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Coefficient Bounds And Fekete-Szegö Inequalities For New Families of Bi-Starlike And Bi-Convex Functions Associated With The q-Bernoulli Polynomial*

Abbas Kareem Wanas†, Samer Chyad Khachi‡

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Abstract

In the present article, we use the q-Bernoulli polynomial and define two certain families $\mathcal{S}^*_\Sigma(q;x)$ and $\mathcal{C}_\Sigma(q;x)$ of normalized holomorphic and bi-univalent functions which are defined in the open unit disk $\mathbb{U}$. We establish upper bounds for the initial Taylor-Maclaurin coefficients and the Fekete-Szegö type inequalities of functions in these families.

1 Introduction

Denote by $\mathcal{A}$ the collection of all analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

having the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, assume that $\mathcal{S}$ stands for the sub-collection of the set $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$.

A function $f \in \mathcal{S}$ is called starlike of order $\gamma (0 \le \gamma < 1)$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \qquad (z \in \mathbb{U})$$

and a function $f \in \mathcal{S}$ is called convex of order $\gamma (0 \le \gamma < 1)$ if

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \gamma, \qquad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ the families of functions which are starlike of order γ and convex of order γ in $\mathbb{U}$, respectively.

According to the Koebe one-quarter theorem [6], every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

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†Department of Mathematics, College of Science,University of Al-Qadisiyah, Al Diwaniyah 58001, Al-Qadisiyah, Iraq
‡Department of Mathematics, College of Science, University of Al-Qadisiyah, Al-Qadisiyah, Iraq

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

For $f \in \mathcal{A}$, if both f and it inverse f^{-1} are univalent in $\mathbb{U}$, we say that f is a bi-univalent function in $\mathbb{U}$. We indicate by Σ the family of all bi-univalent functions in $\mathbb{U}$ given by (1). For a brief historical account and for several interesting examples of functions in the family Σ, one may see the pioneering work on this subject by Srivastava *et al.* [47]. In a considerably large number of sequels to the aforementioned work of Srivastava *et al.* [47], very large number of works related to bi-univalent functions were introduced and studied for several different subfamilies analogously by many authors (see, for example, [1, 5, 11, 21, 29, 30, 31, 41, 42, 43, 44, 45, 48, 49, 50, 51, 52]). From the work of Srivastava *et al.* [47], we choose to recall the following examples of functions in the family Σ :

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

We notice that the family Σ is not empty. However, the Koebe function is not a member of Σ.

The problem to find the general coefficient bounds on the Taylor-Maclaurin coefficients

$$|a_n| \qquad (n \in \mathbb{N};\ n \geqq 3)$$

for functions $f \in \Sigma$ is still not completely addressed for many of the subfamilies of the bi-univalent function family Σ.

The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for $f \in \mathcal{S}$ is well known for its rich history in the field of Geometric Function Theory. Its origin was in the disproof by Fekete and Szegö [8] of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by unity. The functional has since received great attention, particularly in the study of many subfamilies of the family of univalent functions. This topic has become of considerable interest among researchers in Geometric Function Theory of Complex Analysis.

With a view to recalling the principle of subordination between holomorphic functions, let the functions f and g be holomorphic in $\mathbb{U}$. We say that the function f is subordinate to g, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\omega(z)\big).$$

This subordination is denoted by

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

It is well known that, if the function g is univalent in $\mathbb{U}$, then (see [25])

$$f \prec g \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

For $0 < q < 1$, the q-factorial denoted by $[n]_q!$ is defined by (see [14])

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \cdots [2]_q[1]_q, & \text{if } n = 1, 2, 3, ..., \\ 1, & \text{if } n = 0, \end{cases}$$

where $[n]_q$, called the q-analogue of $n \in \mathbb{N}$, is given by

$$[n]_q = \frac{1-q^n}{1-q} \quad \text{for} \quad n \in \mathbb{N}.$$

Jackson [13, 14] introduced the q-derivative operator $\mathfrak{D}_q$ of a function f as follows:

$$\mathfrak{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \qquad (0 < q < 1;\ z \neq 0).$$

It is clear that

$$\lim_{q \to 1-} \mathfrak{D}_q f(z) = f'(z) \quad \text{and} \quad \mathfrak{D}_q f(0) = f'(0).$$

For more conceptual details on the q-derivative operator $\mathfrak{D}_q$, see [7, 9, 10].

