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A Characterization Of 0-Completeness In Dislocated b-Metric Spaces And Its Application In Fixed Point Theory*

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Abstract

The main aim of this paper is to introduce the concept of σ_b-point in a dislocated b-metric space and extend Weston's characterization of metric completeness to dislocated b-metric spaces in terms of σ_b-point. We use this new characterization to obtain some fixed point results including the celebrated Banach Contraction Principle in the framework of 0-complete dislocated metric spaces.

1 Introduction

In 1977, J. D. Weston [17] had characterized metric completeness in terms of the notion of d-point for lower semicontinuous functions. After that, several authors successfully characterized metric completeness in terms of fixed point theory (see [8, 9, 10, 11, 12, 15, 16]). In recent investigations, there exist a lot of generalizations of the concept of metric spaces such as b-metric space, introduced by Bakhtin [2], partial metric space by Matthews [7], and dislocated metric space by Hitzler et al. [5]. Combining the notions of b-metric and dislocated metric, Alghamdi et al. [1] introduced another generalization which is called a dislocated b-metric. They established some fixed point results in dislocated b-metric spaces. In this study, our main purpose is to introduce the concept of σ_b-point in dislocated b-metric spaces and extend Weston's characterization [17] in such spaces in terms of σ_b-point. Finally, we apply this new characterization to obtain some important fixed point results in 0-complete dislocated metric spaces.

2 Some Basic Concepts

This section begins with some definitions, basic facts and properties which will be needed in the sequel.

Definition 1 ([2, 4]) *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

It is worth noting that the class of b-metric spaces is effectively larger than that of the ordinary metric spaces.

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Definition 2 ([14]) *A partial b-metric on a nonempty set X is a function $p_b : X \times X \to [0, \infty)$ such that for some real number $s \geq 1$ and all x, y, $z \in X$:*

(p_{b1}) $p_b(x, x) = p_b(y, y) = p_b(x, y) \Longleftrightarrow x = y$;

(p_{b2}) $p_b(x, x) \leq p_b(x, y)$;

(p_{b3}) $p_b(x, y) = p_b(y, x)$;

(p_{b4}) $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y)] - p_b(z, z)$.

The pair (X, p_b) is called a partial b-metric space.

Taking $s = 1$ in the above definition, we get the definition of a partial metric space [7]. It is obvious that if $p_b(x, y) = 0$, then from (p_{b1}) and (p_{b2}), it follows that $x = y$. However, $x = y$ does not imply $p_b(x, y) = 0$.

Definition 3 ([5]) *Let X be a nonempty set. A function $\sigma : X \times X \to [0, \infty)$ is said to be a dislocated metric (or a metric-like) on X if for any x, y, $z \in X$, the following conditions hold:*

(σ_1) $\sigma(x, y) = 0 \Longrightarrow x = y$;

(σ_2) $\sigma(x, y) = \sigma(y, x)$;

(σ_3) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is then called a dislocated metric (or metric-like) space.

It is valuable to note that a partial metric is also a dislocated metric but the converse is not true, in general. The following example supports the above remark.

Example 1 ([13]) *Let $X = [0, 1]$. Then the mapping $\sigma : X \times X \to [0, \infty)$ defined by $\sigma(x, y) = x + y - xy$ is a dislocated metric on X. But σ is not a partial metric on X since $p(\frac{1}{2}, \frac{1}{2}) = \frac{3}{4} \nleq p(\frac{1}{2}, \frac{1}{4}) = \frac{5}{8}$.*

Definition 4 ([1]) *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $\sigma_b : X \times X \to [0, \infty)$ is said to be a dislocated b-metric (or a b-metric-like) on X if the following conditions hold:*

(*i*) $\sigma_b(x, y) = 0 \Longrightarrow x = y$;

(*ii*) $\sigma_b(x, y) = \sigma_b(y, x)$ *for all* x, $y \in X$;

(*iii*) $\sigma_b(x, y) \leq s\,(\sigma_b(x, z) + \sigma_b(z, y))$ *for all* x, y, $z \in X$.

The pair (X, σ_b) is then called a dislocated b-metric (or a b-metric-like) space.

