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A Note On Laurent Type Hypergeometric Generating Relations*

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Abstract

In this paper, a Laurent type hypergeometric generating relation is derived by using series rearrangement technique. Some special cases are obtained as generating functions of the Bessel functions of different kinds. Finally explicit forms of these Bessel functions are obtained as applications.

1 Introduction and Preliminaries

Korsch et al. [4, p.14948, Eq. (2)] have discussed the general properties of two-dimensional generalized Bessel functions $J_n^{p,q}(u,v)$. They have given the following generating function for the two-dimensional Bessel functions:

$$\exp\left[\frac{u}{2}\left(t^p-\frac{1}{t^p}\right)+\frac{v}{2}\left(t^q-\frac{1}{t^q}\right)\right]=\sum_{n=-\infty}^{\infty}J_n^{p,q}(u,v)\ t^n, \tag{1}$$

where (p,q) being relatively prime positive integers and $u,v\in\mathbb{R}$.

$$J_n^{\mu p,\mu q}(u,v)=\begin{cases}J_{\frac{n}{\mu}}^{p,q}(u,v), & \text{for } \frac{n}{\mu}\in\mathbb{Z}\\ 0, & \text{else,}\end{cases} \tag{2}$$

$\mu\neq 0,1$.

The two-dimensional Bessel functions $J_n^{p,q}(u,v)$ have the following bounds:

$$|J_0^{p,q}(u,v)|\leq 1 \text{ and } |J_n^{p,q}(u,v)|\leq\frac{1}{\sqrt{2}} \text{ for } n\neq 0.$$

Miller introduced a new class of Bessel functions $J_n^{(p,q)}(x)$ with generating function [5, p. 497, Eq. (19)]:

$$\exp\left[\frac{ix}{p+q}(t^p+t^{-q})\right]=\sum_{n=-\infty}^{\infty}J_n^{(p,q)}(x)t^n, \tag{3}$$

where (p,q) being relatively prime positive integers.

Also, the two variable Bessel functions $\mathcal{D}_n^{(p,m)}(x,y)$ possess the generating function [1, p. 116, Eq. (2.15)]:

$$\exp\left[xt^p-\frac{y}{t^m}\right]=\sum_{n=-\infty}^{\infty}\mathcal{D}_n^{(p,m)}(x,y)t^n, \quad 0<|t|<\infty, \quad x,y\in\mathbb{R}, \tag{4}$$

where (p,m) being relatively prime positive integers.

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The $\mathcal{D}_n^{(p,m)}(x,y)$ is given by the following series:

$$\mathcal{D}_n^{(p,m)}(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{\left(\frac{n+mr}{p}\right)} y^r}{\Gamma\left(\frac{n+mr}{p}+1\right) r!}. \tag{5}$$

The following special cases of the $\mathcal{D}_n^{(p,m)}(x,y)$ are as follows:

$$\mathcal{D}_n^{(1,1)}\left(\frac{x}{2},\frac{x}{2}\right) = J_n(x) \text{ (Bessel functions)} \tag{6}$$

$$\mathcal{D}_n^{(1,1)}\left(\frac{x}{2},-\frac{x}{2}\right) = I_n(x) \text{ (Modified Bessel functions)} \tag{7}$$

$$\mathcal{D}_n^{(1,1)}(1,y) = C_n(y) \text{ (Bessel Clifford or Tricomi functions)} \tag{8}$$

$$\mathcal{D}_n^{(1,m)}(1,y) = W_n^m(y) \text{ (Wright functions).} \tag{9}$$

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$${}_pF_q\left[\begin{array}{lc}(\alpha_p); & \\ & z\\ (\beta_q); & \end{array}\right] = {}_pF_q\left[\begin{array}{lc}\alpha_1,\alpha_2,\ldots,\alpha_p; & \\ & z\\ \beta_1,\beta_2,\ldots,\beta_q; & \end{array}\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!} \tag{10}$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function [10]. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\ldots \;;\quad j=1,2,\ldots,q.$$

In contracted notation, the sequence of p numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ is denoted by (α_p) with similar interpretation for others throughout this paper.