For a function $f \in \mathcal{A}$ defined by (1), we deduce that

$$\mathfrak{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

As $q \longrightarrow 1-$, then we have $[n]_q \longrightarrow n$ and $[0]_q = 0$.

The q-exponential function e_q is defined by the power series expansion (see [20])

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \quad (z \in \mathbb{U}).$$

We note that

$$e(z) = \lim_{q \to 1-} e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The q-exponential function e_q is a unique function that satisfies the condition

$$\frac{\mathfrak{D}_q e(z)}{\mathfrak{D}_q z} = \sum_{n=0}^{\infty} \frac{\mathfrak{D}_q z^n}{[n]_q!} = \sum_{n=1}^{\infty} \frac{[n]_q z^{n-1}}{[n]_q!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{[n-1]_q!} = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = e_q(z), \quad (z \in \mathbb{U}).$$

In recent years, several authors studied many applications of the q-calculus associated with various families of analytic and univalent (or multivalent) functions (see, for example, [4, 12, 15, 16, 17, 22, 26, 27, 28, 32, 34, 35, 36, 39, 54]). In his recently-published survey-cum-expository review article, Srivastava [37] explored the mathematical applications of the q-calculus, the fractional q-calculus and the fractional q-derivative operators in Geometric Function Theory of Complex Analysis. Srivastava [37] also exposed the not-yet-widely-understood fact that the so-called (p, q)-variation of the classical q-calculus a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [37, p. 340]).

The q-Bernoulli polynomials $\mathfrak{B}_{q,n}(x)$ in Geometric Function Theory of Complex Analysis are given by the following linear homogeneous recurrence relation remains true(see, for instance, [3, 24]):

$$\mathfrak{B}_{q,n}(x) = q^n \left(x - \frac{1}{q[2]_q} \right) \mathfrak{B}_{q,n-1}(x) - \frac{1}{[n]_q} \sum_{j=0}^{n-2} \binom{n}{j}_q q^{j-1} b_{n-j,q} \mathfrak{B}_{n,q}(x), \tag{3}$$

with

$$\mathfrak{B}_{q,0}(x) = 1, \quad \mathfrak{B}_{q,1}(x) = \frac{[2]_q x - q}{[2]_q}, \quad \text{and} \quad \mathfrak{B}_{q,2}(x) = x(x-1) + \frac{q}{[2]_q [3]_q}.$$

The generating function of the q-Bernoulli polynomials $\mathfrak{B}_{q,n}(x)$ is given as follows (see [3]):

$$\mathfrak{B}_q(x, h) = \frac{h}{e_q(h) - 1} e_q(hx) = \sum_{n=0}^{\infty} \mathfrak{B}_{q,n}(x) \frac{h^n}{[n]_q!}, \qquad |h| < 2\pi. \tag{4}$$

The families of orthogonal polynomials and other special functions and specific polynomials, as well as their extensions and generalizations, are potentially useful in a variety of disciplines in many branches of science, especially in the mathematical, statistical and physical sciences. The relationship between bi-univalent functions and orthogonal polynomials has recently come under the scrutiny of various authors (see, for example, [2, 18, 19, 21, 40, 41, 53]).

2 Main Results

Using the q-Bernoulli polynomials, we now define the following the families $\mathcal{S}^*_\Sigma(q;x)$ and $\mathcal{C}_\Sigma(q;x)$ of holomorphic bi-starlike and bi-convex functions.

Definition 1 *A function $f \in \Sigma$ is said to be in the family $S^*_\Sigma(q;x)$ if it fulfills the following subordination conditions:*

$$\frac{zf'(z)}{f(z)} \prec \mathfrak{B}_q(x,z)$$

and

$$\frac{wg'(w)}{g(w)} \prec \mathfrak{B}_q(x,w),$$

where $z,w \in U$, $x \in [-\pi,\pi]$ and the function $g = f^{-1}$ is given by (2).

