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THE CONDITION FOR GENERALIZING INVERTIBLE SUBSPACES IN CLIFFORD ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a universal Clifford algebra induced by m-dimensional real linear space with basis $\{e_1,e_2...,e_m\}$. The necessary and sufficient condition for the subspaces of form $L_1=lin\{e_0,e_1,...,e_m,e_{m+1},...,e_{m+s}\}$ to be invertible is $m\equiv 2 \pmod 4$, $s=1$ and $e_{m+1}=e_{12...m}$ (see [2]). In this paper we improve this assertion for the subspaces of the form $L=lin\{e_0,e_{A_1},...,e_{A_m},e_{A_{m+1}},...,e_{A_{m+s}}\}$, where $A_i\subseteq\{1,2,...,m\}$ $(i=1,2,...,m+s)$.

1. Introduction

Let V_m be an $m-$dimensional $(m \geq 1)$ real linear space with basis $\{e_1, e_2, \ldots, e_m\}$. Consider the 2^m-dimensional real space $\mathcal{A}$ with basis

$$E = \{e_\emptyset, e_{\{1\}}, \ldots, e_{\{m\}}, e_{\{1,2\}}, \ldots, e_{\{m-1,m\}}, \ldots, e_{\{1,2,\ldots,m\}}\},$$

where $e_{\{i\}} := e_i \ \ (i = 1, 2, \ldots, m)$.

In the following, for each $K = \{k_1, k_2, \ldots, k_t\} \subseteq \{1, 2, \ldots, m\}$ we write $e_K = e_{k_1k_2\ldots k_t}$ with $e_\emptyset = e_0$, and so

$$E = \{e_0, e_1, \ldots, e_m, e_{12}, \ldots, e_{m-1m}, \ldots, e_{12\ldots m}\}.$$

The product of two elements $e_A, e_B \in E$ is given by

$$(1) \qquad e_Ae_B = (-1)^{\sharp(A\cap B)}(-1)^{p(A,B)}e_{A\triangle B}; \quad A, B \subset \{1, 2, \ldots, m\},$$

where

$$\begin{cases} p(A, B) = \sum\limits_{j\in B} p(A, j), \\ p(A, j) = \sharp\{i \in A : i > j\}, \\ A \triangle B = (A\backslash B) \cup (B\backslash A) \end{cases}$$

and $\sharp A$ denotes the number of elements of A.

Each element $a = \sum_A a_A e_A \in \mathcal{A}$ is called a Clifford number. The product of two Clifford numbers $a = \sum_A a_A e_A;\ b = \sum_B b_B e_B$ is defined by the formula

$$ab = \sum_A \sum_B a_A b_B e_A e_B.$$

It is easy to check that in this way $\mathcal{A}$ is turned into a linear associative non-commutative algebra over $\mathbf{R}$. It is called the Clifford algebra over V_m.

It follows at once from the multiplication rule (1) that $e_\emptyset$ is identity element, which is denoted by e_0 and in particular

$$e_i e_j + e_j e_i = 0 \text{ for } i \neq j; \quad e_j^2 = -1 \quad (i, j = 1, 2, \dots, m)$$

and

$$e_{k_1 k_2 \dots k_t} = e_{k_1} e_{k_2} \dots e_{k_t}; \quad 1 \leq k_1 < k_2 < \dots < k_t \leq m.$$

The involution for basic vectors is given by

$$\bar{e}_{k_1 k_2 \dots k_t} = (-1)^{\frac{t(t+1)}{2}} e_{k_1 k_2 \dots k_t}.$$

For any $a = \sum_A a_A e_A \in \mathcal{A}$, we write $\bar{a} = \sum_A a_A \bar{e}_A$. For any Clifford number $a = \sum_A a_A e_A$, we write $|a| = \left(\sum_A a_A^2\right)^{\frac{1}{2}}$.

2. Result and Proof

We use the following definitions.

