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Nonexistence For The "Missing" Similarity Boundary-Layer Flow*

Joseph Edward Paullet†

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Abstract

This note considers the boundary value problem

$$\phi''(\eta) + \lambda\phi'(\eta) + \phi(\eta)^2 = 0, \quad \eta \geq 0, \quad \lambda > 0,$$

subject to

$$\phi(0) = 1 \text{ and } \phi(\infty) = 0,$$

which arises in certain situations of boundary layer flow. Previous work on the problem established the existence of a $\lambda_{\min} \in [1, 2/\sqrt{3}]$ such that solutions exist for $\lambda \geq \lambda_{\min}$. It has been conjectured that for $\lambda < \lambda_{\min}$ no solution exists. We partially resolve this conjecture by proving that for $\lambda \leq \sqrt{2/3} \approx .8165$ no solution to the boundary value problem exists.

1 Introduction

In [1] and [2], Magyari *et al.* consider the boundary value problem (BVP):

$$\phi''(\eta) + \lambda\phi'(\eta) + \phi(\eta)^2 = 0 \text{ for } \eta \geq 0, \tag{1}$$

subject to

$$\phi(0) = 1 \text{ and } \phi(\infty) = 0. \tag{2}$$

This BVP arises in two distinct physical situations. One is in steady boundary-layer flow due to a moving permeable flat surface in a quiescent viscous fluid [1]. The other is in free convection boundary-layer flow of a Darcy-Boussinesq fluid from a heated vertical permeable plate [2]. In both of these situations, the usual similarity variable transformation produces a valid reduced model most of the time. However, in the first situation, when the surface is stretching with inverse-linear velocity, Magyari *et al.* [1] show that a logarithmic term in the wall coordinate must be added to the usual expression for the stream function in order to obtain a correct reduction; that given by (1-2). A similar term must be included in the second situation when the wall temperature distribution is inverse-linear [2]; again resulting in the BVP (1-2).

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†School of Science, Pennsylvania State University at Erie, The Behrend College, Erie, Pennsylvania 16563-0203, USA.

Magyari *et al.* call these two special cases "missing" boundary-layer flows because the usual similarity variable reduction misses valid and physically relevant results.

For the BVP (1-2), Magyari *et al.* show that no solution exists for $\lambda \leq 0$. Numerically, they find a value $\lambda_{\min} \approx 1.079131$ such that a unique solution exists for $\lambda = \lambda_{\min}$ and multiple solutions exist for all $\lambda > \lambda_{\min}$. Recently, Zhang [3] proved that there exists a $\lambda_{\min} \in [1, 2/\sqrt{3}]$ such that for $\lambda \geq \lambda_{\min}$ a solution to the BVP (1-2) exists. Existence or nonexistence of solutions for $0 < \lambda < \lambda_{\min}$ remains an open question. We partially resolve this question by proving that for $0 < \lambda \leq \sqrt{2/3} \approx 0.8165$ no solution to the BVP (1-2) exists.

2 Nonexistence Result

The following Theorem is our main result.

THEOREM. For $0 < \lambda \leq \sqrt{2/3}$ no solution to the boundary value problem (1-2) exists.

PROOF. Consider the initial value problem (IVP) given by

$$\phi''(\eta) + \lambda\phi'(\eta) + \phi(\eta)^2 = 0 \text{ for } \eta \geq 0, \tag{3}$$

subject to

$$\phi(0) = 1 \text{ and } \phi'(0) = \alpha, \tag{4}$$

where α is a free parameter. By standard existence and uniqueness theory, the IVP (3-4) will have a unique local solution for any value of α. We will show that there is no value of α such that the solution of the IVP (3-4) will exist for all $\eta \geq 0$ and satisfy the desired boundary condition at infinity, $\phi(\infty) = 0$.