Definition 2 *A function $f \in \Sigma$ is said to be in the family $C_\Sigma(q;x)$ if it fulfills the following subordination conditions:*

$$1 + \frac{zf''(z)}{f'(z)} \prec \mathfrak{B}_q(x,z)$$

and

$$1 + \frac{wg''(w)}{g'(w)} \prec \mathfrak{B}_q(x,w),$$

where $z,w \in U$, $x \in [-\pi,\pi]$ and the function $g = f^{-1}$ is given by (2).

Theorem 1 *Let $f \in \mathcal{A}$ be in the family $\mathcal{S}^*_\Sigma(q;x)$. Then*

$$|a_2| \leqq \min\left\{\sqrt{\frac{|[2]_q x - q|}{[2]_q}}, \frac{|[2]_q x - q|\sqrt{|[2]_q x - q|}}{[2]_q\sqrt{\left|\left([2]_q - 1\right)x^2 + (1-2q)x + \frac{q(q[3]_q - 1)}{[2]_q[3]_q}\right|}}\right\}$$

and

$$|a_3| \leqq \min\left\{\frac{3\left|[2]_q x - q\right|}{2[2]_q} + \frac{|x(x-1)|}{[2]_q} + \frac{q}{[2]_q^2[3]_q}, \frac{|[2]_q x - q|}{2[2]_q} + \frac{\left([2]_q x - q\right)^2}{[2]_q^2}\right\}.$$

Proof. Suppose that $f \in \mathcal{S}^*_\Sigma(q;x)$. Then there are two holomorphic functions $u, v : \mathbb{U} \longrightarrow \mathbb{U}$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{5}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \qquad (w \in \mathbb{U}), \tag{6}$$

with

$$u(0) = v(0) = 0 \qquad \text{and} \qquad \max\{|u(z)|, |v(w)|\} < 1 \quad (z,w \in \mathbb{U}),$$

such that

$$\frac{zf'(z)}{f(z)} = \mathfrak{B}_q(x, u(z))$$

and

$$\frac{wg'(w)}{g(w)} = \mathfrak{B}_q(x, v(w)),$$

or, equivalently, that

$$\frac{zf'(z)}{f(z)} = 1 + \mathfrak{B}_{q,1}(x)u(z) + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)u^2(z) + \cdots \tag{7}$$

and

$$\frac{wg'(w)}{g(w)} = 1 + \mathfrak{B}_{q,1}(x)v(w) + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)v^2(w) + \cdots . \tag{8}$$

Combining (5), (6), (7) and (8), we find that

$$\frac{zf'(z)}{f(z)} = 1 + \mathfrak{B}_{q,1}(x)u_1 z + \left[\mathfrak{B}_{q,1}(x)u_2 + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)u_1^2\right] z^2 + \cdots \tag{9}$$

and

$$\frac{wg'(w)}{g(w)} = 1 + \mathfrak{B}_{q,1}(x)v_1 w + \left[\mathfrak{B}_{q,1}(x)v_2 + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)v_1^2\right] w^2 + \cdots . \tag{10}$$

It is well-known that, if

$$\max\{|u(z)|, |v(w)|\} < 1 \qquad (z, w \in \mathbb{U}),$$

then

$$|u_j| \leqq 1 \qquad \text{and} \qquad |v_j| \leqq 1 \qquad (\forall\, j \in \mathbb{N}). \tag{11}$$

Now, by comparing the corresponding coefficients in (9) and (10), and after some simplification, we have

$$a_2 = \mathfrak{B}_{q,1}(x)u_1, \tag{12}$$

$$2a_3 - a_2^2 = \mathfrak{B}_{q,1}(x)u_2 + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)u_1^2, \tag{13}$$

$$-a_2 = \mathfrak{B}_{q,1}(x)v_1 \tag{14}$$

and

$$\left(3a_2^2 - 2a_3\right) = \mathfrak{B}_{q,1}(x)v_2 + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)v_1^2. \tag{15}$$

It follows from (12) and (14) that

$$u_1 = -v_1 \tag{16}$$

and

$$2a_2^2 = \mathfrak{B}_{q,1}^2(x)(u_1^2 + v_1^2). \tag{17}$$

If we add (13) to (15), we find that

$$2a_2^2 = \mathfrak{B}_{q,1}(x)(u_2 + v_2) + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)(u_1^2 + v_1^2). \tag{18}$$

