alpha < 1$.

Noticing that (1.6.34) holds true for any $v \in H_0^1(B_\delta)$, and $Dw = Du$ in B_δ, therefore we obtain

$$\int_{B_\delta} [a_{ij}(x,u)D_j u - A_{ij}(x,u)D_j\eta - a_{ij}(x,u)D_j\eta u - b_i u\eta]D_i v dx$$

$$-\int_{B_\delta} [\frac{\partial A_{ij}}{\partial x_i} D_j\eta + A_{ij}(x,u)D_{ij}\eta + f\eta - b_i D_i\eta u - c(x,u,\nabla u)\eta] v dx$$

$$= 0, \quad \forall v \in H_0^1(B_\delta).$$

Thus, u restricting on B_δ is a weak solution of the equation

$$D_i A_i(x,u,\nabla u) + B(x,u,\nabla u) = 0, \quad x \in B_\delta(x_0)$$

where

$$\begin{cases} A_i(x,u,\nabla u) = a_{ij}(x,u)D_j u - A_{ij}(x,u)D_j\eta - a_{ij}(x,u)D_j\eta u - b_i u\eta \\ \\ B(x,u,\nabla u) = \frac{\partial A_{ij}}{\partial x_i} D_j\eta + A_{ij}(x,u)D_{ij}\eta + f\eta - b_i D_i\eta u - c(x,u,\nabla u)\eta \end{cases}$$

According to the assumptions, it is easy to see that $A_i, B \in c^1(\overline{B}_\delta \times R \times R^n)$, and $u \in W^{1,2}(B_\delta) \cap c^\alpha(\overline{B}_\delta)$. By means of the H^2-regularity of quasilinear elliptic equations (see [LU]), we derive that $u \in H^2_{loc}(B_\delta)$. Thus the theorem is proven.

Next, we consider the interior $W^{2,p}$-regularity of (1.6.25). Assume that $a_{ij}(x,z) \in c^1(\overline{\Omega})$ are independent of z, and

$$|c(x,z,\xi)| \le c(|z|^k + |\xi|^q + 1). \tag{1.6.35}$$

where $0 \le k, 0 \le q < 2$.

Theorem 1.6.19. Let (1.6.35) be satisfied, and $b_i \in c^1(\overline{\Omega}), f \in L^{k^*}(\Omega), k^* = k+1/k$. If $u \in \widetilde{X} \cap L^{k+1}(\Omega)$ is a weak solution of (1.6.25), then $u \in W^{2,p}_{loc}(\Omega \backslash \Omega'), p = min\{2, k+1/k, 2/q\}$. Furthermore, if $a_{ij}, b, c \in c^\infty(\overline{\Omega} \times R \times R^n)$, and $np/(n-2p) > k+1$, $np/(n-p) > 2$, then $u \in c^\infty(\Omega \backslash \Omega')$.

Proof. As the proof of Theorem 1.6.18, we can get that $w = \eta \cdot u \in W^{1,2}(B_{2\delta}) \cap L^{k+1}(B_{2\delta})$ is a weak solution of the equation

$$\begin{cases} -D_i(a_{ij}(x)D_j w) = g - D_i g_i, \quad x \in B_{2\delta} \\ \\ w|_{\partial B_{2\delta}} = 0 \end{cases}$$

where

$$\begin{cases} g_i = 2a_{ij}(x)D_j\eta u + b_i\eta \cdot u, \\ \\ g = D_i a_{ij} D_j \eta u + a_{ij} D_{ij}\eta u + f\eta - b_i D_i \eta u - c(x, u, \nabla u)\eta \end{cases}$$

By (1.6.35) and $\widetilde{X} \cap L^{k+1}(\Omega) \hookrightarrow W^{1,2}(B_{2\delta}) \cap L^{k+1}(B_{2\delta})$, we can see that $g \in L^{\widetilde{k}}(B_{2\delta}), D_i g_i \in L^2(B_{2\delta}), \widetilde{k} = min\{\frac{k+1}{k}, \frac{2}{q}\}$. According to the L^p-estimates (Theorem 1.1.11), ones obtain that $w \in W^{2,p}(B_{2\delta})$, i.e. $u \in W^{2,p}(B_\delta), p = min\{2, (k+1)/k, 2/q\}$. The first conclusion is proven.

By iteration, similar to the proof of Theorem 1.5.3, one can derive the second conclusion of this theorem. The proof is complete.

1.7. The Boundary Value Problem of the Equations with Nonnegative Characteristic Form

1.7.1. Formulation of the boundary value problem

For second order equations with nonnegative characteristic form, Keldys[Ke] and Fichera[Fi] presented a kind of boundary condition, with that the associated problem is of well posedness. However, for higher order ones, the discussion of well posed boundary value problem has not been seen. Here we shall give a kind of boundary value condition, which is consistent with Dirichlet problem if the equations are elliptic, and coincident with Keldys-Fichera boundary value problem when the equations are of second order.

We consider the linear partial differential operator

$$\begin{aligned} Lu = & \sum_{|\alpha|=|\beta|=m, |\gamma|=m-1} (-1)^m D^\alpha (a_{\alpha\beta}(x) D^\beta u + b_{\alpha\gamma}(x) D^\gamma u) \\ & + \sum_{|\theta|, |\lambda| \le m-1} (-1)^{|\theta|} D^\theta (d_{\theta\lambda}(x) D^\lambda u). \end{aligned} \tag{1.7.1}$$

where $x \in \Omega, \Omega \subset R^n$ is an open set, the coefficients of L are bounded measurable functions, and $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$.