Supposing that none of numerator parameters is zero or a negative integer and for $\beta_j \neq 0,\,-1,\,-2,\ldots;$ $j=1,\,2,\ldots,\,q$, we note that the ${}_pF_q$ series defined by equation (10):

(i) converges for $|z|<\infty$, if $p\leq q$,

(ii) converges for $|z|<1$, if $p=q+1$ and

(iii) diverges for all z, $z\neq 0$, if $p>q+1$.

A multivariable hypergeometric function provides an interesting and useful unification of the generalized hypergeometric function ${}_pF_q$ of one variable (with p numerator and q denominator parameters).

The following generalization of the hypergeometric function in several variables has been given by Srivastava and Daoust [11], which is referred to in the literature as the generalized Lauricella function of several variables:

$$F^{A:B';B''^{(n)}}_{C:D';D''^{(n)}}\left(\begin{array}{llllll}[(a):\theta',\theta''^{(n)}]: & [(b'):\phi']; & [(b''):\phi'']; & \cdots; & [(b^{(n)}):\phi^{(n)}]; & \\ & & & & & z_1,z_2,\cdots,z_n\\ \quad : & [(d'):\delta']; & [(d''):\delta'']; & \cdots; & [(d^{(n)}):\delta^{(n)}]; & \end{array}\right)$$

$$= \sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\Omega(m_1,m_2,...,m_n)\,\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}\cdots\frac{z_n^{m_n}}{m_n!}, \tag{11}$$

where

$$\Omega(m_1,m_2,...,m_n)$$

$$: \quad = \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+m_2\theta''_j+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\prod_{j=1}^{B''}(b''_j)_{m_2\phi''_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+m_2\psi''_j+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\prod_{j=1}^{D''}(d''_j)_{m_2\delta''_j}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

and the coefficients

$$\theta_j^{(k)}, j=1,2,...,A;\quad \phi_j^{(k)}, j=1,2,...,B^{(k)};\quad \psi_j^{(k)}, j=1,2,...,C;\quad \delta_j^{(k)}, j=1,2,...,D^{(k)};$$

for all $k \in \{1,2,...,n\}$ are real and positive, (a) abbreviates the array of A parameters $a_1, a_2, ..., a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}, j=1,2,...,B^{(k)}$; for all $k \in \{1,2,...,n\}$ with similar interpretations for (c) and $(d^{(k)})$, $k=1,2,...,n$; *et cetera.*

A significant progress has been made in the study of generalized Bessel functions. Notably, fractional integral operators involving generalized Bessel functions are obtained in [6]. In [3], differential subordinations and superordinations for generalized Bessel functions are given. In [9], the inclusion properties of new subclasses of analytic functions are established by using the generalized Bessel functions of the first kind. The computation of image formulas of generalized fractional hypergeometric operators, involving the product of multivariable Srivastava polynomial and multi-index Bessel function is a recent investigation, see for example [2].

Recently, Qureshi *et al.* [7, 8] have obtained explicit expressions of some hybrid special functions related to the Bessel and Tricomi functions. In this article, our main motive is to derive a Laurent type hypergeometric generating relation. In Section 2, a general series identity is derived. Using the general series identity, a Laurent type hypergeometric generating relation is obtained. In Section 3, some special cases of the obtained results are presented in terms of the generating relations of the Bessel functions.

2 Main Results

In this section, we derive a general series identity in the form of the following lemma:

Lemma 1 *Let $\{\Omega_1(\ell)\}$, $\{\Omega_2(\ell)\}$, $\{\Omega_3(\ell)\}$ and $\{\Omega_4(\ell)\}$; $\ell \in \{1,2,3,\cdots\}$ are four bounded sequences of arbitrary complex numbers and $\Omega_i(0) \neq 0$ $(i=1,2,3,4)$. Then*