It is clear that every partial b-metric space is a dislocated b-metric space with the same coefficient s and every b-metric space is also a dislocated b-metric space with the same coefficient s. However, the reverse implications need not hold true, in general.

Example 2 ([6]) *Let $X = [0, \infty)$, $p > 1$ a constant, and $\sigma_b : X \times X \to [0, \infty)$ be defined by*

$$\sigma_b(x, y) = (x + y)^p, \ \forall x,\, y \in X.$$

Then (X, σ_b) is a dislocated b-metric space with coefficient $s = 2^{p-1}$, but it is neither a partial b-metric space nor a b-metric space. Indeed, for any $0 < y < x$, we have

$$0 \neq \sigma_b(x, x) = (x + x)^p > (x + y)^p = \sigma_b(x, y).$$

Example 3 ([6]) *Let (X, σ) be a dislocated metric space and $\sigma_b(x, y) = (\sigma(x, y))^p$, where $p > 1$ is a real number. Then σ_b is a dislocated b-metric with coefficient $s = 2^{p-1}$.*

Definition 5 ([1]) *Let (X, σ_b) be a dislocated b-metric space with coefficient s, and let (x_n) be a sequence in X and $x \in X$. Then*

- (i) *(x_n) converges to x if and only if $\lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x)$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.*
- (ii) *(x_n) is Cauchy if $\lim_{n,m\to\infty} \sigma_b(x_n, x_m)$ exists and is finite.*
- (iii) *(X, σ_b) is said to be a complete dislocated b-metric space if for every Cauchy sequence (x_n) in X, there exists $x \in X$ such that*
$$\lim_{n,m\to\infty} \sigma_b(x_n, x_m) = \lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x).$$

Definition 6 *A sequence (x_n) in a dislocated b-metric space (X, σ_b) is called 0-Cauchy if*

$$\lim_{n,m\to\infty} \sigma_b(x_n, x_m) = 0.$$

The space (X, σ_b) is said to be a 0-complete dislocated b-metric space if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $\sigma_b(x, x) = 0$, i.e.,

$$\lim_{n,m\to\infty} \sigma_b(x_n, x_m) = \lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x) = 0.$$

Remark 1 *The definition of a 0-complete dislocated metric space (X, σ) can be obtained from the above definition by taking $s = 1$.*

Remark 2 *If (X, σ_b) is complete, then it is 0-complete.*

The converse assertion of the above remark may not hold good, in general. The following example supports the above remark.

Example 4 *The space $X = [0, \infty) \cap \mathbb{Q}$ with $\sigma_b(x, y) = \max\{x, y\}$ is a 0-complete dislocated b-metric space with coefficient $s = 1$, but it is not complete. Moreover, the sequence (x_n) with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X, σ_b), but it is not a 0-Cauchy sequence.*

3 A Characterization of 0-Completeness

Definition 7 *Let (X, σ_b) be a dislocated b-metric space. A function $\varphi : X \to \mathbb{R}$ is called lower semicontinuous if, for each sequence $(x_n) \subseteq X$ converges to a point $x \in X$ with $\sigma_b(x, x) = 0$, we have*

$$\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n).$$

Definition 8 *Let (X, σ_b) be a dislocated b-metric space with coefficient $s \geq 1$ and $h : X \to \mathbb{R}$ be a function. A point $x_0 \in X$ is called a σ_b-point for h if for every point $x \in X$ other than x_0,*

$$h(x_0) - h(x) < \frac{1}{s}\sigma_b(x_0, x).$$

Taking $s = 1$ in the above definition, we get σ-point in dislocated metric space (X, σ). In case of metric spaces (X, d), the above σ_b-point reduces to d-point.