(i) An element $a \in \mathcal{A}$ is said to be invertible if there exists an element a^{-1} such that $aa^{-1} = a^{-1}a = e_0$; a^{-1} is said to be the inverse of a.

(ii) A subspace $X \subset \mathcal{A}$ is said to be invertible if every non-zero element in X is invertible in $\mathcal{A}$.

(iii) $L(u_1, u_2, \dots, u_n) = lin\{u_1, u_2, \dots u_n\}, u_i \in \mathcal{A} \quad (i = 1, 2, \dots, n)$.

It is well-known (see [1]) that for any special Clifford number of the form $a = \sum_{i=0}^{m} a_i e_i \neq 0$ we have $a^{-1} = \frac{\bar{a}}{|a|^2}$. So $L(e_0, e_1, \dots, e_m)$ is invertible, and if $m \equiv 2 \pmod 4$ (see [2]), then every $a = \sum_{i=0}^{m+1} a_i e_i \neq 0$, where $e_{m+1} = e_{12\dots m}$ is invertible and $a^{-1} = \dfrac{\bar{a}}{|a|^2}$. So $L(e_0, e_1, \dots, e_m, e_{m+1})$ is invertible.

We shall need the following lemmas.

Lemma 1. (see Lemma 1 [3]) *If* $L(e_{A_1}, e_{A_2}, \ldots, e_{A_k})$, *where* $e_{A_i} \in E, e_{A_i} \neq e_{A_j}$ *for* $i \neq j,\ \ i, j \in \{1, 2, \ldots, k\}$, *is invertible if and only if* $L(e_{A_1}\bar{e}_{A_k}, e_{A_2}\bar{e}_{A_k}, \ldots, e_{A_k}\bar{e}_{A_k})$ *is invertible.*

By Lemma 1 we shall study subspaces of $\mathcal{A}$ in the form $L(e_0, e_{A_1}, \ldots, e_{A_l})$.

Lemma 2. (see Lemma 3 [3]) $L(e_0, e_{A_1}, \ldots, e_{A_l})$, $e_{A_i} \in E, e_{A_i} \neq e_{A_j}$ *for* $i \neq j$, *is invertible if and only if*

$$e_{A_i}\bar{e}_{A_j} + e_{A_j}\bar{e}_{A_i} = 0 \text{ for } i \neq j, \quad i, j \in \{0, 1, 2, \ldots, m\}, \text{ where } e_{A_0} = e_0.$$

Lemma 3. (see Theorem [3]) *If* $L(e_0, e_{A_1}, e_{A_2}, \ldots, e_{A_l})$, $e_{A_i} \in E, e_{A_i} \neq e_{A_j}$ *for* $i \neq j,\ \ i, j \in \{1, 2, \ldots. l\}$ *is invertible, then*

(i) $l \leq m + 1$.

(ii) *If* $l = m + 1$, *then*

$$\text{either } e_{A_l} = e_{A_1}e_{A_2}\ldots e_{A_{l-1}} \qquad \text{or} \qquad e_{A_l} = -e_{A_1}e_{A_2}\ldots e_{A_{l-1}}.$$

The purpose of this paper is to prove the following.

Theorem. $L(e_0, e_{A_1}, \ldots, e_{A_m}, e_{A_{m+1}}, \ldots, e_{A_{m+s}})$ *is invertible if and only if the following conditions simultaneously hold:*

(1) $e_{A_i}\bar{e}_{A_j} + e_{A_j}\bar{e}_{A_i} = 0$ *for* $i \neq j \quad i, j \in \{0, 1, 2, \ldots m\}$, *where* $e_{A_0} = e_0$,

(2) $m \equiv 2 \pmod 4$,

(3) $s = 1$,

(4) *Either* $e_{A_{m+1}} = e_{A_1}e_{A_2}\ldots e_{A_m}$ *or* $e_{A_{m+1}} = -e_{A_1}e_{A_2}\ldots e_{A_m}$.

