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Inequality Involving The p-Deformed Jacobi Polynomial*

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Abstract

We obtain an inequality involving the p-deformed Jacobi polynomial: $P_{n,p}^{(\alpha,\beta)}(x)$ with the help of the contour integration and the theory of equations. The inequalities for the Gegenbauer polynomial, the Legendre polynomial and the Chebyshev polynomial are illustrated as the special cases.

1 Introduction

Numerous scientific phenomena and numerical analysis use the Jacobi polynomial. On the real half-line, it is employed to obtain the approximate solution of the characteristic equation with the Cauchy kernel [11, Eq. (1.1), page 326]:

$$a\phi(x) + \frac{1}{\pi i}\int_0^{+\infty} \frac{b\phi(\sigma)}{\sigma - x}\, d\sigma = f(x), \quad x \in (0, +\infty),$$

where a, b are given complex numbers with $a^2 - b^2 \neq 0$, $b \neq 0$, $f(x)$ is the given complex valued Hölder continuous function and $\phi(x)$ is unknown function. The Jacobi polynomial also solves the differential equation of heat conduction in non-homogeneous moving rectangular parallelepiped [10]. The singular differential equations are numerically solved by Jacobi-Gauss type interpolations [1]. Moreover, this polynomial is involved in achieving the coupled system's numerical solution for fractional differential equations [9]. In 2007, Rafael Díaz and Eddy Pariguan [6] introduced the function (Euler integral form):

$$\Gamma_p(z) = \int_0^{\infty} e^{-t^p/p}\, t^{z-1}\, dt,$$

where $z \in \mathbb{C}$, $\Re(z) > 0$, $p > 0$. We note that the special case: $\Gamma_2(x)$ represents the Gaussian integral [6, p. 183]. In fact, the occurrence of the product of the form $x(x + p)(x + 2p) \cdots (x + (n - 1)p)$ in combinatorics of creation and annihilation operators [4], [5] and the perturbative computation of Feynman integrals [3] led Rafael Diaz et al. to generalize the gamma function in the above form. Diaz at el.[6] defined the Pochhammer p-symbol for $z \in \mathbb{C}$, $p \in \mathbb{R}$ and $n \in \mathbb{N}$, which is given by

$$(z)_{n,p} = z(z + p)(z + 2p) \cdots (z + (n - 1)p).$$

The following are some basic formulas follow from the above p-Gamma function and the p-Pochhammer symbol [6, Prop. 6, p. 183].

$$\Gamma_p(z + p) = z\, \Gamma_p(z),\ (z)_{n,p} = \Gamma_p(z + np)/\Gamma_p(z),\ \Gamma_p(p) = 1.$$

In [8], it is shown that for $\alpha, \beta \in (-1, \infty)$, $p \in (0, \infty)$ and $n \in \mathbb{N} \cup \{0\}$,

$$\int_{-1}^{1} (1 - x)^{\frac{\alpha+1}{p}-1}(1 + x)^{\frac{\beta+1}{p}-1} P_{n,p}^{(\alpha,\beta)}(x) P_{m,p}^{(\alpha,\beta)}(x)\, dx$$
$$= \frac{p^{n+1} 2^{\frac{\alpha+\beta+2}{p}-1}\Gamma_p(\alpha + np + 1)\Gamma_p(\beta + np + 1)}{n!(\alpha + \beta + 2 + (2n - 1)p)\Gamma_p(\alpha + \beta + 2 + np - p)}\ \delta_{mn}, \tag{1}$$

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wherein $P_{n,p}^{(\alpha,\beta)}(x)$ is the p-deformed Jacobi polynomial defined explicitly by [8]

$$P_{n,p}^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_{n,p}}{n!} \sum_{k=0}^{n} \frac{(-n)_k(\alpha+\beta+2+np-p)_{k,p}}{(1+\alpha)_{k,p}k!} \left(\frac{1-x}{2}\right)^k. \tag{2}$$

The orthogonal polynomials are known to satisfy specific inequalities. For instance, the Jacobi polynomial fulfills the inequality that appeared in [7], and the Legendre polynomial satisfies the inequalities derived in [12, Theorem 60 and 61, p.172-173].

Here our objective is to investigate the inequality relation for the p-Jacobi polynomial $P_{n,p}^{(\alpha,\beta)}(x)$ as stated in (2).