Example 5 *Let $X = [0, \infty)$ and let $\sigma_b(x, y) = (x + y)^2$, $\forall\, x, y \in X$. Then (X, σ_b) is a dislocated b-metric space with coefficient $s = 2$. Let $h : X \to \mathbb{R}$ be defined by $h(x) = x^2$ for all $x \in X$. Then, $h(0) - h(x) = -x^2 < \frac{1}{2}\sigma_b(0, x)$ for every $x \in X$ with $x \neq 0$. Therefore, 0 is a σ_b-point for h. We now consider another mapping $g : X \to \mathbb{R}$ defined by $g(x) = \frac{x^2}{5}$ for all $x \in X$. Then it is easy to verify that every point of X is a σ_b-point for g.*

Theorem 1 *If the dislocated b-metric space (X, σ_b) with coefficient $s \geq 1$ is 0-complete then any lower semicontinuous function $h : X \to \mathbb{R}$ which is bounded below has a σ_b-point.*

Proof. For any point $x_1 \in X$, we can construct a sequence (x_n) in the following way:

For each $n \in \mathbb{N}$, let

$$c_n = \inf\{h(x) : h(x_n) - h(x) \geq \frac{1}{s^2}\sigma_b(x_n, x),\ x_n \neq x\}$$

and let x_{n+1} be a point such that

$$h(x_n) - h(x_{n+1}) \geq s^{n+1}\sigma_b(x_n, x_{n+1}) \tag{1}$$

and

$$h(x_{n+1}) < c_n + n^{-1}. \tag{2}$$

We now clarify that if the set $\{h(x) : h(x_n) - h(x) \geq \frac{1}{s^2}\sigma_b(x_n, x),\ x_n \neq x\}$ is empty, then for every $x \in X$ other than x_n,

$$h(x_n) - h(x) < \frac{1}{s^2}\sigma_b(x_n, x) \leq \frac{1}{s}\sigma_b(x_n, x).$$

So, in this case x_n becomes a σ_b-point for h and the theorem is proved. Therefore, we assume that the set

$$\{h(x) : h(x_n) - h(x) \geq \frac{1}{s^2}\sigma_b(x_n, x),\ x_n \neq x\}$$

is nonempty.

In other words, in above construction, we have considered none of x_n as a σ_b-point for h. Because, if x_n is a σ_b-point for h, then we have nothing to prove. It follows from condition (1) that the sequence $(h(x_n))$ is nonincreasing in $\mathbb{R}$. Also, it is bounded below by assumed hypothesis. So, the sequence $(h(x_n))$ is convergent and hence it is Cauchy.

For $m \geq n$, we have

$$\begin{aligned} h(x_n) - h(x_m) &= h(x_n) - h(x_{n+1}) + h(x_{n+1}) - h(x_{n+2}) \\ &\quad + \cdots + h(x_{m-2}) - h(x_{m-1}) + h(x_{m-1}) - h(x_m) \\ &\geq s^n[s\sigma_b(x_n, x_{n+1}) + s^2\sigma_b(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + s^{m-n-1}\sigma_b(x_{m-2}, x_{m-1}) + s^{m-n}\sigma_b(x_{m-1}, x_m)] \\ &\geq s\sigma_b(x_n, x_{n+1}) + s^2\sigma_b(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + s^{m-n-1}\sigma_b(x_{m-2}, x_{m-1}) + s^{m-n-1}\sigma_b(x_{m-1}, x_m) \\ &\geq \sigma_b(x_n, x_m). \end{aligned} \tag{3}$$

Hence,

$$\sigma_b(x_n, x_m) \leq h(x_n) - h(x_m) \to 0 \text{ as } m,\, n \to \infty.$$

This proves that the sequence (x_n) is 0-Cauchy in (X, σ_b). By 0-completeness of (X, σ_b), it follows that the sequence (x_n) converges to a point $x_0 \in X$ such that $\sigma_b(x_0, x_0) = 0$. Thus, $\sigma_b(x_n, x_0) \to \sigma_b(x_0, x_0) = 0$. We now compute that for each $y \in X$,

$$\sigma_b(x_0, y) \leq s[\sigma_b(x_0, x_n) + \sigma_b(x_n, y)].$$

This implies that

$$\sigma_b(x_0, y) \leq s \limsup_{n\to\infty}[\sigma_b(x_0, x_n) + \sigma_b(x_n, y)] = s \limsup_{n\to\infty} \sigma_b(x_n, y).$$