Upon substituting the value of $u_1^2 + v_1^2$ from (17) into the right-hand side of (18), we deduce that

$$a_2^2 = \frac{\mathfrak{B}_{q,1}^3(x)(u_2 + v_2)}{2\left[\mathfrak{B}_{q,1}^2(x) - \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)\right]}. \tag{19}$$

By further computations using (3), (11), (17) and (19), we obtain

$$|a_2| \leqq \sqrt{\frac{|[2]_q x - q|}{[2]_q}}, \qquad |a_2| \leqq \frac{|[2]_q x - q| \sqrt{|[2]_q x - q|}}{[2]_q \sqrt{\left|\left([2]_q - 1\right)x^2 + (1 - 2q)x + \frac{q(q[3]_q - 1)}{[2]_q [3]_q}\right|}}.$$

Next, if we subtract (15) from (13), we can easily see that

$$4\left(a_3 - a_2^2\right) = \mathfrak{B}_{q,1}(x)(u_2 - v_2) + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)(u_1^2 - v_1^2). \tag{20}$$

In view of (16) and substituting the value of a_2^2 from (17) into (20), we find that

$$a_3 = \frac{\mathfrak{B}_{q,1}(x)(u_2 - v_2)}{4} + \frac{\mathfrak{B}_{q,1}^2(x)(u_1^2 + v_1^2)}{2}.$$

Thus, by applying (3), we obtain

$$|a_3| \leqq \frac{|[2]_q x - q|}{2[2]_q} + \frac{([2]_q x - q)^2}{[2]_q^2}.$$

In addition, substituting the value of a_2^2 from (18) into (20), we deduce that

$$a_3 = \frac{\mathfrak{B}_{q,1}(x)(u_2 - v_2)}{4} + \frac{\mathfrak{B}_{q,1}(x)(u_2 + v_2)}{2} + \frac{\mathfrak{B}_{q,2}(x)(u_1^2 + v_1^2)}{2[2]_q},$$

and we have

$$|a_3| \leqq \frac{3\,|[2]_q x - q|}{2[2]_q} + \frac{|x(x-1)|}{[2]_q} + \frac{q}{[2]_q^2[3]_q}.$$

This completes the proof of Theorem 1. ■

Theorem 2 *Let $f \in \mathcal{A}$ be in the family $\mathcal{C}_\Sigma(q;x)$. Then*

$$|a_2| \leqq \min\left\{\frac{1}{2}\sqrt{\frac{|[2]_q x - q|}{[2]_q}}, \frac{|[2]_q x - q|\sqrt{|[2]_q x - q|}}{[2]_q\sqrt{2\left|([2]_q - 2)\,x^2 + 2(1-q)x + \frac{q(q[3]_q - 2)}{[2]_q[3]_q}\right|}}\right\}$$

and

$$|a_3| \leqq \min\left\{\frac{2\,|[2]_q x - q|}{3[2]_q} + \frac{|x(x-1)|}{2[2]_q} + \frac{q}{2[2]_q^2[3]_q}, \frac{|[2]_q x - q|}{6[2]_q} + \frac{([2]_q x - q)^2}{4[2]_q^2}\right\}.$$

Proof. Suppose that $f \in \mathcal{C}_\Sigma(q;x)$. Then there are two holomorphic functions $u, v : \mathbb{U} \longrightarrow \mathbb{U}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \mathfrak{B}_q(x, u(z))$$

and

$$1 + \frac{wg''(w)}{g'(w)} = \mathfrak{B}_q(x, v(w)),$$

where u and v have the forms (5) and (6). We have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \mathfrak{B}_{q,1}(x)u(z) + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)u^2(z) + \cdots \tag{21}$$

and

$$1 + \frac{wg''(w)}{g'(w)} = 1 + \mathfrak{B}_{q,1}(x)v(w) + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)v^2(w) + \cdots. \tag{22}$$