2 Inequality

We establish the inequality in the following

Theorem 1 *For $\alpha, \beta \in [0,\infty)$, $p \in [1,\infty)$, $n \in \mathbb{N}$ and $x \in (-1,1)$,*

$$\left|P_{n,p}^{(\alpha,\beta)}(x)\right| \leq \left[\frac{2^{\gamma+\delta-2np}p^{np}\gamma^{\gamma}\ \delta^{\delta}}{n^{np}(\gamma+\delta-np)^{\gamma+\delta-np}}\right]^{1/2p} (1-x)^{\frac{np-\gamma}{2p}}(1+x)^{\frac{np-\delta}{2p}},$$

where $\gamma = 1+\alpha+(n-1)p$ and $\delta = 1+\beta+(n-1)p$.

Proof. The Rodrigues formula for the polynomial (2) is [8]

$$(1-x)^{\frac{\alpha+1}{p}-1}(1+x)^{\frac{\beta+1}{p}-1}P_{n,p}^{(\alpha,\beta)}(x) = \left(-\frac{p}{2}\right)^n \frac{(-1)^n}{n!} D^n\left[(1-x)^{\frac{\alpha+1}{p}+(n-1)}(1+x)^{\frac{\beta+1}{p}+(n-1)}\right],$$

where $D = d/dx$. In view of the contour integral formula for the n-th derivative of an analytic function [2, Eq. (5), p. 167], we have

$$\begin{aligned} n!\left(\frac{2}{p}\right)^n (1-x)^{\frac{\alpha+1}{p}-1}(1+x)^{\frac{\beta+1}{p}-1}P_{n,p}^{(\alpha,\beta)}(x) &= D^n\left[(1-x)^{\frac{\alpha+1}{p}+(n-1)}(1+x)^{\frac{\beta+1}{p}+(n-1)}\right] \\ &= \frac{n!}{2\pi i}\int_C \frac{(1-z)^{\frac{\alpha+1}{p}+(n-1)}(1+z)^{\frac{\beta+1}{p}+(n-1)}}{(z-x)^{n+1}}\,dz, \end{aligned}$$

where C is any positively oriented simple closed contour enclosing the point x in the z-plane. Hence,

$$\left(\frac{2}{p}\right)^n (1-x)^{\frac{\alpha+1}{p}-1}(1+x)^{\frac{\beta+1}{p}-1}P_{n,p}^{(\alpha,\beta)}(x) = \frac{1}{2\pi i}\int_C \frac{(1-z)^{\frac{\alpha+1}{p}+(n-1)}(1+z)^{\frac{\beta+1}{p}+(n-1)}}{(z-x)^{n+1}}dz = I_{n,p}^{(\alpha,\beta)}(x), \tag{3}$$

say. The circle $C(x;r)$, interior to C, with its center at point x and radius r, can be used to substitute the closed contour C using the principle deformation of path [2, Corollary, p. 159]; in particular, we choose $C(0;r) = \mathfrak{C}$.

We will make use of the following substitutes in the present work.

$$\frac{\alpha-p+1}{n} = a \quad \text{and} \quad \frac{\beta-p+1}{n} = b. \tag{4}$$

Using these in (3), and putting $z = x+s$, we get

$$I_{n,p}^{(\alpha,\beta)}(x) = \frac{1}{2\pi i}\int_{\mathfrak{C}} \left(\frac{(1-x-s)^{\frac{a}{p}+1}(1+x+s)^{\frac{b}{p}+1}}{s}\right)^n \frac{ds}{s}.$$

For determining the radius r of the circle C, we take the logarithm of the integrand which is the function of s, given by

$$\left(\frac{a+p}{p}\right)\log(1-x-s)+\left(\frac{b+p}{p}\right)\log(1+x+s)-\log s=\xi(s),$$

say, then

$$\xi^{'}(s)=-\frac{a+p}{p(1-x-s)}+\frac{b+p}{p(1+x+s)}-\frac{1}{s}.$$

Now if

$$\frac{(a+p)}{p(1-x-s)}-\frac{(b+p)}{p(1+x+s)}+\frac{1}{s}=\frac{As^2+Bs+C}{s(1+x+s)(1-x-s)},$$

then it follows that

$$A\equiv A(x)=\frac{a+b+p}{p},\ B\equiv B(x)=\frac{(a+b)x}{p}+\frac{(a-b)}{p},\ C\equiv C(x)=1-x^2.$$