Let $\{g_{\alpha\beta}(x)\}$ be a series of functions with $g_{\alpha\beta} = g_{\beta\alpha}, |\alpha| = |\beta| = k$. If in certain order we put all multiple indexes α with $|\alpha| = k$ into a row$\{\alpha^1, \cdots, \alpha^{N_k}\}$, then $\{g_{\alpha\beta}(x)\}$ can be made into a symmetric matrix $(g_{\alpha^i\alpha^j})$. By this rule, we get a symmetric matrix

$$M(x) = (a_{\alpha^i\alpha^j}(x))_{i,j=1,\cdots,N_m} \tag{1.7.2}$$

Suppose that the matrix $M(x)$ is semi-positive, i.e.

$$0 \le a_{\alpha^i\alpha^j}(x)\xi_i\xi_j, \ \ \forall x \in \overline{\Omega}, \xi \in R^{N_m}, \tag{1.7.3}$$

and the odd order part of (1.7.1) can be written as

(1.7.4)

$$\sum_{|\alpha|=m,|\gamma|=m-1} (-1)^m D^\alpha(b_{\alpha\gamma}(x)D^\gamma u) = \sum_{i=1}^{n} \sum_{|\lambda|=|\theta|=m-1} (-1)^m D^{\lambda+\delta_i}(b^i_{\lambda\theta}(x)D^\theta u)$$

where $\delta_i = \{\delta_{i1}, \cdots, \delta_{in}\}, \delta_{ij}$ is the Kronecker symbol. Assume that for all $1 \le i \le n$, we have

$$b^i_{\lambda\theta}(x) = b^i_{\theta\lambda}(x), \quad x \in \Omega$$

We introduce another symmetric matrix

$$B(x) = \left(\sum_{k=1}^{n} b^k_{\lambda^i\lambda^j}(x) \cdot n_k\right)_{i,j=1,\cdots,N_{m-1}}, \quad x \in \partial\Omega \tag{1.7.5}$$

where $\vec{n} = \{n_1, \cdots, n_n\}$ is the outward normal at $x \in \partial\Omega$. Let the following matrices be orthogonal

$$C^M(x) = (C^M_{ij}(x))_{i,j=1,\cdots,N_m}, \quad x \in \Omega$$

$$C^B(x) = (C^B_{ij}(x))_{i,j=1,\cdots,N_{m-1}}, \quad x \in \partial\Omega$$

satisfying

$$C^M(x)M(x)C^M(x)' = (e_i(x)\delta_{ij})_{i,j=1,\cdots,N_m}$$

$$C^B(x)B(x)C^B(x)' = (h_i(x)\delta_{ij})_{i,j=1,\cdots,N_{m-1}}.$$

where $C(x)'$ is the transposed matrix of $C(x), e_i(x)$ are the eigenvalues of $M(x)$ and $h_i(x)$ the eigenvalues of $B(x)$. Denote by

$\sum_i^M = \{x \in \partial\Omega | e_i(x) > 0\}, \quad 1 \le i \le N_m$

$\sum_i^B = \{x \in \partial\Omega | h_i(x) > 0\}, \quad 1 \le i \le N_{m-1}$

$\sum_1^C = \partial\Omega \setminus \sum_i^B; \quad 1 \le i \le N_{m-1}.$

For multiple indices $\alpha, \beta, \alpha \le \beta$ means that $\alpha_i \le \beta_i, \forall 1 \le i \le n$. Now let us consider the following boundary value problem,

$$Lu = f(x), \quad x \in \Omega \tag{1.7.6}$$

$$D^\alpha u|_{\partial\Omega} = 0, \quad |\alpha| \le m-2 \tag{1.7.7}$$

$$\sum_{j=1}^{N_{m-1}} C^B_{ij}(x) D^{\lambda^j} u|_{\sum_i^B} = 0, \quad |\lambda^j| = m-1, \;\; 1 \le i \le N_{m-1} \tag{1.7.8}$$

$$\sum_{j=1}^{N_m} C^m_{ij}(x) D^{\alpha^i - \delta_{k_j}} u \cdot n_{k_j}|_{\sum_i^M} = 0, \tag{1.7.9}$$

for all $\delta_{k_j} \le \alpha^j, |\alpha^j| = m$, and $1 \le i \le N_m$ where $\delta_{k_j} = \{\underbrace{0, \cdots, 1}_{k_j}, \cdots, 0\}$.

We can see that the item (1.7.9) of boundary value condition is determined by the leading term matrix (1.7.2), and (1.7.8) is defined by the odd term matrix (1.7.5). Moreover, if the operator L is not elliptic, then the item (1.7.7) implies that the operator

$$L'u = \sum_{|\theta|, |\lambda| \le m-1} (-1)^{|\theta|} D^\theta (d_{\theta\lambda}(x) D^\lambda u)$$

is elliptic.

In order to illustrate the boundary value condition (1.7.7)-(1.7.9), in following we give an example.

Example 1.7.1. Give the differential equation

$$\frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + \frac{\partial^3 u}{\partial x_2^3} - \triangle u = f, \quad x \in \Omega \subset R^2. \tag{1.7.10}$$

here $\Omega = \{(x_1, x_2) \in R^2 | 0 < x_1 < 1, 0 < x_2 < 1\}$. let $\alpha^1 = \{2, 0\}, \alpha^2 = \{1, 1\}, \alpha^3 = \{0, 2\}$ and $\lambda^1 = \{1, 0\}, \lambda^2 = \{0, 1\}$, then the leading and odd term matrices of (1.7.10) respectively are

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 0 \\ 0 & n_2 \end{pmatrix}$$

and the orthogonal matrices are

$$C^M = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}, \quad C^B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We can see that $\sum_1^M = \partial\Omega, \sum_2^M = \partial\Omega, \sum_3^M = \phi$, and $\sum_1^B = \phi, \sum_2^B$ as shown in Fig 1.2 below

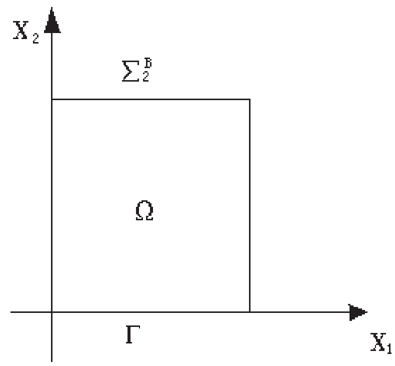

Fig. 1.2

The item (1.7.8) is

$$\sum_{j=1}^{2} C_{2j}^{B} D^{\lambda^j} u|_{\sum_2^B} = D^{\lambda^2} u|_{\sum_2^B} = \frac{\partial u}{\partial x_2}|_{\sum_2^B} = 0$$

and the item (1.7.9) is

$$\sum_{j=1}^{3} C_{1j}^{M} D^{\alpha^i - \delta_{k_j}} u \cdot n_{k_j}|_{\sum_1^M} = D^{\alpha^1 - \delta_{k_1}} u n_{k_1}|_{\sum_1^M} = 0$$

$$\sum_{j=1}^{3} C_{2j}^{M} D^{\alpha^j - \delta_{k_j}} u \cdot n_{k_j}|_{\sum_2^M} = D^{\alpha^2 - \delta_{k_2}} u \cdot n_{k_2}|_{\sum_2^M} = 0$$