Proof. First we prove the sufficiency. From $e_{A_i}\bar{e}_{A_j} + e_{A_j}\bar{e}_{A_i} = 0$ for $i \neq j, \quad i, j \in \{0, 1, \ldots m\}$ we have

$$\bar{e}_{A_i}\bar{e}_{A_j} + \bar{e}_{A_j}\bar{e}_{A_i} = 0 \text{ and } e_{A_i} + \bar{e}_{A_i} = 0 \text{ for } i \neq j, \quad i, j \in \{1, \ldots m\}.$$

We shall prove that $e_{A_k}\bar{e}_{A_{m+1}} + e_{A_{m+1}}\bar{e}_{A_k} = 0$ for $k \in \{0, 1, \ldots m\}$. For $k = 0$, by $\overline{ab} = \bar{b}\bar{a}$ and by $m \equiv 2 \pmod 4$, we get that

$$\begin{aligned} e_0\bar{e}_{A_{m+1}} + e_{A_{m+1}}\bar{e}_0 &= \overline{e_{A_1}e_{A_2}\ldots e_{A_m}} + e_{A_1}e_{A_2}\ldots e_{A_m} \\ &= \bar{e}_{A_m}\bar{e}_{A_{m-1}}\ldots\bar{e}_{A_1} + e_{A_1}e_{A_2}\ldots e_{A_m} \\ &= (-1)^m e_{A_m}e_{A_{m-1}}\ldots e_{A_1} + e_{A_1}e_{A_2}\ldots e_{A_m} \end{aligned}$$

$$= (-1)^m (-1)^{\frac{m(m-1)}{2}} e_{A_1} e_{A_2} \dots e_{A_m} + e_{A_1} e_{A_2} \dots e_{A_m}$$
$$= -e_{A_1} e_{A_2} \dots e_{A_m} + e_{A_1} e_{A_2} \dots e_{A_m} = 0.$$

For $k \in \{1, 2, \dots, m\}$ we have

$$e_{A_k} \bar{e}_{A_{m+1}} + e_{A_{m+1}} \bar{e}_{A_k} = e_{A_k} \bar{e}_{A_m} \dots \bar{e}_{A_k} \dots \bar{e}_{A_1} + e_{A_1} \dots e_{A_k} \dots e_{A_m} \bar{e}_{A_k}$$
$$= (-1)^{m-k} \bar{e}_{A_m} \dots e_{A_k} \bar{e}_{A_k} \dots \bar{e}_{A_1} + (-1)^{m-k} e_{A_1} \dots e_{A_k} \bar{e}_{A_k} \dots e_{A_m}$$
$$= (-1)^{m-k} \big[(-1)^{m-1} e_{A_m} \dots e_{A_{k+1}} e_{A_{k-1}} \dots e_{A_1} + e_{A_1} \dots e_{A_{k-1}} e_{A_{k+1}} \dots e_{A_m} \big]$$
$$= (-1)^{m-k} \big[-(-1)^{\frac{(m-1)(m-2)}{2}} e_{A_1} \dots e_{A_{k-1}} e_{A_{k+1}} \dots e_{A_m}$$
$$+ e_{A_1} \dots e_{A_{k-1}} e_{A_{k+1}} \dots e_{A_m} \big] = 0.$$

Take $0 \neq a = a_0 e_0 + \sum\limits_{i=1}^{m+1} a_i e_{A_i} \in L(e_0, e_{A_1}, \dots e_{A_m}, e_{A_{m+1}})$.