From (21) and (22), we deduce that

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \mathfrak{B}_{q,1}(x)u_1 z + \left[\mathfrak{B}_{q,1}(x)u_2 + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)u_1^2\right]z^2 + \cdots \tag{23}$$

and

$$1 + \frac{wg''(w)}{g'(w)} = 1 + \mathfrak{B}_{q,1}(x)v_1 w + \left[\mathfrak{B}_{q,1}(x)v_2 + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)v_1^2\right]w^2 + \cdots. \tag{24}$$

Now, by comparing the corresponding coefficients in (23) and (24), and after some simplification, we have

$$2a_2 = \mathfrak{B}_{q,1}(x)u_1, \tag{25}$$

$$6a_3 - 4a_2^2 = \mathfrak{B}_{q,1}(x)u_2 + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)u_1^2, \tag{26}$$

$$-2a_2 = \mathfrak{B}_{q,1}(x)v_1 \tag{27}$$

and

$$8a_2^2 - 6a_3 = \mathfrak{B}_{q,1}(x)v_2 + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)v_1^2. \tag{28}$$

It follows from (25) and (27) that

$$u_1 = -v_1 \tag{29}$$

and

$$8a_2^2 = \mathfrak{B}_{q,1}^2(x)(u_1^2 + v_1^2). \tag{30}$$

If we add (26) to (28), we find that

$$4a_2^2 = \mathfrak{B}_{q,1}(x)(u_2 + v_2) + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)(u_1^2 + v_1^2). \tag{31}$$

Upon substituting the value of $u_1^2 + v_1^2$ from (30) into the right-hand side of (31), we deduce that

$$a_2^2 = \frac{\mathfrak{B}_{q,1}^3(x)(u_2 + v_2)}{4\left[\mathfrak{B}_{q,1}^2(x) - \frac{2}{[2]_q}\mathfrak{B}_{q,2}(x)\right]}. \tag{32}$$

By further computations using (3), (11), (30) and (32), we obtain

$$|a_2| \leqq \frac{1}{2}\sqrt{\frac{|[2]_q x - q|}{[2]_q}}, \qquad |a_2| \leqq \frac{|[2]_q x - q|\sqrt{|[2]_q x - q|}}{[2]_q\sqrt{2\left|\left([2]_q - 2\right)x^2 + 2(1-q)x + \frac{q(q[3]_q - 2)}{[2]_q[3]_q}\right|}}.$$

Next, if we subtract (28) from (26), we can easily see that

$$12\left(a_3 - a_2^2\right) = \mathfrak{B}_{q,1}(x)(u_2 - v_2) + \frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)(u_1^2 - v_1^2). \tag{33}$$

In view of (29) and substituting the value of a_2^2 from (30) into (33), we find that

$$a_3 = \frac{\mathfrak{B}_{q,1}(x)(u_2 - v_2)}{12} + \frac{\mathfrak{B}_{q,1}^2(x)(u_1^2 + v_1^2)}{8}.$$

Thus, by applying (3), we obtain

$$|a_3| \leqq \frac{|[2]_q x - q|}{6[2]_q} + \frac{\left([2]_q x - q\right)^2}{4[2]_q^2}.$$

In addition, substituting the value of a_2^2 from (31) into (33), we deduce that

$$a_3 = \frac{\mathfrak{B}_{q,1}(x)(u_2 - v_2)}{12} + \frac{\mathfrak{B}_{q,1}(x)(u_2 + v_2)}{4} + \frac{\mathfrak{B}_{q,2}(x)(u_1^2 + v_1^2)}{4[2]_q},$$

and we have

$$|a_3| \leqq \frac{2\left|[2]_q x - q\right|}{3[2]_q} + \frac{|x(x-1)|}{2[2]_q} + \frac{q}{2[2]_q^2[3]_q}.$$

This completes the proof of Theorem 2. ■

In the next theorems, we present the Fekete-Szegö type inequalities for the families $\mathcal{S}_\Sigma^*(q;x)$ and $\mathcal{C}_\Sigma(q;x)$.