$$\sum_{i,j,k,\ell=0}^{\infty}\Omega_1(i)\Omega_2(j)\Omega_3(k)\Omega_4(\ell)\frac{(\alpha t^p)^i\ (\lambda t^q)^j\ \left(\frac{\beta}{t^r}\right)^k\ \left(\frac{\mu}{t^s}\right)^\ell}{i!\ j!\ k!\ \ell!}$$
$$= \sum_{n=-\infty}^{\infty}\frac{\alpha^{\frac{n}{p}}}{\Gamma\left(1+\frac{n}{p}\right)}\sum_{j,k,\ell=0}^{\infty}\Omega_1\left(\frac{n-qj+rk+s\ell}{p}\right)\Omega_2(j)\Omega_3(k)\Omega_4(\ell)$$
$$\frac{1}{\left(1+\frac{n}{p}\right)_{\frac{r}{p}k+\frac{s}{p}\ell-\frac{q}{p}j}}\frac{\left(\frac{\lambda}{\alpha^{\frac{q}{p}}}\right)^j}{j!}\ \frac{\left(\beta\alpha^{\frac{r}{p}}\right)^k}{k!}\ \frac{\left(\mu\alpha^{\frac{s}{p}}\right)^\ell}{\ell!}\ t^n, \tag{12}$$

where (p,r), (p,s), (q,r), (q,s) being relatively prime positive integers and each of the multiple series involved is absolutely convergent.

Proof. Suppose the l.h.s. of equation (12) is denoted by Δ. Then, we have

$$\Delta = \sum_{i,j,k,\ell=0}^{\infty}\Omega_1(i)\Omega_2(j)\Omega_3(k)\Omega_4(\ell)\ \frac{\alpha^i\ \lambda^j\ \beta^k\ \mu^\ell}{i!\ j!\ k!\ \ell!}\ t^{pi+qj-rk-s\ell}. \tag{13}$$

Putting $pi+qj-rk-s\ell=n$ or $i=\frac{n-qj+rk+s\ell}{p}$ in equation (13), we get

$$\Delta=\sum_{n=-\infty}^{\infty}\sum_{j,k,\ell=0}^{\infty}\Omega_1\left(\frac{n-qj+rk+s\ell}{p}\right)\Omega_2(j)\Omega_3(k)\Omega_4(\ell)\frac{\alpha^{\left(\frac{n-qj+rk+s\ell}{p}\right)}\,\lambda^j\,\beta^k\,\mu^\ell}{\Gamma\left(\frac{n-qj+rk+s\ell}{p}+1\right)\,j!\,k!\,\ell!}\,t^n. \tag{14}$$

On simplification, we get Lemma 1. ■

Theorem 2 *The following Laurent type hypergeometric generating relation holds true:*

$$ {}_AF_B\left[\begin{array}{ll}(a_A)\;; & \\ & \alpha t^p \\ (b_B)\;; & \end{array}\right] {}_CF_D\left[\begin{array}{ll}(c_C)\;; & \\ & \lambda t^q \\ (d_D)\;; & \end{array}\right] {}_GF_H\left[\begin{array}{ll}(g_G)\;; & \\ & \frac{\beta}{t^r} \\ (h_H)\;; & \end{array}\right] {}_VF_W\left[\begin{array}{ll}(v_V)\;; & \\ & \frac{\mu}{t^s} \\ (w_W)\;; & \end{array}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{\prod_{m=1}^{A}(a_m)_{\frac{n}{p}}}{\prod_{m=1}^{B}(b_m)_{\frac{n}{p}}}\frac{\alpha^{\frac{n}{p}}}{\Gamma\left(1+\frac{n}{p}\right)}\times$$

$$F^{A:C;G;V}_{B+1:D;H;W}\left(\begin{array}{ll}\left[(a_A)+\frac{n}{p}:-\frac{q}{p},\frac{r}{p},\frac{s}{p}\right] & :\;[(c_C):1]\;; \\ \left[(b_B)+\frac{n}{p}:-\frac{q}{p},\frac{r}{p},\frac{s}{p}\right]\;\left[1+\frac{n}{p}:-\frac{q}{p},\frac{r}{p},\frac{s}{p}\right] & :\;[(d_D):1]\;;\end{array}\right.$$

$$\left.\begin{array}{ll}[(g_G):1]\;; & [(v_V):1]\;; \\ [(h_H):1]\;; & [(w_W):1]\;;\end{array}\;\lambda\alpha^{-\frac{q}{p}},\quad\beta\alpha^{\frac{r}{p}},\quad\mu\alpha^{\frac{s}{p}}\right)t^n,\quad t\neq 0, \tag{15}$$

where (p,r), (p,s), (q,r), (q,s) being relatively prime positive integers and each of the multiple series involved is absolutely convergent.