Let $a^{-1} = \dfrac{1}{|a|^2} \left(a_0 e_0 + \sum\limits_{i=1}^{m+1} a_i \bar{e}_{A_i} \right)$. Then

$$a \cdot a^{-1} = \frac{1}{|a|^2} \left(a_0 e_0 + \sum_{i=1}^{m+1} a_i e_{A_i} \right) \left(a_0 e_0 + \sum_{i=1}^{m+1} a_j \bar{e}_{A_j} \right)$$
$$= \frac{1}{|a|^2} \left[a_0^2 e_0 + a_0 \left(\sum_{i=1}^{m+1} a_i e_{A_i} + \sum_{j=1}^{m+1} a_j \bar{e}_{A_j} \right) + \sum_{i=1}^{m+1} a_i^2 e_{A_i} \bar{e}_{A_i} \right.$$
$$\left. + \sum_{i<j} a_i a_j (e_{A_i} \bar{e}_{A_j} + e_{A_j} \bar{e}_{A_i}) \right] = \frac{1}{|a|^2} \left(\sum_{i=0}^{m+1} a_i^2 \right) e_0 = e_0.$$

Similarly, one can check the equality $a^{-1} \cdot a = e_0$.

Now we prove the necessity. By Lemma 2 we have $e_{A_i} \bar{e}_{A_j} + e_{A_j} \bar{e}_{A_i} = 0$ for $i \neq j;\ \ i, j \in \{0, 1, \dots, m\}$ and by Lemma 3 we get that $s = 1$ and

$$\text{either} \quad e_{A_{m+1}} = e_{A_1} e_{A_2} \dots e_{A_m} \quad \text{or} \quad e_{A_{m+1}} = -e_{A_1} e_{A_2} \dots e_{A_m}.$$

We shall prove that $m \equiv 2 \pmod 4$. From $e_{A_i} \bar{e}_{A_j} + e_{A_j} \bar{e}_{A_i} = 0$ for $i \neq j;\ \ i, j \in \{0, 1, \dots, m\}$ one gets

$e_{A_i} + \bar{e}_{A_i} = 0, \;\; i \in \{1, 2, \ldots, m\}$. Hence either $\sharp A_i = 4p_i + 1$ or $\sharp A_i = 4p_i + 2 \;\; (p_i \in I\!N)$, $i \in \{1, 2, \ldots, m\}$. So $e_{A_i} e_{A_i} = -e_0 \quad (i = 1, 2, \ldots, m)$.

Let $m \equiv 0 \pmod 4$ or $m \equiv 3 \pmod 4$. Choosing $a = e_0 + e_{A_{m+1}}$ and $b = e_0 - e_{A_{m-1}}$ we find

$$ab = e_0 + e_{A_{m+1}} - e_{A_{m+1}} - e_{A_{m+1}} e_{A_{m+1}} = e_0 - e_{A_1} \ldots e_{A_m} . e_{A_1} \ldots e_{A_m}$$

$$= e_0 - \left[(-1)^m (-1)^{\frac{m(m-1)}{2}} e_0 \right] = e_0 - (-1)^{\frac{m(m+1)}{2}} e_0 = e_0 - e_0 = 0.$$

Hence the non-zero numbers a and b are not invertible.

Let $m \equiv 1 \pmod 4$. Choosing $a = e_{A_1} + e_{A_{m+1}}$ and $b = e_{A_1} - e_{A_{m+1}}$ we get

$$ab = (e_{A_1} + e_{A_{m+1}})(e_{A_1} - e_{A_{m+1}})$$

$$= e_{A_1} e_{A_1} - e_{A_1} e_{A_{m+1}} + e_{A_{m+1}} e_{A_1} - e_{A_{m+1}} e_{A_{m+1}}$$

$$= -e_0 - e_{A_1} e_{A_1} e_{A_2} \ldots e_{A_m} + e_{A_1} e_{A_2} \ldots e_{A_m} e_{A_1} - (-1)^{\frac{m(m+1)}{2}} e_0$$

$$= e_{A_2} \ldots e_{A_m} + (-1)^{m-1} e_{A_1} e_{A_1} e_{A_2} \ldots e_{A_m} = e_{A_2} \ldots e_{A_m} - e_{A_2} \ldots e_{A_m} = 0.$$

Hence a and b are not invertible. So $m \equiv 2 \pmod 4$. The theorem is proved.

References

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