Theorem 3 *For $\mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the family $\mathcal{S}_{\Sigma}^{*}(q;x)$. Then*

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{|[2]_q x-q|}{2[2]_q}; \\ |\varphi-1| \leqq \frac{[2]_q^2\left|([2]_q-1)x^2+(1-2q)x+\frac{q(q[3]_q-1)}{[2]_q[3]_q}\right|}{2([2]_q x-q)^2}, \\ \frac{|[2]_q x-q|^3|\mu-1|}{[2]_q^3\left|([2]_q-1)x^2+(1-2q)x+\frac{q(q[3]_q-1)}{[2]_q[3]_q}\right|}; \\ |\varphi-1| \geqq \frac{[2]_q^2\left|([2]_q-1)x^2+(1-2q)x+\frac{q(q[3]_q-1)}{[2]_q[3]_q}\right|}{2([2]_q x-q)^2}. \end{cases}$$

Proof. It follows from (19) and (20) that

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{\mathfrak{B}_{q,1}(x)(u_2-v_2)}{4}+(1-\mu)\,a_2^2 \\ &= \frac{\mathfrak{B}_{q,1}(x)(u_2-v_2)}{4}+\frac{\mathfrak{B}_{q,1}^3(x)(u_2+v_2)\,(1-\mu)}{2\left[\mathfrak{B}_{q,1}^2(x)-\frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)\right]} \\ &= \frac{\mathfrak{B}_{q,1}(x)}{2}\left[\left(\varphi(\mu,x)+\frac{1}{2}\right)u_2+\left(\varphi(\mu,x)-\frac{1}{2}\right)v_2\right], \end{aligned}$$

where

$$\varphi(\mu,x)=\frac{\mathfrak{B}_{q,1}^2\,(1-\mu)}{\mathfrak{B}_{q,1}^2(x)-\frac{1}{[2]_q}\mathfrak{B}_{q,2}(x)}.$$

Thus, according to (3), we have

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{|[2]_q x-q|}{2[2]_q}, & 0 \leqq |\varphi(\mu,x)| \leqq \frac{1}{2}, \\ \frac{|[2]_q x-q|.|\varphi(\mu,x)|}{[2]_q}, & |\varphi(\mu,x)| \geqq \frac{1}{2}. \end{cases}$$

After simple computation, we deduce that

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{|[2]_q x-q|}{2[2]_q}; \\ |\varphi-1| \leqq \frac{[2]_q^2\left|([2]_q-1)x^2+(1-2q)x+\frac{q(q[3]_q-1)}{[2]_q[3]_q}\right|}{2([2]_q x-q)^2}, \\ \frac{|[2]_q x-q|^3|\mu-1|}{[2]_q^3\left|([2]_q-1)x^2+(1-2q)x+\frac{q(q[3]_q-1)}{[2]_q[3]_q}\right|}; \\ |\varphi-1| \geqq \frac{[2]_q^2\left|([2]_q-1)x^2+(1-2q)x+\frac{q(q[3]_q-1)}{[2]_q[3]_q}\right|}{2([2]_q x-q)^2}. \end{cases}$$

This completes the proof of Theorem 3. ■

By putting $\mu = 1$ in Theorem 3, we obtain the following result.

Corollary 1 *If $f \in \mathcal{A}$ be in the family $\mathcal{S}_{\Sigma}^{*}(q;x)$, then*

$$\left|a_3-a_2^2\right| \leqq \frac{|[2]_q x-q|}{2[2]_q}.$$

Theorem 4 *For $\mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the family $\mathcal{C}_\Sigma(q;x)$. Then*

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{|[2]_q x-q|}{6[2]_q}; \\ |\psi-1| \leqq \frac{[2]_q^2\left|([2]_q-2)x^2+2(1-q)x+\frac{q(q[3]_q-2)}{[2]_q[3]_q}\right|}{3([2]_q x-q)^2}, \\ \frac{|[2]_q x-q|^3|\mu-1|}{2[2]_q^3\left|([2]_q-2)x^2+2(1-q)x+\frac{q(q[3]_q-2)}{[2]_q[3]_q}\right|}; \\ |\psi-1| \geqq \frac{[2]_q^2\left|([2]_q-2)x^2+2(1-q)x+\frac{q(q[3]_q-2)}{[2]_q[3]_q}\right|}{3([2]_q x-q)^2}. \end{cases}$$