Since $A(x)s^2+B(x)s+C(x)$ is a quadratic polynomial in s, we have

$$\begin{aligned}\Delta(x) &= [B(x)]^2-4A(x)C(x)\\ &= \left(\frac{a+b+2p}{p}\right)^2x^2+\frac{2(a^2-b^2)}{p^2}x+\frac{(a-b)^2}{p^2}-\frac{4(a+b+p)}{p},\end{aligned}$$

which is a polynomial in x having two roots x^+ and x^-, say, where

$$x^{\pm}=\frac{-\frac{2(a^2-b^2)}{p^2}\pm\sqrt{\frac{4(a^2-b^2)^2}{p^4}-4\left(\frac{a+b+2p}{p}\right)^2\left(\frac{(a-b)^2}{p^2}-\frac{4(a+b+p)}{p}\right)}}{2\left(\frac{a+b+2p}{p}\right)^2}.$$

Thus, $\Delta(x)=(x-x^-)(x-x^+)$ and since $x\in(-1,1)$, $-1<x^-<x^+<1$. But then $\Delta(x)<0,\forall\, x\in(x^-,x^+)$. This indicates that the equation: $A(x)s^2+B(x)s+C(x)=0$ has two conjugate solutions $s_1,\bar{s_1}=s_2$, say.

We note from the relation between roots and coefficients, that

$$|s_1|^2=|s_2|^2=s_1s_2=\frac{C(x)}{A(x)}=\frac{p(1-x^2)}{a+b+p}.$$

Hence, suggested by this, we choose the radius

$$r=\sqrt{\frac{p(1-x^2)}{a+b+p}}. \tag{5}$$

Now, with $s=re^{i\theta}$,

$$\left|I_{n,p}^{(\alpha,\beta)}(x)\right|\le\frac{1}{2\pi}\int_0^{2\pi}(|(1-x-re^{i\theta})^{1+\frac{a}{p}}(1+x+re^{i\theta})^{1+\frac{b}{p}}r^{-1}|)^n\ d\theta.$$

Here the expression:

$$|(1-x-re^{i\theta})^{1+\frac{a}{p}}(1+x+re^{i\theta})^{1+\frac{b}{p}}r^{-1}|,$$

with $e^{i\theta}=\cos\theta+i\sin\theta$, transforms to

$$\left[(r^2+(1-x)^2-2r(1-x)\cos\theta)\right]^{\frac{a+p}{2p}}\left[(r^2+(1+x)^2+2r(1+x)\cos\theta)\right]^{\frac{b+p}{2p}}r^{-1}. \tag{6}$$

Next, we put

$$\left|(1-x-re^{i\theta})^{1+\frac{a}{p}}(1+x+re^{i\theta})^{1+\frac{b}{p}}r^{-1}\right|=e^{\Phi(cos\theta)}. \tag{7}$$

Then taking logarithm and putting $\cos\theta = t$, we get

$$\frac{a+p}{2p}\ \ln(r^2+(1-x)^2-2r(1-x)t)+\frac{b+p}{2p}\ \ln(r^2+(1+x)^2+2r(1+x)t)-\ln(r)=\Phi(t),$$

where $t\in(-1,1)$. Let

$$t_1=-\frac{r^2+(1+x)^2}{2r(1+x)},\quad t_2=\frac{r^2+(1-x)^2}{2r(1-x)}. \tag{8}$$

Then since $[r-(1\pm x)]^2\geq 0$, it follows that $t_1\leq -1$ and $1\leq t_2$. Hence,

$$\Phi(t)=\frac{a+p}{2p}\ \ln(t_2-t)+\frac{b+p}{2p}\ \ln(t-t_1)+k, \tag{9}$$

where

$$k=\frac{a+p}{2p}\ \ln(1-x)+\frac{b+p}{2p}\ \ln(1+x)+\frac{a+b+2p}{2p}\ \ln(2)+\frac{a+b}{2p}\ \ln r, \tag{10}$$

which is independent of t. Using (5) in (8), we have

$$t_1=\frac{-(a+b+2p)-(a+b)x}{2\sqrt{p(a+b+p)(1-x^2)}},\quad t_2=\frac{(a+b+2p)-(a+b)x}{2\sqrt{p(a+b+p)(1-x^2)}},$$

and

$$t_2-t_1=\frac{(a+b+2p)}{\sqrt{p(a+b+p)(1-x^2)}}. \tag{11}$$

From (9),

$$\Phi^{'}(t)=-\frac{a+p}{2p(t_2-t)}+\frac{b+p}{2p(t-t_1)}=\frac{(a+b+2p)(t_0-t)}{2p(t-t_1)(t_2-t)},$$