We begin by listing some properties that such a solution must satisfy. First note that the ODE (3) implies that $\phi(\eta)$ cannot have a minimum. Thus, if $\alpha \leq 0$ gives a solution to the BVP, then $\phi(\eta)$ is monotonically decreasing for all $\eta > 0$ and tends to zero as $\eta \to \infty$. If $\alpha > 0$ gives a solution, then $\phi(\eta)$ must attain a positive maximum and then monotonically decrease to zero as η goes to infinity.

A differentiation of (3) yields

$$\phi'''(\eta) + \lambda\phi''(\eta) + 2\phi(\eta)\phi'(\eta) = 0.$$

Note that after ϕ is ultimately decreasing, ϕ' cannot have a maximum. Thus for a solution, ϕ' is *ultimately* monotonically increasing and bounded above by zero. Thus $\phi'(\infty) \leq 0$ exists, and since $\phi(\infty)$ also exists, we must then have $\phi'(\infty) = 0$.

Next we derive several integral relationships that any solution must satisfy. An integration of the ODE (3) from 0 to η gives

$$\phi'(\eta) - \alpha + \lambda\phi(\eta) - \lambda + \int_0^\eta \phi(t)^2 dt = 0.$$

Letting η tend to infinity and solving for the integral results in

$$\int_0^\infty \phi(t)^2 dt = \alpha + \lambda. \tag{5}$$

Multiplying the ODE (3) by ϕ' and integrating from 0 to η we obtain

$$\frac{\phi'(\eta)^2 - \alpha^2}{2} + \lambda \int_0^\eta \phi'(t)^2\, dt + \frac{\phi(\eta)^3 - 1}{3} = 0.$$

Again letting η tend to infinity and solving for the integral gives

$$\int_0^\infty \phi'(t)^2 dt = \frac{2 + 3\alpha^2}{6\lambda}. \tag{6}$$

Finally, multiplying the ODE (3) by ϕ and integrating, by parts where necessary, from 0 to η we obtain

$$\phi(\eta)\phi'(\eta) - \alpha - \int_0^\eta \phi'(t)^2 dt + \frac{\lambda\left(\phi(\eta)^2 - 1\right)}{2} + \int_0^\eta \phi(t)^3 dt = 0.$$

Letting η tend to infinity results in

$$\int_0^\infty \phi(t)^3 dt = \alpha + \frac{\lambda}{2} + \int_0^\infty \phi'(t)^2 dt. \tag{7}$$

Now, if $\alpha \le 0$ gives a solution to the BVP (1-2), then the solution is monotonically decreasing and $0 < \phi(\eta) < 1$ for all $\eta > 0$. Thus $\phi(\eta)^3 < \phi(\eta)^2$ for all $\eta > 0$. Using this fact along with (5), (6) and (7) we obtain

$$\alpha + \frac{\lambda}{2} + \frac{2 + 3\alpha^2}{6\lambda} < \alpha + \lambda, \tag{8}$$

or, after rearranging terms, $2 + 3\alpha^2 < 3\lambda^2$, which cannot hold if $\lambda \le \sqrt{2/3}$. Thus for $\lambda \le \sqrt{2/3}$ no solution to the BVP (1-2) exists for which $\alpha \le 0$.

Next consider the possibility that $\alpha > 0$ gives a solution. As noted earlier, such a solution must attain a positive maximum, necessarily above one, and then decrease monotonically toward zero. Thus there exists a point $\eta_0 > 0$ at which $\phi(\eta)$ decreases through one. In the above expressions, we can integrate from η_0 to $\eta > \eta_0$ and in (5), (6) and (7) replace the lower limit of integration with η_0 and replace α with $\phi'(\eta_0)$. Thus, for $\eta > \eta_0$ we again have $0 < \phi(\eta) < 1$ and the exact same argument now implies that

$$2 + 3\phi'(\eta_0)^2 < 3\lambda^2, \tag{9}$$

which is again contradicted if $\lambda \le \sqrt{2/3}$, proving the theorem.

References

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