$\forall \delta_{k_1} \leq \alpha^1$ and $\delta_{k_2} \leq \alpha^2$. Since only $\delta_{k_1} = \{1,0\} \leq \alpha^1 = \{2,0\}$, hence we have

$$D^{\alpha^1 - \delta_{k_1}} u \cdot n_{k_1}|_{\sum_1^M} = \frac{\partial u}{\partial x_1} \cdot n_1|_{\partial\Omega} = 0$$

however, $\delta_{k_2} = \{1,0\} < \alpha^2 = \{1,1\}$ and $\delta_{k_2} = \{0,1\} < \alpha^2$, therefore

$$D^{\alpha^2 - \delta_{k_2}} u \cdot n_{k_2}|_{\sum_2^M} = \begin{cases} \frac{\partial u}{\partial x_2} \cdot n_1|_{\partial\Omega} = 0 \\ \\ \frac{\partial u}{\partial x_1} \cdot n_2|_{\partial\Omega} = 0 \end{cases}$$

Thus the associated boundary value condition of (1.7.10) is as follows

$$u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial x_2}|_{\partial\Omega/\Gamma} = 0, \quad \frac{\partial u}{\partial x_1}|_{\partial\Omega} = 0$$

which implies that $\frac{\partial u}{\partial x_2}$ is free on $\Gamma = \{(x_1, x_2) \in \partial\Omega | 0 < x_1 < 1, x_2 = 0\}$.

Remark 1.7.2. In general the matrices $M(x)$ and $B(x)$ arranged are not unique, hence the boundary value conditions relating to the operator L may be not unique.

Remark 1.7.3. When all leading terms of L are zero, the equation (1.7.6) is an odd order one. In this case, only (1.7.7) and (1.7.8) remain.

Now we return to discuss the relations between the condition (1.7.7)-(1.7.9) with Dirichlet and Keldys-Fichera boundary value conditions.

It is easy to verify that the problem (1.7.6)-(1.7.9) is the Dirichlet problem provided the operator L being elliptic. In this case, $\sum_i^M = \partial\Omega$ for all $1 \le i \le N_m$. Besides, (1.6.9) run over all $1 \le i \le N_m$ and $\delta_{k_j} \le \alpha^j$, moreover $C^M(x)$ is non-degenerate for any $x \in \partial\Omega$. Solving the system of equations, we get $D^\alpha u|_{\partial\Omega} = 0, \forall |\alpha| = m - 1$.

When $m = 1$, namely L is of second order, the condition (1.7.8) is the form
$u|_{\sum^B} = 0, \quad \sum^B = \{x \in \partial\Omega | \sum_{i=1}^n b_i(x) n_i > 0\}$.
and (1.7.9) is

$$\sum_{j=1}^n C_{ij}^M(x) n_j u|_{\sum_i^M} = 0, \quad 1 \le i \le n$$

Noticing

$$\sum_{i,j=1}^n a_{ij}(x) n_i n_j = \sum_{i=1}^n e_i(x) (\sum_{j=1}^n C_{ij}^M(x) n_j)^2$$

thus the condition (1.7.9) is the form
$u|_{\sum^M} = 0, \quad \sum^M = \{x \in \partial\Omega | a_{ij}(x) n_i n_j > 0\}$.
It shows that when $m = 1$, (1.7.8) and (1.7.9) are coincident with Keldys-Fichera boundary value condition.

Next, we shall give the definition of weak solutions of (1.7.6)-(1.7.9). Let

$$X = \{v \in C^\infty(\overline{\Omega}) \quad | D^\alpha v|_{\partial\Omega} = 0, \quad |\alpha| \le m - 2, \ \textit{and} \quad v \ \textit{satisfy} \ (1.7.9), \ \|v\|_2 < \infty\}. \tag{1.7.10}$$

where $\|\cdot\|_2$ is defined by

$$\|v\|_2 = [\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha v|^2 dx + \int_{\partial\Omega} \sum_{|\gamma|=m-1} |D^\gamma v|^2 ds]^{\frac{1}{2}}$$

We denote by X_2 the completion of X under the norm $\|\cdot\|_2$ and by X_1 the completion of X with the following norm

$$\|v\|_1 = [\int_\Omega (\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\alpha v D^\beta v + \sum_{|\gamma|\leq m-1} |D^\gamma v|^2) dx$$

$$+ \int_{\partial\Omega} \sum_{i=1}^{N_{m-1}} |h_i(x)| (\sum_{j=1}^{N_{m-1}} C^B_{ij} D^{\gamma^j} v)^2 ds]^{\frac{1}{2}}$$

Definition 1.7.4. $u \in X_1$ is a weak solution of (1.7.6)-(1.7.9) if for any $v \in X_2$, the following equality holds

$$\int_\Omega [\sum_{|\alpha|=|\beta|=m, |\gamma|=m-1} (a_{\alpha\beta}(x) D^\beta u + b_{\alpha\gamma}(x) D^\gamma u) D^\alpha v$$

$$+ \sum_{|\theta|,|\lambda|\leq n-1} d_{\theta,\lambda}(x) D^\lambda u D^\theta v] dx - \sum_{i=1}^{N_{m-1}} \int_{\sum_i^C} h(x) \cdot \tag{1.7.11}$$

$$\times (\sum_{j=1}^{N_{m-1}} C^B_{ij} D^{\gamma^j} u)(\sum_{j=1}^{N_{m-1}} C^B_{ij} D^{\gamma^j} v) ds = \int_\Omega f(x) \cdot v dx$$

We need to check the reasonableness of the boundary value problem (1.7.6)-(1.7.9) under the definition of weak solutions, i.e., the solutions in the classical sense are necessarily the solutions in weak sense, and conversely when a weak solution satisfies certain regularity conditions, it will surely satisfy the given boundary value conditions. Here we assume that all coefficients of L are sufficiently smooth.