Proof. Taking

$$\Omega_1(i)=\frac{\prod_{m=1}^{A}(a_m)_i}{\prod_{m=1}^{B}(b_m)_i},\quad\Omega_2(j)=\frac{\prod_{m=1}^{C}(c_m)_j}{\prod_{m=1}^{D}(d_m)_j},\quad\Omega_3(k)=\frac{\prod_{m=1}^{G}(g_m)_k}{\prod_{m=1}^{H}(h_m)_k},\quad\Omega_4(\ell)=\frac{\prod_{m=1}^{V}(v_m)_\ell}{\prod_{m=1}^{W}(w_m)_\ell},$$

in general series identity (12), applying some algebraic properties of Pochhammer symbols and after simplification, we obtain:

$$ {}_AF_B\left[\begin{array}{ll}(a_A)\;; & \\ & \alpha t^p \\ (b_B)\;; & \end{array}\right] {}_CF_D\left[\begin{array}{ll}(c_C)\;; & \\ & \lambda t^q \\ (d_D)\;; & \end{array}\right] {}_GF_H\left[\begin{array}{ll}(g_G)\;; & \\ & \frac{\beta}{t^r} \\ (h_H)\;; & \end{array}\right] {}_VF_W\left[\begin{array}{ll}(v_V)\;; & \\ & \frac{\mu}{t^s} \\ (w_W)\;; & \end{array}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{\alpha^{\frac{n}{p}}}{\Gamma\left(1+\frac{n}{p}\right)}\sum_{j,k,\ell=0}^{\infty}\frac{\prod_{m=1}^{A}(a_m)_{\frac{n-qj+rk+s\ell}{p}}}{\prod_{m=1}^{B}(b_m)_{\frac{n-qj+rk+s\ell}{p}}}\frac{\prod_{m=1}^{C}(c_m)_j}{\prod_{m=1}^{D}(d_m)_j}\frac{\prod_{m=1}^{G}(g_m)_k}{\prod_{m=1}^{H}(h_m)_k}\frac{\prod_{m=1}^{V}(v_m)_\ell}{\prod_{m=1}^{W}(w_m)_\ell}$$

$$\frac{1}{\left(1+\frac{n}{p}\right)_{-\frac{q}{p}j+\frac{r}{p}k+\frac{s}{p}\ell}}\frac{\left(\frac{\lambda}{\alpha^{\frac{q}{p}}}\right)^j}{j!}\frac{\left(\beta\alpha^{\frac{r}{p}}\right)^k}{k!}\frac{\left(\mu\alpha^{\frac{s}{p}}\right)^\ell}{\ell!}\,t^n$$

$$=\sum_{n=-\infty}^{\infty}\frac{\prod_{m=1}^{A}(a_m)_{\frac{n}{p}}}{\prod_{m=1}^{B}(b_m)_{\frac{n}{p}}}\frac{\alpha^{\frac{n}{p}}}{\Gamma\left(1+\frac{n}{p}\right)}\sum_{j,k,\ell=0}^{\infty}\frac{\prod_{m=1}^{A}\left(a_m+\frac{n}{p}\right)_{\frac{-qj+rk+s\ell}{p}}}{\prod_{m=1}^{B}\left(b_m+\frac{n}{p}\right)_{\frac{-qj+rk+s\ell}{p}}}\frac{\prod_{m=1}^{C}(c_m)_j}{\prod_{m=1}^{D}(d_m)_j}\frac{\prod_{m=1}^{G}(g_m)_k}{\prod_{m=1}^{H}(h_m)_k}\frac{\prod_{m=1}^{V}(v_m)_\ell}{\prod_{m=1}^{W}(w_m)_\ell}$$

$$\frac{1}{\left(1+\frac{n}{p}\right)_{-\frac{q}{p}j+\frac{r}{p}k+\frac{s}{p}\ell}} \frac{\left(\frac{\lambda}{\alpha^{\frac{q}{p}}}\right)^j}{j!} \frac{\left(\beta\alpha^{\frac{r}{p}}\right)^k}{k!} \frac{\left(\mu\alpha^{\frac{s}{p}}\right)^\ell}{\ell!} t^n. \tag{16}$$