Therefore, for each $y \in X$, we have

$$\limsup_{n\to\infty} \sigma_b(x_n, y) \geq \frac{1}{s}\sigma_b(x_0, y). \tag{4}$$

From condition (3), it follows that

$$h(x_m) \leq h(x_n) - \sigma_b(x_n, x_m) \tag{5}$$

for all $m \geq n$. By using conditions (4), (5) and lower semicontinuity of the function h, one can obtain that

$$\begin{aligned} h(x_0) &\leq \liminf_{m\to\infty} h(x_m) \\ &\leq \liminf_{m\to\infty} [h(x_n) - \sigma_b(x_n, x_m)] \\ &= h(x_n) - \limsup_{m\to\infty} \sigma_b(x_n, x_m) \\ &\leq h(x_n) - \frac{1}{s}\sigma_b(x_n, x_0) \end{aligned}$$

for all $n \geq 1$. Thus,

$$h(x_n) - h(x_0) \geq \frac{1}{s}\sigma_b(x_n, x_0) \text{ for all } n \geq 1. \tag{6}$$

If x_0 is not a σ_b-point for h, then for some $x(\neq x_0) \in X$, we have

$$h(x_0) - h(x) \geq \frac{1}{s}\sigma_b(x_0, x) > 0. \tag{7}$$

Using conditions (6) and (2), we obtain

$$h(x) \leq h(x_{n+1}) + h(x) - h(x_0) < c_n + n^{-1} + h(x) - h(x_0). \tag{8}$$

In view of condition (7), we can choose n in such a way that condition (8) ensures that $h(x) < c_n$.

From conditions (6) and (7), it follows that

$$\begin{aligned} h(x_n) - h(x) &= h(x_n) - h(x_0) + h(x_0) - h(x) \\ &\geq \frac{1}{s}[\sigma_b(x_n, x_0) + \sigma_b(x_0, x)] \\ &> 0, \end{aligned}$$

which implies that $h(x_n) > h(x)$. So, $x_n \neq x$. Moreover,

$$h(x_n) - h(x) \geq \frac{1}{s}[\sigma_b(x_n, x_0) + \sigma_b(x_0, x)] \geq \frac{1}{s^2}\sigma_b(x_n, x),\ x_n \neq x.$$

It now follows from the definition of c_n that $h(x) \geq c_n$, which contradicts the fact that $h(x) < c_n$. Thus, x_0 is a σ_b-point for h. ■

The following theorem is an immediate consequence of Theorem 1.

Theorem 2 *If the dislocated metric space (X, σ) is 0-complete then any lower semicontinuous function $h : X \to \mathbb{R}$ which is bounded below has a σ-point.*

Proof. The proof can be obtained from Theorem 1 by taking $s = 1$. ■

The following is the result of Weston [17].

Theorem 3 *If the metric space (X, d) is complete then any lower semicontinuous function $X \to \mathbb{R}$ which is bounded below has a d-point.*

Proof. The result follows from Theorem 1 by taking $\sigma_b = d$. ■

Remark 3 *It is worthy to mention that the main result is obtained under the weaker assumption that the given dislocated b-metric space is 0-complete. Moreover, the result remains valid in complete b-metric spaces and 0-complete partial b-metric spaces.*

4 An Application in Fixed Point Theory

In this section, we give an application of our main Theorem 1 in fixed point theory. We assume that (X, σ) is a dislocated metric space and $h : X \to \mathbb{R}$ is a function.

Remark 4 *When σ and h are given, a relation " $\ll$ " can be defined on X as follows:*

$$x \ll y \text{ if and only if } h(y) - h(x) \geq \sigma(x, y).$$

This relation orders X. In fact, " $\ll$ " is transitive, antisymmetric, but it is not reflexive.