Let u be a classical solution of (1.7.6)-(1.7.9). Denote by $< Lu, v >$ the left part of (1.7.11), we want to show

$$< Lu, v > = \int_\Omega Lu \cdot v dx, \quad \forall v \in X \tag{1.7.12}$$

Thanks to integration by part, we have

$$\int_\Omega Lu\cdot vdx=\int_\Omega[\sum_{|\alpha|=|\beta|=m,|\gamma|=m-1}(a_{\alpha\beta}(x)D^\beta u+b_{\alpha\gamma}(x)D^\gamma u)D^\alpha v$$

$$+\sum_{|\theta|,|\lambda|\leq m-1}d_{\theta\lambda}(x)D^\lambda uD^\theta v]dx-\int_{\partial\Omega}[\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(x)D^\beta u\cdot$$

$$\times D^{\alpha-\delta_k}v\cdot n_k+\sum_{|\lambda|=|\theta|=m-1}\sum_{i=1}^n b^i_{\lambda\theta}(x)\cdot n_iD^\theta uD^\lambda v]ds$$

Since $v\in X$, we have

$$\int_{\partial\Omega}\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(x)D^\beta uD^{\alpha-\delta_k}v\cdot n_kds$$

$$=\int_{\partial\Omega}\sum_{i=1}^{N_m}e_i(x)(\sum_{j=1}^{N_m}C^M_{ij}D^{\alpha^j}u)(\sum_{j=1}^{N_m}C^M_{ij}D^{\alpha^j-\delta_k}v\cdot n_k)ds=0$$

Because u satisfies (1.7.8),

$$\int_{\partial\Omega}\sum_{|\lambda|=|\theta|=m-1}\sum_{i=1}^n b^i_{\lambda\theta}(x)\cdot n_iD^\theta uD^\lambda vds$$

$$=\int_{\partial\Omega}\sum_{i=1}^{N_{m-1}}h_i(x)(\sum_{j=1}^{N_{m-1}}C^B_{ij}D^{\gamma^j}u)(\sum_{j=1}^{N_{m-1}}C^B_{ij}D^{\gamma^j}v)ds$$

$$=\sum_{i=1}^{N_{m-1}}\int_{\sum_i^C}h_i(x)(\sum_{j=1}^{N_{m-1}}C^B_{ij}D^{\gamma^j}u)(\sum_{j=1}^{N_{m-1}}C^B_{ij}D^{\gamma^j}v)ds$$

From the three equalities above we obtain (1.7.12).

Let $u\in X_1$ be a weak solution of (1.7.6)-(1.7.9). Then the boundary value condition (1.7.7) and (1.7.9) can be reflected by the space X_1. In fact, we can show that if $u\in X_1$, then u satisfies

$$\sum_{i=1}^{N_m}\int_{\sum_i^M}e_i(x)(\sum_{j=1}^{N_m}C^M_{ij}D^{\alpha^j-\delta_{k_j}}u\cdot N_{k_j})\times \tag{1.7.13}$$

$$\times(\sum_{j=1}^{N_m}C^M_{ij}D^{\alpha^j}v)ds=0,\quad \forall v\in X_1\cap W^{m+1,2}(\Omega).$$

Evidently, when $u \in X, v \in X_1 \cap W^{m+1,2}(\Omega)$, we have

$$\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\beta u D^\alpha v dx$$

$$= -\int_\Omega \sum_{|\alpha|=m,|\beta|=m} D_i(a_{\alpha,\beta} D^\alpha v) D^{\beta-\delta_i} u dx \tag{1.7.14}$$

If we can verify that for any $u \in X_1$, (1.6.14) holds true, then we get

$$\int_{\partial\Omega} \sum_{|\alpha|=|\beta|=n} a_{\alpha\beta}(x) D^\alpha v D^{\beta-\delta_i} u \cdot N_i ds = 0$$

which means that (1.7.13) holds true. Since X is dense in X_1, for $u \in X_1$ given, let $u_k \in X$ and $u_k \to u$ in X_1. Then

$$\lim_{k\to\infty} \int_\Omega \sum_{|\alpha|=m,|\beta|=m} a_{\alpha\beta} D^\beta u_k D^\alpha v dx = \int_\Omega \sum_{|\alpha|=m,|\beta|=m} a_{\alpha\beta} D^\beta u D^\alpha v dx$$

$$\lim_{k\to\infty} \int_\Omega \sum_{|\alpha|=m,|\beta|=m} D_i(a_{\alpha\beta} D^\alpha v) D^{\beta-\delta_i} u_k dx = \int_\Omega \sum_{|\alpha|=m,|\beta|=m} D_i(a_{\alpha\beta} D^\alpha v) D^{\beta-\delta_i} u dx$$

Due to u_k satisfying (1.7.14), hence $u \in X_1$ satisfies (1.7.14) Thus (1.7.13) is verified.

Remark 1.7.5. When (1.7.2) is a diagonal matrix, then (1.7.9) is the form

$$D^\gamma u|_{\sum_\gamma^M} = 0, \qquad for\ |\gamma| = m-1$$

where $\sum_\gamma^M = \{x \in \partial\Omega | \sum_{i=1}^n a_{\gamma+\delta_i\gamma+\delta_i}(x) \cdot n_i^2 > 0\}$. In this case, the corresponding trace embedding theorems can be set, and the boundary value condition (1.7.9) is naturally satisfied. On the other hand, if the weak solution u of (1.7.6)-(1.7.9) belong to $X_1 \cap W^{m,p}(\Omega)$ for some $p > 1$, then by the trace embedding theorems, the condition (1.7.9) also holds true.

It remains to verify the condition (1.7.8). Let $u_0 \in X_1 \cap W^{m+1,2}(\Omega)$ satisfy (1.7.11). Since $W^{m+1,2}(\Omega) \hookrightarrow X_2$, hence we have

$$\int_\Omega [\sum_{|\alpha|=|\beta|=m,|\gamma|=m-1} (a_{\alpha\beta}(x) D^\beta u_0 + b_{\alpha\gamma}(x) D^\gamma u_0) D^\alpha u_0$$

$$+ \sum_{|\theta|,|\lambda|\le m-1} d_{\theta\lambda}(x) D^\lambda u_0 D^\theta u_0 - f u_0] dx- \tag{1.7.15}$$

$$- \sum_{i=1}^{N_{m-1}} \int_{\sum_i^C} h_i(x)(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} u_0)^2 ds = 0$$

On the other hand, by (1.7.11), for any $v \in C_0^\infty(\Omega)$, we get

$$\int_\Omega [- \sum_{|\alpha|=|\beta|=m} D_i(a_{\alpha\beta}(x) D^\alpha u_0) D^{\beta-\delta_i} v + \sum_{|\theta|,|\lambda| \le m-1} d_{\theta\lambda}(x) D^\lambda u_0 D^\theta v$$
$$-fv - D_i(\sum_{|\theta|=|\gamma|=m-1} b_{\theta\gamma}^i(x) D^\gamma u_0) D^\theta v] dx = 0 \tag{1.7.16}$$