On using the definition of the Srivastava-Daoust hypergeometric functions (11) in the r.h.s. of equation (16), we obtain assertion (15). ■

3 Applications

I. Taking $A = B = C = D = G = H = V = W = 0$, $\alpha = \frac{u}{2}$, $\lambda = \frac{v}{2}$, $\beta = -\frac{u}{2}$, $r = p$, $\mu = -\frac{v}{2}$, $s = q$ in Theorem 2, we get

$$\begin{aligned} &\exp\left[\frac{u}{2}\left(t^p - \frac{1}{t^p}\right) + \frac{v}{2}\left(t^q - \frac{1}{t^q}\right)\right] \\ &= \sum_{n=-\infty}^{\infty} \frac{\left(\frac{u}{2}\right)^{\frac{n}{p}}}{\Gamma\left(1+\frac{n}{p}\right)} \times \\ &\times F^{0:0;0;0}_{1:0;0;0}\left(\begin{array}{c} - : - ; - ; - ; \\ \left[1+\frac{n}{p} : -\frac{q}{p}, \frac{r}{p}, \frac{s}{p}\right] : - ; - ; - ; \end{array} \frac{v}{2}\left(\frac{u}{2}\right)^{-\frac{q}{p}}, \ -\frac{u^2}{4}, \ -\frac{v}{2}\left(\frac{u}{2}\right)^{\frac{q}{p}}\right) t^n. \end{aligned} \tag{17}$$

On comparison of equations (17) and (1), the two-dimensional Bessel functions $J_n^{p,q}(u,v)$ have the following explicit representation

$$\begin{aligned} J_n^{p,q}(u,v) &= \frac{\left(\frac{u}{2}\right)^{\frac{n}{p}}}{\Gamma\left(1+\frac{n}{p}\right)} \times \\ &\times F^{0:0;0;0}_{1:0;0;0}\left(\begin{array}{c} - : - ; - ; - ; \\ \left[1+\frac{n}{p} : -\frac{q}{p}, \frac{r}{p}, \frac{s}{p}\right] : - ; - ; - ; \end{array} \frac{v}{2}\left(\frac{u}{2}\right)^{-\frac{q}{p}}, \ -\frac{u^2}{4}, \ -\frac{v}{2}\left(\frac{u}{2}\right)^{\frac{q}{p}}\right). \end{aligned}$$

II. Taking $A = B = C = D = G = H = V = W = 0$, $\alpha = x$, $\lambda = 0$, $\beta = -y$, $r = m$, $\mu = 0$ in Theorem 2, we get

$$\exp\left[xt^p - \frac{y}{t^m}\right] = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k x^{\frac{n+mk}{p}} y^k}{\Gamma\left(\frac{n+mk}{p}+1\right) k!} t^n. \tag{18}$$

On comparison of equations (18) and (4), we get the explicit representation of the two-variable Bessel functions $\mathcal{D}_n^{(p,m)}(x,y)$ defined by equation (5). In view of equations (6)–(9), we can find the explicit representations of the special cases of the two-variable Bessel functions $\mathcal{D}_n^{(p,m)}(x,y)$ by suitable substitutions of the variable and indices in equation (18).

III. Taking $A = B = C = D = G = H = V = W = 0$, $\alpha = \frac{ix}{p+q}$, $\lambda = 0$, $\beta = \frac{ix}{p+q}$, $r = q$, $\mu = 0$ in Theorem 2, we get

$$\exp\left[\frac{ix}{p+q}(t^p + t^{-q})\right] = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{\left(\frac{ix}{p+q}\right)^{\frac{n}{p}+\left(1+\frac{r}{p}\right)k}}{\Gamma\left(\frac{n+qk}{p}+1\right) k!} t^n. \tag{19}$$

On comparison of equations (19) and (3), the Bessel functions $J_n^{(p,q)}(x)$ have the following explicit representation:

$$J_n^{(p,q)}(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{ix}{p+q}\right)^{\frac{n}{p}+\left(1+\frac{r}{p}\right)k}}{\Gamma\left(\frac{n+qk}{p}+1\right)\ k!}.$$

4 Conclusion

We conclude our present investigation by observing that several other Laurent type hypergeometric generating relations for the complex special functions can also be deduced in an analogous manner.

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