Definition 9 *A point x_0 in (X, σ) is said to be a minimal point w.r.t. $\ll$ if and only if $x \ll x_0$ implies $x = x_0$.*

Theorem 4 *A point of X is a σ-point for h if and only if it is a minimal point w.r.t. $\ll$.*

Proof. Let $x_0 \in X$ be a σ-point for h. Then,

$$h(x_0) - h(x) < \sigma(x, x_0), \ \forall \ x \in X \text{ and } x \neq x_0. \tag{9}$$

Now $x \ll x_0$ implies that $h(x_0) - h(x) \geq \sigma(x, x_0)$. This gives that $x = x_0$. Because if $x \neq x_0$, then by condition (9) it follows that $h(x_0) - h(x) < \sigma(x, x_0)$, a contradiction. Therefore, x_0 is a minimal point w.r.t. $\ll$.

Conversely, let x_0 be a minimal point w.r.t. $\ll$. Then $x \ll x_0$ implies that $x = x_0$. That is, $x \ll x_0$ does not hold for all $x \in X$ with $x \neq x_0$. Therefore, $h(x_0) - h(x) < \sigma(x, x_0)$ for all $x \in X$ with $x \neq x_0$. This gives that x_0 is a σ-point for h. ■

Theorem 5 *If a function $f : X \to X$ is such that it may be possible to choose σ and h so that the relation $\ll$ has the property that $fx \neq x$ implies $fx \ll x$, then any σ-point for h is a fixed point for f.*

Proof. Let $x_0 \in X$ be a σ-point for h. Then,

$$h(x_0) - h(x) < \sigma(x, x_0), \ \forall \ x \in X \text{ and } x \neq x_0. \tag{10}$$

If $fx_0 \neq x_0$, then by hypothesis $fx_0 \ll x_0$ which implies that

$$h(x_0) - h(fx_0) \geq \sigma(fx_0, x_0),$$

which contradicts the condition (10). So, it must be the case that $fx_0 = x_0$. This shows that x_0 is a fixed point of f. ■

We now apply Theorems 2 and 5 to prove Banach Contraction Principle in 0-complete dislocated metric spaces.

Theorem 6 *Let (X, σ) be a 0-complete dislocated metric space and let $f : X \to X$ be a mapping satisfying the following condition:*

$$\sigma(fx, fy) \leq \alpha \, \sigma(x, y) \tag{11}$$

for all $x, y \in X$, where $0 \leq \alpha < 1$ is a constant. Then f has a unique fixed point u(say) in X with $\sigma(u, u) = 0$.

Proof. Let $h(x) = \beta\,\sigma(fx, x)$, where $\beta = \frac{1}{1-\alpha} > 0$ and $x \in X$. We first show that $h : X \to \mathbb{R}$ is a lower semicontinuous function. Let $y_n \to y$ in (X, σ) with $\sigma(y, y) = 0$. Then, $\lim_{n\to\infty} \sigma(y, y_n) = \sigma(y, y) = 0$. We have to show that

$$h(y) \leq \liminf_{n\to\infty} h(y_n).$$

By using condition (11), we have

$$\begin{aligned} h(y) &= \beta\,\sigma(fy, y) \leq \beta\,[\sigma(fy, y_n) + \sigma(y_n, y)] \\ &\leq \beta\,[\sigma(fy, fy_n) + \sigma(fy_n, y_n) + \sigma(y_n, y)] \\ &\leq \beta\,[\alpha\,\sigma(y, y_n) + \sigma(fy_n, y_n) + \sigma(y_n, y)] \\ &= \beta(\alpha + 1)\,\sigma(y, y_n) + h(y_n). \end{aligned}$$

This gives that,

$$h(y) \leq \liminf_{n\to\infty} h(y_n).$$

Thus, h is a lower semicontinuous function on a 0-complete dislocated metric space (X, σ) which is also bounded below. Therefore, Theorem 2 ensures the existence of a σ-point u(say) for h.