Because the coefficients of L are sufficiently smooth, and $C_0^\infty(\Omega)$ is dense in $W_0^{m-1,2}(\Omega)$, the equality (1.7.16) also holds for any $v \in W_0^{m-1,2}(\Omega)$. Therefore, due to $u_0 \in W_0^{m-1,2}(\Omega)$ we have

$$\int_\Omega [- \sum_{|\alpha|=|\beta|=m} D_i(a_{\alpha\beta}(x) D^\beta u_0) D^{\alpha-\delta_i} u_0 + \sum_{|\theta|,|\lambda| \le m-1} d_{\theta\lambda}(x) D^\lambda u_0 D^\theta u_0$$
$$-fu_0 - D_k(\sum_{|\theta|=|\gamma|=m-1} b_{\theta\gamma}^k(x) D^\gamma u_0) D^\theta u_0] dx = 0 \tag{1.7.17}$$

From (1.7.13) one derives

$$- \int_\Omega \sum_{|\alpha|=|\beta|=m} D_i(a_{\alpha\beta}(x) D^\alpha u_0) D^{\beta-\delta_i} u_0 dx = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\alpha u_0 D^\beta u_0 dx$$

Furthermore

$$- \int_\Omega D_k(\sum_{|\theta|=|\gamma|=m-1} b_{\theta\gamma}^k(x) D^\gamma u_0) D^\theta u_0 dx = \int_\Omega \sum_{|\alpha|=m,|\gamma|=m-1} b_{\alpha\gamma}(x) D^\gamma u_0 D^\alpha u_0 dx$$
$$- \sum_{i=1}^{N_{m-1}} \int_{\sum_i^C \cup \sum_i^B} h_i(x)(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} u_0)^2 ds$$

From (1.7.15) and (1.7.17) one can see that

$$\sum_{i=1}^{N_{m-1}} \int_{\sum_i^B} h_i(x)(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} u_0)^2 ds = 0$$

Noticing $h_i(x) > 0$ in $\sum_i^B$, one deduces that u_0 satisfies (1.6.8) provided $u_0 \in X_1 \cap W^{m+1,2}(\Omega)$.

Finally, we discuss the well posedness of the boundary value problem (1.7.6)-(1.7.9).

Theorem 1.7.6(Existence Theorem). Let $\Omega \subset R^n$ be an arbitrary open set, $f \in L^2(\Omega)$ and $b_{\alpha\gamma} \in C^1(\overline{\Omega})$. If there exist a constant $C > 0$ and $g \in L^1(\Omega)$ such that

$$C \sum_{|\gamma|=m-1} |\xi_\gamma|^2 + C|\xi_i|^2 - g \leq \sum_{|\lambda|,|\theta|\leq m-1} d_{\theta\lambda}(x)\xi_\theta\xi_\lambda - \frac{1}{2}\sum_{i=1}^{n} \sum_{|\gamma|=|\beta|=m-1} D_i b^i_{\gamma\beta}(x)\xi_\gamma\xi_\beta. \tag{1.7.18}$$

where ξ_α is the component of $\xi \in R^{N_{m-1}}$ corresponding to $D^\alpha u$, then the problem (1.7.6)-(1.7.9) has a weak solution in X_1.

Proof. Let $< Lu, v >$ be the inner product as in (1.7.12). It is easy to verify that $< Lu, v >$ defines a bounded linear operator $L : X_1 \to X_2^*$. Hence L is weakly continuous. From (1.7.18), for $u \in X$ we drive that

$$\begin{aligned}
(1.7.19)\qquad < Lu, u >= &\int_\Omega [\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)D^\alpha u D^\beta u + \\
&+\sum_{i=1}^{n} \sum_{|\lambda|=|\theta|=m-1} b^i_{\lambda\theta}(x)D^\theta u D^{\lambda+\delta_i}u + \sum_{|\gamma|,|\alpha|\leq m-1} d_{\gamma\alpha}(x)D^\gamma u D^\alpha u]dx \\
&- \sum_{i=1}^{N_{m-1}} \int_{\sum_i^C} h_i(x)(\sum_{j=1}^{N_{m-1}} C^B_{ij} D^{\gamma^j} u)^2 ds \\
= &\int_\Omega [\sum_{|\alpha|=|\beta|=n} a_{\alpha\beta}(x)D^\alpha u D^\beta u + \sum_{|\gamma|,|\alpha|\leq m-1} d_{\gamma\alpha}(x)D^\gamma u D^\alpha u \\
&-\frac{1}{2}\sum_{i=1}^{n} \sum_{|\gamma|=|\beta|=m-1} D_i b^i_{\gamma\beta}(x)D^\gamma u D^\beta u]dx + \\
&+\frac{1}{2}\sum_{i=1}^{N_{m-1}} [\int_{\sum_i^B} - \int_{\sum_i^C} h_i(x)(\sum_{j=1}^{N_{m-1}} C^B_{ij} D^{\gamma^j} u)^2 ds \\
\geq &\int_\Omega [\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)D^\alpha u D^\beta u + C\sum_{|\gamma|=m-1} |D^\gamma u|^2 + Cu^2 - g(x)]dx \\
&+\frac{1}{2}\sum_{i=1}^{N_{m-1}} [\int_{\sum_i^B \cup \sum_i^C} |h_i(x)|(\sum_{j=1}^{N_{m-1}} C^B_{ij}(x) D^{\gamma^j} u)^2]ds
\end{aligned}$$

Hence we obtain

$$< Lu, u > \geq C\|u\|_1^2 - C, \quad \forall u \in X$$

Thus by Holder inequality we have

$$< Lu - f, u > \geq 0, \quad \forall u \in X, \quad \|u\|_1 = R \textit{ great enough}.$$

By Theorem 1.3.2, the theorem is proven.

Theorem 1.7.7(Uniqueness Theorem). Under the assumptions of Theorem 1.7.6 with $g(x) = 0$ in (1.7.18). If the problem (1.7.6)-(1.7.9) has a weak solution in $X_1 \cap W^{m,p}(\Omega) \cap W^{m-1,q}(\Omega)(\frac{1}{p} + \frac{1}{q} = 1)$, then such a solution is unique. Moreover, if $b_{\alpha\gamma}(x) = 0$ in $L, \forall|\alpha| = m, |\gamma| = m - 1$, then the weak solution $u \in X_1$ of (1.7.6)-(1.7.7) is unique.