Proof. It follows from (32) and (33) that

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{\mathfrak{B}_{q,1}(x)(u_2-v_2)}{12}+(1-\mu)\,a_2^2 \\ &= \frac{\mathfrak{B}_{q,1}(x)(u_2-v_2)}{12}+\frac{\mathfrak{B}_{q,1}^3(x)(u_2+v_2)\,(1-\mu)}{4\left[\mathfrak{B}_{q,1}^2(x)-\frac{2}{[2]_q}\mathfrak{B}_{q,2}(x)\right]} \\ &= \frac{\mathfrak{B}_{q,1}(x)}{4}\left[\left(\psi(\mu,x)+\frac{1}{3}\right)u_2+\left(\psi(\mu,x)-\frac{1}{3}\right)v_2\right], \end{aligned}$$

where

$$\psi(\mu,x)=\frac{\mathfrak{B}_{q,1}^2\,(1-\mu)}{\mathfrak{B}_{q,1}^2(x)-\frac{2}{[2]_q}\mathfrak{B}_{q,2}(x)}.$$

Thus, according to (3), we have

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{|[2]_q x-q|}{6[2]_q}, & 0 \leqq |\psi(\mu,x)| \leqq \frac{1}{3}, \\ \frac{|[2]_q x-q|.|\psi(\mu,x)|}{2[2]_q}, & |\psi(\mu,x)| \geqq \frac{1}{3}. \end{cases}$$

After simple computation, we deduce that

$$\left|a_3-\mu a_2^2\right| \leqq \begin{cases} \frac{|[2]_q x-q|}{6[2]_q}; \\ |\psi-1| \leqq \frac{[2]_q^2\left|([2]_q-2)x^2+2(1-q)x+\frac{q(q[3]_q-2)}{[2]_q[3]_q}\right|}{3([2]_q x-q)^2}, \\ \frac{|[2]_q x-q|^3|\mu-1|}{2[2]_q^3\left|([2]_q-2)x^2+2(1-q)x+\frac{q(q[3]_q-2)}{[2]_q[3]_q}\right|}; \\ |\psi-1| \geqq \frac{[2]_q^2\left|([2]_q-2)x^2+2(1-q)x+\frac{q(q[3]_q-2)}{[2]_q[3]_q}\right|}{3([2]_q x-q)^2}. \end{cases}$$

This completes the proof of Theorem 4. ■

By putting $\mu = 1$ in Theorem 4, we obtain the following result.

Corollary 2 *If $f \in \mathcal{A}$ be in the family $\mathcal{C}_\Sigma(q;x)$, then*

$$\left|a_3-a_2^2\right| \leqq \frac{|[2]_q x-q|}{6[2]_q}.$$

3 Conclusion

The fact that we can find many unique and effective usages of a large variety of interesting special functions and specific polynomials in Geometric Function Theory of Complex Analysis provided the primary inspiration and motivation for our analysis in this article. Our main objective was to define a new families $\mathcal{S}^*_\Sigma(q;x)$ and $\mathcal{C}_\Sigma(q;x)$ of normalized holomorphic and bi-univalent functions which are defined by means of the q-Bernoulli polynomial $\mathfrak{B}_{q,n}(x)$ given by the recurrence relation (3) and by generating function $\mathfrak{B}_q(x,h)$ in (4). We have established inequalities for the initial Taylor-Maclaurin coefficients and Fekete-Szegö problem of functions belonging to these newly-introduced families.

It should be remarked that, in many recent investigations dealing with some of the topics of our presentation in this paper, the basic or quantum (or q-) calculus was extensively used (see, for example, [23], [33] and [46]).

We deduce the present article by recalling a recently-published survey-cum-expository review article in which Srivastava [37] explored the mathematical applications of the q-calculus, the fractional q-calculus and the fractional q-derivative operators in Geometric Function Theory of Complex Analysis, especially in the study of Fekete-Szegö functional. Srivastava [37] also exposed the not-yet-widely-understood fact that the so-called (p,q)-variation of the classical q-calculus is, in fact, a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [37, p. 340]; see also [38, pp. 1511–1512]).

As future research directions, the contents of the paper on a q-Bernoulli polynomial could inspire further research related to other families.

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