where

$$t_0=\frac{(a+p)t_1+(b+p)t_2}{(a+b+2p)}=\frac{-a+b-(a+b)x}{2\sqrt{p(a+b+p)(1-x^2)}}$$

is such that $t_1<t_0<t_2$. These inequalities are implied by the fact that $a+b+2p>\pm(a+b)$. Also, since $a+p>0,\ b+p>0$, we have

$$\Phi^{''}(t)=-\frac{a+p}{2p(t_2-t)^2}-\frac{b+p}{2p(t-t_1)^2}<0.$$

From this, it follows that $\Phi(t)$ is concave and since $\phi'(t_0)=0$, $\phi(t)$ has a global maximum at t_0. From (6) and (7), we thus obtain the inequality:

$$\left|I_{n,p}^{(\alpha,\beta)}(x)\right|\leq\frac{1}{2\pi}\int_0^{2\pi}e^{n\Phi(\cos\theta)}d\theta=\frac{1}{\pi}\int_0^{\pi}e^{n\Phi(\cos\theta)}d\theta\leq e^{n\Phi(t_0)}, \tag{12}$$

in view of the property:

$$f(x)=f(2a-x)\implies\int_0^{2a}f(x)\ dx=2\int_0^{a}f(2a-x)\ dx,$$

of definite integrals. Since,

$$t_2-t_0=\frac{(a+p)(t_2-t_1)}{a+b+2p}\ \text{ and }\ t_0-t_1=\frac{(b+p)(t_2-t_1)}{a+b+2p}, \tag{13}$$

we find from (9) and (13) that

$$\Phi(t_0) = \frac{a+p}{2p}\ \ln\left(\frac{(a+p)(t_2-t_1)}{a+b+2p}\right) + \frac{b+p}{2p}\ \ln\left(\frac{(b+p)(t_2-t_1)}{a+b+2p}\right) + k.$$

From (10) and (11), we further find that

$$\Phi(t_0) = \ln\left[\frac{2^{a+b+2p}(a+p)^{a+p}(b+p)^{b+p}(1-x)^a(1+x)^b}{p(a+b+p)^{a+b+p}}\right]^{\frac{1}{2p}}.$$

Hence,

$$\begin{aligned} e^{n\Phi(t_0)} &= \left[\frac{2^{a+b+2p}(a+p)^{a+p}(b+p)^{b+p}(1-x)^a(1+x)^b}{p^p(a+b+p)^{a+b+p}}\right]^{\frac{n}{2p}} \\ &= \left[\frac{2^{a+b+2p}(a+p)^{a+p}(b+p)^{b+p}}{p^p(a+b+p)^{a+b+p}}\right]^{\frac{n}{2p}} (1-x)^{\frac{an}{2p}}(1+x)^{\frac{bn}{2p}}. \end{aligned} \tag{14}$$

From (12) and (14), we obtain

$$\left|I_{n,p}^{\alpha,\beta}(x)\right| \le \left[\frac{2^{a+b+2p}(a+p)^{a+p}(b+p)^{b+p}}{p^p(a+b+p)^{a+b+p}}\right]^{\frac{n}{2p}} (1-x)^{\frac{an}{2p}}(1+x)^{\frac{bn}{2p}}.$$

In view of (3), we have

$$\begin{aligned} &\left(\frac{2}{p}\right)^n (1-x)^{\frac{\alpha+1}{p}-1}(1+x)^{\frac{\beta+1}{p}-1}\left|P_{n,p}^{(\alpha,\beta)}(x)\right| \\ \le\ & \left[\frac{2^{a+b+2p}(a+p)^{a+p}(b+p)^{b+p}}{p^p(a+b+p)^{a+b+p}}\right]^{n/2p} (1-x)^{an/2p}(1+x)^{bn/2p}. \end{aligned}$$

Further simplification using (4), leads us to the theorem. ∎

3 Particular Cases

For $p = 1$ and $n \in \mathbb{N}$, the inequality reduces to

$$\left|P_n^{(\alpha,\beta)}(x)\right| \le \left[2^{\alpha+\beta}\frac{(\alpha+n)^{\alpha+n}(\beta+n)^{\beta+n}}{n^n(\alpha+\beta+n)^{\alpha+\beta+n}}\right]^{1/2} (1-x)^{-\alpha/2}(1+x)^{-\beta/2}. \tag{15}$$