Proof. Let $u_0 \in X_1 \cap W^{m,p}(\Omega) \cap W^{m-1,q}(\Omega)$ be a weak solution of (1.7.6)-(1.7.9). We can see that (1.7.11) holds for all $v \in X_1 \cap W^{m,p}(\Omega) \cap W^{m-1,q}(\Omega)$. Hence $< Lu_0, u_0 >$ is well defined. Let $u_1 \in X_1 \cap W^{m,p}(\Omega) \cap W^{m-1,q}(\Omega)$. Then from (1.7.19) it follows that $< Lu_1 - Lu_0, u_1 - u_0 >= 0 \Rightarrow u_1 = u_0$, which means that the solution of (1.7.6)-(1.7.9) in $X_1 \cap W^{m,p}(\Omega) \cap W^{m-1,q}(\Omega)$ is unique. If all the odd terms $b_{\alpha\gamma}(x) = 0$ of L, then (1.7.11) holds for all $v \in X_1$, in the same fashion we know that the weak solution of (1.7.6)-(1.7.9) in X_1 is unique. The proof is complete.

Remark 1.7.8. In subsection 1.7.3, we can see that under certain assumptions, the weak solutions of degenerate elliptic equations are in $X_1 \cap W^{m,p}(\Omega) \cap W^{m-1,q}(\Omega)(\frac{1}{p} + \frac{1}{q} = 1)$.

1.7.2. Existence of higher order quasilinear equations

Give the quasilinear differential operator

$$\begin{aligned}(1.7.20) \qquad Au = & \sum_{|\alpha|=|\beta|=m, |\gamma|=m-1} (-1)^m D^\alpha (a_{\alpha\beta}(x, \bigwedge u) D^\beta u + b_{\alpha\gamma}(x) D^\gamma u) \\ & + \sum_{|\gamma|=|\theta|=m-1} (-1)^{m-1} D^\gamma (d_{\gamma\theta}(x, \bigwedge u) D^\theta u) \\ & + \sum_{|\lambda| \leq m-1} (-1)^{|\lambda|} D^\lambda g_\lambda (x, \bigwedge u)\end{aligned}$$

where $m \geq 2$ and $\bigwedge u = \{D^\alpha u\}_{|\alpha| \leq m-2}$.

Let $a_{\alpha\beta}(x,\xi) = a_{\beta\alpha}(x,\xi)$, the odd order part of (1.7.20) be as that in (1.7.4), $b_{\alpha\gamma} \in C^1(\overline{\Omega})$ and $\sum_i^B, \sum_i^C$ be the same as those in subsection 1.7.1. The leading matrix is

$$M(x,\xi) = (a_{\alpha^i\alpha^j}(x,\xi))_{i,j=1,\cdots,N_m}$$

and the eigenvalues are $e_i(x,\xi)$. We denote

$\sum_i^M = \{x \in \partial\Omega | e_i(x,0) > 0\}, \quad 1 \le i \le N_m.$

We consider the following problem

$$Au = f(x), \quad x \in \Omega \tag{1.7.21}$$

$$\bigwedge u|_{\partial\Omega} = 0. \tag{1.7.22}$$

$$\sum_{j=1}^{N_{m-1}} C_{ij}^B(x) D^{\lambda^j} u|_{\sum_i^B} = 0, \quad |\lambda^j| = m-1, 1 \le i \le N_{m-1}. \tag{1.7.23}$$

$$\sum_{j=1}^{N_m} C_{ij}^M(x,0) D^{\alpha^j - \delta_{k_j}} u \cdot N_{k_j}|_{\sum_i^M} = 0, \tag{1.7.24}$$

$\forall \delta_{k_j} \le \alpha^j, |\alpha^j| = n, 1 \le i \le N_m, \delta_{k_j} = \{\underbrace{0,\cdots,1}_{k_j},\cdots,0\}.$

Denote the anisotropic Sobolev space by

$$W_{|\alpha|\le k}^{p_\alpha}(\Omega) = \{u \in L^{p_0}(\Omega) | p_0 \ge 1, D^\alpha u \in L^{p_\alpha}(\Omega), \ \forall 1 \le |\alpha| \le k, \ and\, p_\alpha \ge 1, \ or \ = 0\}.$$

whose norm is

$$\|u\| = \sum_{|\alpha|\le k} sign\ p_\alpha \|D_\alpha u\|_{L^{p_\alpha}}.$$

when all $p_\alpha = p$ for $|\alpha| = k$, then the space is denoted by $W_{k,|\alpha|\le k-1}^{p,p_\alpha}(\Omega)$.

$q_\theta(|\theta| \le k)$ is termed the critical embedding exponent from $W_{|\alpha|\le k}^{p_\alpha}(\Omega)$ to $L^p(\Omega)$, if q_θ is the largest number of the exponent p in where $D_\theta u \in L^p(\Omega), \forall u \in W_{|\alpha|\le k}^{p_\alpha}(\Omega)$, and the embedding is continuous.

For example, when Ω is bounded, the space $X = \{u \in L^k(\Omega) | k \ge 1, D_i u \in L^2(\Omega), 1 \le i \le n\}$ with norm $\|u\| = \|\nabla u\|_{L^2} + \|u\|_{L^k}$ is an anisotropic Sobolev space, and the critical embedding exponents from X to $L^p(\Omega)$ are $q_i = 2 (1 \le i \le n), q_0 = max\{k, \frac{2n}{n-2}\}$.

Suppose that
(A_1). The coefficients of the leading term of A satisfy one of the following two conditions

i). $a_{\alpha\beta}(x,\eta) = a_{\alpha\beta}(x)$

ii). $a_{\alpha\beta}(x,\eta) = 0$, as $\alpha \neq \beta$.

(A_2). There is a constant $M > 0$ such that

$$0 \leq M \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,0)\xi_\alpha\xi_\beta \leq \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,\eta)\xi_\alpha\xi_\beta \leq M^{-1} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,0)\xi_\alpha\xi_\beta. \tag{1.7.25}$$

(A_3). There are functions $G_i(x,\eta)(i=0,1,\cdots,n)$ with $G_i(x,0)=0$, $\forall 1 \leq i \leq n$, such that

$$\sum_{|\gamma|=m-1} g_\gamma(x,\bigwedge u)D_\gamma u = \sum_{i=1}^{n} D_i G_i(x,\bigwedge u) + G_0(x,\bigwedge u).$$

(A_4). There is a constant $c > 0$ such that

$$c|\xi|^2 \leq \sum_{|\alpha|=|\beta|=m-1} [d_{\alpha\beta}(x)\xi_\alpha\xi_\beta - \frac{1}{2}\sum_{i=1}^{n} D_i b^i_{\alpha\beta}(x)\xi_\alpha\xi_\beta] \tag{1.7.26}$$

$$c \sum_{|\lambda|\leq m-1} sign p_\lambda |\eta_\lambda|^{p_\lambda} - f_1 \leq \sum_{|\theta|\leq m-2} g_\theta(x,\eta)\eta_\theta + G_0(x,\eta). \tag{1.7.27}$$

where $f_1 \in L^1(\Omega), p_0 > 1, p_\lambda > 1$ or $= 0, \forall 1 \leq |\lambda| \leq m-2$.