We now show that $fx \neq x$ implies $fx \ll x$. Let $fx \neq x$. By using condition (11), we obtain

$$\begin{aligned} h(x) - h(fx) &= \beta\,[\sigma(fx, x) - \sigma(f^2x, fx)] \\ &\geq \beta\,[\sigma(fx, x) - \alpha\,\sigma(fx, x)] \\ &= \beta\,(1 - \alpha)\,\sigma(fx, x)] \\ &= \sigma(fx, x). \end{aligned}$$

Thus f satisfies the condition that $fx \neq x$ implies $fx \ll x$. By applying Remark 5, it follows that the σ-point u for h is a fixed point for f in X. For uniqueness, let $v \in X$ be another fixed point of f. Then, by condition (11), we get

$$\sigma(u, v) = \sigma(fu, fv) \leq \alpha\,\sigma(u, v).$$

Since $0 \leq \alpha < 1$, it follows that $\sigma(u, v) = 0$ and hence $u = v$.

Moreover, $\sigma(u, u) = \sigma(fu, fu) \leq \alpha\,\sigma(u, u)$ gives that $\sigma(u, u) = 0$. ■

We now give an example in support of the above theorem.

Example 6 *Let $X = [0, 1]$ and $\sigma : X \times X \to [0, \infty)$ be defined by*

$$\sigma(x, y) = x + y,\ \forall x,\, y \in X.$$

Then (X, σ) is a 0-complete dislocated metric space. Let $f : X \to X$ be defined by

$$fx = \frac{x^2}{1 + x},\ \forall x \in X.$$

Then,

$$\begin{aligned} \sigma(fx, fy) &= fx + fy = \frac{x^2}{1 + x} + \frac{y^2}{1 + y} \\ &= \frac{x}{1 + x}x + \frac{y}{1 + y}y \\ &\leq \frac{1}{2}(x + y) \\ &= \frac{1}{2}\sigma(x, y) \end{aligned}$$

for all x, $y \in X$. Therefore, all the conditions of Theorem 6 hold good and 0 is the unique fixed point of f in X with $\sigma(0, 0) = 0$.

Remark 5 *It is valuable to note that the last result of this section is obtained under the weaker assumption that the given dislocated metric space is* 0*-complete.*

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References

[1] M. A. Alghamdi, N. Hussain and P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, J. Ineqal. Appl., 402(2013), 1–25.

[2] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, Funct. Anal. Gos. Ped. Inst. Unianowsk, 30(1989), 26–37.

[3] M. Bukatin, R. Kopperman, S. Matthews and H. Pajoohesh, Partial metric spaces, Am. Math. Mon., 116(2009), 708–718.

[4] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav, 1(1993), 5–11.

[5] P. Hitzler and A. K. Seda, Dislocated topologies, J. Electr. Eng., 51(2000), 3–7.

[6] N. Hussain, J. R. Roshan, V. Parvaneh and Z. Kadelburg, Fixed points of contractive mappings in b-metric-like spaces, The Scientific World Journal, 2014(2014), 15 pages.

[7] S. Matthews, Partial metric topology, Ann. N. Y. Acad. Sci., 728(1994), 183–197.

[8] S. K. Mohanta, A fixed point theorem via generalized w-distance, Bull. Math. Anal. Appl., 3(2011), 134–139.

[9] S. K. Mohanta, Generalized w-distance and a fixed point theorem, Int. J. Contemp. Math. Sciences, 6(2011), 853–860.

[10] S. K. Mohanta and R. Maitra, A characterization of completeness in cone metric spaces, J. Nonlinear Anal. Appl., 6(2013), 227–233.

[11] H. K. Nashine and Z. Kadelburg, Cyclic contractions and fixed point results via control functions on partial metric spaces, International J. Anal., 2013(2013).

[12] S. Park, Characterizations of metric completeness, Colloq. Math., 49(1984), 21–26.

[13] N. Shobkolaei, S. Sedghi, J. R. Roshan and N. Hussain, Suzuki type fixed point results in metric-like spaces, J. Func. Spaces Appl., 2013(2013), 9 pages.

[14] S. Shukla, Partial b-metric spaces and fixed point theorems, Mediterr. J. Math., 11(2014), 703–711.

[15] T. Suzuki and W. Takahashi, Fixed point theorems and characterizations of metric completeness, Topolo. Method. Nonlinear Anal., 8(1996), 371–382.

[16] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc., 136(2008), 1861–1869.

[17] J. D. Weston, A characterization of metric completeness, Proc. Amer. Math. Soc., 64(1977), 186–188.