Here, the choice $\alpha = 0 = \beta$ yields the inequality for the Legendre polynomial [12, Theorem 60, p. 172]:

$$|P_n(x)| < 1, \quad -1 < x < 1.$$

Next, if we put $\alpha = \beta = \nu - 1/2$, then $P_n^{(\nu-1/2,\nu-1/2)}(x) = C_n^\nu(x)$, the Gegenbauer polynomial [12, Eq. (1), p. 276]. Hence from (15), we find the corresponding inequality:

$$|C_n^\nu(x)| \le \frac{(2\nu)_n}{(\nu+1/2)_n}\left[\frac{(2\nu+2n-1)^{2\nu+2n-1}}{n^n 2^{2n}(2\nu+n-1)^{2\nu+n-1}}\right]^{1/2}(1-x^2)^{(-2\nu+1)/4}, \quad \nu > 0.$$

The Chebyshev polynomial of first kind [12, Eq.(1), p. 301] is expressed as $T_n(x) = P_n^{(-1/2,-1/2)}(x)$. Hence, for $n \ge 2$, we find from (15), the inequality:

$$|T_n(x)| = \frac{n!}{(1/2)_n}\left|P_n^{(-1/2,-1/2)}(x)\right| \le \frac{n!}{(1/2)_n}\left[\frac{(2n-1)^{2n-1}}{n^n 2^{2n-1}(n-1)^{n-1}}\right]^{1/2}(1-x^2)^{1/4}.$$

Since $2n-1<2n$, this simplifies to the form:

$$|T_n(x)| < \frac{n!}{(1/2)_n}\left[\frac{n}{n-1}\right]^{(n-1)/2}(1-x^2)^{1/4},\ n\geq 2.$$

Similarly, the Chebyshev polynomial of second kind: $U_n(x)$ is the case: $P_n^{(1/2,1/2)}(x)$ [12, Eq.(2), p. 301]. From (15), the inequality assumes the form:

$$|U_n(x)| = \frac{(n+1)!}{(3/2)_n}\left|P_n^{(1/2,1/2)}(x)\right| \leq \frac{(n+1)!}{(3/2)_n}\left[\frac{(2n+1)^{2n+1}}{n^n 2^{2n+1}(n+1)^{n+1}}\right]^{1/2}(1-x^2)^{-1/4}.$$

Further, using the relation: $2n+1<2n+2$, the above inequality takes the elegant form:

$$|U_n(x)| < \frac{\sqrt{2}(n+1)!}{(3/2)_n}\left[\frac{n+1}{n}\right]^{n/2}(1-x^2)^{-1/4}.$$

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References

[1] B.-T. Guo and L.-L., Wang, Jacobi interpolation approximations and their applications to singular differential equations, Adv. Comput. Math., 14(2001), 227–276.

[2] J. W. Brown and R. V. Churchill, Complex Variables and Applications, 8^{th} ed., McGraw-Hill Higher Education, New York, 2009.

[3] P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison and E. Witten, Quantum fields and strings: a course for mathematicians, American Mathematical Society, 38(1999), 489–494.

[4] R. Díaz and E. Pariguan, Quantum symmetric functions, Comm. Algebra, 33(2005), 1947–1978.

[5] R. Díaz and C. Teruel, q,k-generalized gamma and beta functions, J. Nonlinear Math. Phys., 12(2005), 118–134.

[6] R. Díaz and E. Pariguan, On hypergeometric function and Pochhammer k-symbol, Divulg. Mat., 15(2007), 179–192.

[7] U. Haagerup and H. Schlichtkrull, Inequalities for jacobi polynomials, Ramanujan J., 33(2014), 227–246.

[8] N. R. Joshi and B. I. Dave, Orthogonality and other properties of a p-deformed Jacobi polynomial, Math. Student, 92(2023), 187–202.

[9] H. Khalil and R. Ali Khan, The use of Jacobi polynomials in the numerical solution of coupled system of fractional differential equations, Int. J. Comput. Math., 92(2015), 1452–1472.

[10] D. Kumar and F. Ayant, Application of Jacobi polynomial and multivariable Aleph-function in heat conduction in non-homogeneous moving rectangular parallelepiped, Kragujevac J. Math., 45(2021), 439–448.

[11] D. Pylak, Application of Jacobi polynomials to the approximate solution of a singular integral equation with Cauchy kernel on the real half-line, Comput. Methods Appl. Math., 6(2006), 326–335.

[12] E. D. Rainville, Special Functions, The Macmillan Company, New York, 1960.