(A_5). There is a constant $c > 0$ such that

$$|a_{\alpha,\beta}(x,\eta)| \leq c \tag{1.7.28}$$

$$|d_{\gamma\theta}(x,\eta)| \leq c[\sum_{|\beta|\leq m-2} |\eta_\beta|^{s_\beta} + 1] \tag{1.7.29}$$

$$|g_\gamma(x,\eta)| \leq c[\sum_{|\beta|\leq m-2} |\eta_\beta|^{\overline{S}_\beta} + 1]. \tag{1.7.30}$$

where $1 \leq S_\beta < q_{\beta/2}, 1 \leq \overline{S}_\beta < q_\beta, q_\beta$ is critical embedding exponent from $W^{2,p_\lambda}_{m-1,|\lambda|\leq m-2}(\Omega)$ to $L^p(\Omega)$.

Let X be defined by (1.7.10) and X_1 be the completion of X under the norm

$$\|v\|_1 = [\int_\Omega (\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,0)D^\alpha v D^\beta v + \sum_{|\gamma|=m-1} |D^\gamma v|^2)dx$$

$$+\int_{\partial\Omega} \sum_{i=1}^{N_{m-1}} |h_i(x)|(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} v)^2 ds]^{\frac{1}{2}} + \sum_{|\gamma|\le m-2} signp_\gamma \|D^\gamma v\|_{L^{p_\gamma}}$$

and X_2 be the completion of X with the norm

$$\|v\|_2 = \|v\|_{W^{m,p}} + \|v\|_{W^{m,2}} + [\int_{\partial\Omega} \sum_{|\gamma|=m-1} |D^\gamma v|^2 ds]^{\frac{1}{2}}$$

where $p \ge max\{2, q_\beta(q_\beta - \overline{S}_\beta)^{-1}, 2q_\beta(q_\beta - 2S_\beta)^{-1}\}$.

$u \in X_1$ is a weak solution of (1.7.21)-(1.7.24), if for any $v \in X_2$, we have

$$\int_\Omega [\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x, \bigwedge u)D^\alpha u D^\beta v + \sum_{|\alpha|=m, |\gamma|=m-1} b_{\alpha\gamma}(x) D^\gamma u D^\alpha v$$

$$+ \sum_{|\gamma|=|\theta|=m-1} d_{\gamma\theta}(x, \bigwedge u)D^\theta u D^\gamma v + \sum_{|\lambda|\le m-1} g_\lambda(x, \bigwedge u) D^\lambda v - fv]dx \tag{1.7.31}$$

$$- \sum_{i=1}^{N_{m-1}} \int_{\sum_i^C} h_i(x)(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} u)(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} v)ds = 0$$

Theorem 1.7.9. Under the condition $(A_1) - (A_5)$, if $f \in L^{p_0'}(\Omega)$, $(\frac{1}{p_0} + \frac{1}{p_0'} = 1)$, then the problem (1.7.21)-(1.7.24) has a weak solution in X_1.

Proof. Denote by $< Au, v >$ the left part of (1.7.31). It is easy to verify that the inner product $< Au, v >$ defines a bounded mapping $A : X_1 \to X_2^*$ by the condition (A_5).

Let $u \in X$, by $(A_2) - (A_4)$, one can deduce that

$$< Au, u > \ge \int_\Omega [M \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,0)D^\alpha u D^\beta u + C \sum_{|\gamma|=m-1} |D^\gamma u|^2$$

$$+C \sum_{|\theta|\le m-2} |D^\theta u|^{p_\theta}]dx + \frac{1}{2} \sum_{i=1}^{N_{m-1}} [\int_{\sum_i^B} - \int_{\sum_i^C} h_i(x) \tag{1.7.32}$$

$$\times(\sum_{j=1}^{N_{m-1}} C_{ij}^B(x) D^{\gamma^j} u)^2 ds] - \int_\Omega [f \cdot u + |f_1|] dx$$

Noticing $h_i|_{\sum_i^B} > 0, h_i|_{\sum_i^C} \le 0, \sum_i^B \cup \sum_i^C = \partial\Omega$, by Holder and Young inequalities, from (1.6.32) we can get

$$< Au, u > \ge 0, \quad \forall u \in X, \|u\|_{X_1} \textit{ large enough.}$$

By using the same method as in Theorem 1.5.1 and Theorem 1.6.3, we can prove that the mapping $A : X_1 \to X_2^*$ is weakly continuous. By Theorem 1.3.2, this theorem is proven.

In the following, we take an example to illustrate the application of Theorem 1.7.9.

Example 1.7.10. We consider the boundary value problem of odd order equation as follows

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} - \triangle u + u^3 = f(x, y), \quad (x, y) \in \Omega \subset R^2 \tag{1.7.33}$$

where Ω is an unit ball in R^2, see Fig. 1.3 below

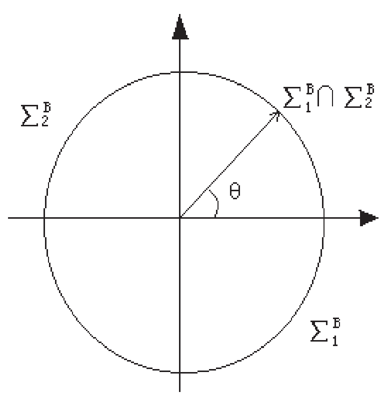

Fig. 1.3

The odd term matrix is

$$B(x, y) = \begin{pmatrix} n_x & 0 \\ 0 & n_y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

It is easy to see that
$\sum_1^B = \{x \in \partial\Omega | n_x = x > 0\} = \{-\frac{\pi}{2} < \theta < \frac{\pi}{2}\}$
$\sum_2^B = \{x \in \partial\Omega | n_y = y > 0\} = \{0 < \theta < \pi\}$.
The boundary value condition associated with (1.7.33) is

$$u|_{\partial\Omega} = 0 \tag{1.7.34}$$

$$\frac{\partial u}{\partial x}|_{\sum_1^B} = \frac{\partial u}{\partial x}(cos\theta, sin\theta) = 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \tag{1.7.35}$$

$$\frac{\partial u}{\partial y}|_{\sum_2^B} = \frac{\partial u}{\partial y}(cos\theta, sin\theta) = 0, \quad 0 < \theta < \pi. \tag{1.7.36}$$

Applying Theorem 1.7.9, if $f \in L^{4/3}(\Omega)$, then the problem (1.7.33)-(1.7.36) has a weak solution $u \in W^{1,2}(\Omega)$.

1.7.3. $W^{m,p}$-solutions of degenerate elliptic equations

In the following, we give some existence theorems of $W^{m,p}$-solutions for the boundary value condition (1.7.22)-(1.7.24) of higher order degenerate elliptic equations.

First we consider the quasilinear equations

$$\sum_{|\alpha|=|\beta|=m, |\gamma|=m-1} (-1)^m D^\alpha(a_{\alpha\beta}(x, Du)D^\beta u + b_{\alpha\gamma}(x)D^\gamma u) + \sum_{|\gamma|\le m-1} (-1)^{|\gamma|} D^\gamma g_\gamma(x, Du) = f(x), \quad x \in \Omega. \tag{1.7.37}$$

where $Du = \{D^\alpha u\}_{|\alpha|\le m-1}$. The boundary value condition associated with the equation (1.7.37) is given by (1.7.22)-(1.7.24). Suppose that $\Omega \subset R^n$ is bounded, and

(B_1) The condition (1.7.25) holds, and there is a continuous function $\lambda(x) \ge 0$ on $\overline{\Omega}$ such that

$$\lambda(x)|\xi|^{2m} \le \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x, 0)\xi^\alpha\xi^\beta, \quad \forall \xi \in R^n$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}, \alpha = (\alpha_1, \cdots, \alpha_n)$.

(B_2).$\Omega' = \{x \in \Omega | \lambda(x) = 0\}$ is a measure zero set in R^n, and there is a sequence of subdomains Ω_k with cone property such that $\Omega_k \subset\subset \Omega/\Omega', \Omega_k \subset \Omega_{k+1}$

and $\cup_k \Omega_k = \Omega/\Omega'$.
(B_3) The positive definite condition

$$C \sum_{|\lambda|\leq m-1} |\xi_\lambda|^{p_\lambda} - f_1 \leq \sum_{|\theta|\leq m-1} g_\theta(x,\xi)\xi_\theta - \frac{1}{2}\sum_{i=1}^{n} \sum_{|\gamma|=|\alpha|=m-1} D_i b^i \xi_\alpha \xi_\gamma$$

where $p_0 > 1, p_\lambda > 1$ or $= 0$ for $1 \leq |\lambda| \leq m-1$; $f_1 \in L^1(\Omega)$
(B_4). The structural conditions

$$|a_{\alpha\beta}(x,\xi)| \leq c$$

$$|g_\gamma(x,\xi)| \leq c[\sum_{|\theta|\leq m-1} |\xi_\theta|^{S_\theta} + 1]$$

where $0 \leq S_\theta < q_\theta, q_\theta$ is the critical embedding exponent from $W^{p_\lambda}_{|\lambda|\leq m-1}(\Omega)$ to $L^p(\Omega)$.
Let X be define by (1.7.10) and $\widetilde{X}_1$ be the completion of X with the norm

$$\|u\| = [\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,0)D^\alpha u D^\beta u dx]^{\frac{1}{2}} + \sum_{|\alpha|\leq m-1} sign p_\alpha \|D^\alpha u\|_{L^{p_\alpha}}$$

$$+[\sum_{i=1}^{N_{m-1}} \int_{\partial\Omega} |h_i(x)|(\sum_{j=1}^{N_{m-1}} C^B_{ij}(x) D^{\gamma^j} u) ds]^{\frac{1}{2}}$$

Theorem 1.7.11. Under the assumptions $(B_1)-(B_4)$, if $f \in L^{p_0'}(\Omega)$, then the problem (1.7.37)(1.7.22)-(1.7.24) has a weak solution $u \in \widetilde{X}_1$. Moreover, if there is a $\delta \geq 1$ such that

$$\int_\Omega |\lambda(x)|^{-\delta} dx < \infty$$

then the weak solution $u \in W^{m,p}(\Omega) \cap \widetilde{X}_1, p = \frac{2\delta}{1+\delta}$.

The proof of Theorem 1.7.11 is parallel to that of Theorem 1.6.15, here we omit the details.

Next, we consider the quasilinear equation

$$\sum_{|\alpha|=|\beta|=m,|\gamma|=m-1} (-1)^m D^\alpha(a_{\alpha\beta}(x,Du)D^\beta u + b_{\alpha\beta}(x)D^\gamma u) \tag{1.7.38}$$

$$+ \sum_{|\gamma|\leq m-1} (-1)^{|\gamma|} D^\gamma g_\gamma(x, \Box u) = f(x), \quad x \in \Omega$$

where $\square u = \{u, \cdots, D^m u\}$.

Suppose that

(B_4') There is a $\delta \geq 1$ such that

$$\int_\Omega |\lambda(x)|^{-\delta} dx < \infty.$$

(B_5') The structural conditions

$$|a_{\alpha\beta}(x, \eta)| \leq c$$

$$|g_\gamma(x, \xi)| \leq c[\sum_{|\theta| \leq m-1} |\xi_\theta|^{S_{\gamma\theta}} + \sum_{|\alpha|=m} |\xi_\alpha|^{t_\gamma} + 1]$$

where $0 \leq S_{\gamma\theta} < \frac{q_\gamma - 1}{q_\gamma} q_\theta, 0 \leq t_\gamma < p(q_\gamma - 1)/q_\gamma, p = 2\delta/1 + \delta, q_\gamma, q_\theta$ are the critical embedding exponents from $W^{p_\lambda}_{|\lambda| \leq m-1}(\Omega)$ to $L^q(\Omega)$.

Theorem 1.7.12. Let the conditions $(B_1) - (B_3)$ and $(B_4')(B_5')$ be satisfied. If $f \in L^{p_0'}(\Omega)$, then the problem (1.7.38)(1.7.22)-(1.7.24) has a weak solution $u \in W^{m,p}(\Omega) cap \widetilde{X}_1, p = 2\delta/(1 + \delta)$.

The proof of Theorem 1.7.12 is parallel to that of Theorem 1.6.17.