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ON POLYNOMIAL VALUES OF THE SUM AND THE PRODUCT OF THE TERMS OF LINEAR RECURRENCES

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Abstract. Let $G^{(i)}=\left\{G_x^{(i)}\right\}_{x=0}^{\infty}$ $(i=1,2,\dots,m)$ linear recursive sequences and let $F(x)=dx^q+d_px^p+d_{p-1}x^{p-1}+\dots+d_0$, where d and d_i's are rational integers, be a polynomial. In this paper we showed that for the equations $\sum_{i=1}^{m} G_{x_i}^{(i)}=F(x)$ and $\prod_{i=1}^{m} G_{x_i}^{(i)}=F(x)$ where x_i-s are non-negative integers, with some restriction, there are no solutions in x_i-s and x if $q>q_0$, where q_0 is an effectively computable positive constant.

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1. Introduction

Let $m \geq 2$ be an integer and define the linear recurrences $G^{(i)} = \left\{G_x^{(i)}\right\}_{x=0}^{\infty}$ $(i = 1,2,\dots,m)$ of order k_i by the recursion

(1) $$G_x^{(i)} = A_1^{(i)}G_{x-1}^{(i)} + A_2^{(i)}G_{x-2}^{(i)} + \dots + A_{k_i}^{(i)}G_{x-k_i}^{(i)} \quad (x \geq k_i \geq 2),$$

where the initial values $G_j^{(i)}$ and the coefficients $A_{j+1}^{(i)}$ $(j = 0,1,\dots,k_i-1)$ are rational integers. Suppose that

$$A_{k_i}^{(i)}\left(\left|G_0^{(i)}\right| + \left|G_1^{(i)}\right| + \dots + \left|G_{k_i-1}^{(i)}\right|\right) \neq 0$$

for any recurrences and denote the distinct roots of the characteristic polynomial

$$g^{(i)}(u) = u^{k_i} - A_1^{(i)}u^{k_i-1} - \dots - A_{k_i}^{(i)}$$

of the sequence $G^{(i)}$ by $\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_{t_i}^{(i)}$ $(t_i \geq 2)$. It is known that there exist uniquely determined polynomials $p_j^{(i)}(u) \in \mathbf{Q}(\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_{t_i}^{(i)})[u]$ $(j = 1,2,\dots,t_i)$ of degree less than the multiplicity $m_j^{(i)}$ of roots $\alpha_j^{(i)}$ such that for $x \geq 0$

(2) $$G_x^{(i)} = p_1^{(i)}(x)\left(\alpha_1^{(i)}\right)^x + p_2^{(i)}(x)\left(\alpha_2^{(i)}\right)^x + \dots + p_{t_i}^{(i)}(x)\left(\alpha_{t_i}^{(i)}\right)^x.$$

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Using the terminology of F. Mátyás [9], we say that $G^{(1)}$ is the dominant sequence among the sequences $G^{(i)}$ $(i = 1, 2, \dots, m)$ if $m_1^{(1)} = 1$, the polynomial $p_1^{(1)}(x) = a$ is a non-zero constant and, using the notation $\alpha_1^{(1)} = \alpha$,

$$|\alpha| = \left|\alpha_1^{(1)}\right| > \left|\alpha_2^{(1)}\right| \geq \dots \geq \left|\alpha_{t_1}^{(1)}\right| \text{ and } |\alpha| \geq \left|\alpha_j^{(i)}\right| \tag{3}$$

for $2 \leq i \leq m$ and $1 \leq j \leq t_i$. (Since $A_{k_1}^{(1)} \in \mathbf{Z} \setminus \{0\}$, therefore $|\alpha| > 1$.) In this case

$$G_x^{(1)} = a\alpha^x + p_2^{(1)}(x)\left(\alpha_2^{(1)}\right)^x + \dots + p_{t_1}^{(1)}(x)\left(\alpha_{t_1}^{(1)}\right)^x. \tag{4}$$

If $\left|\alpha_1^{(i)}\right| > \left|\alpha_j^{(i)}\right|$ $(j = 2, \dots, t_i)$ in a sequence $G^{(i)}$ and $m_1^{(i)} = 1$ then we denote $p_1^{(i)}(x)$ by a_i, in the case $i = 1$ by a.

In the following we assume that

$$F(x) = dx^q + d_p x^p + d_{p-1}x^{p-1} + \dots + d_0, \tag{5}$$

is a polynomial with rational integer coefficients, where $d \neq 0$, $q \geq 2$ and $q > p$.

In the paper we use the following notations:

$$\Sigma_{x_1, x_2, \dots, x_m} = \sum_{i=1}^{m} G_{x_i}^{(i)} \tag{6}$$

and

$$\mathcal{G}_{x_1, x_2, \dots, x_m} = \prod_{i=1}^{m} G_{x_i}^{(i)}, \tag{7}$$

where x_i-s are non-negative integers.

The Diophantine equation

$$G_n = F(x) \tag{8}$$

with positive integer variables n and x was investigated by several authors. It is known that if G is a nondegenerate second order linear recurrence, with some restrictions, and $F(x) = dx^q$ then the equation (8) have finitely many integer solutions in variables $n \geq 1$ and $q \geq 2$.

For general linear recurrences we know a similar result (see [11]). A more general result was proved by I. Nemes and A. Pethő [10], furthermore by P. Kiss [4].

Using some other conditions, B. Brindza, K. Liptai and L. Szalay [2] proved that the equation

$$G_{x_1}^{(1)} G_{x_2}^{(2)} = w^q$$

implies that q is bounded above, while L. Szalay [12] made the following generalization of this problem. Let $d \neq 0$ fixed integer and s a product of powers of given primes. Then, under some conditions, the equation $dG^{(1)}_{x_1}G^{(2)}_{x_2}\dots G^{(m)}_{x_m} = sw^q$ in positive integers $w > 1, q, x_1, \dots, x_m$ implies that q is bounded above by a constant.

The author in [8] showed that for the equation $G^{(1)}_n G^{(2)}_m = F(x)$, with some restriction, there are no solutions in n, m and x if $q > q_0$, where q_0 is an effectively computable positive constant.

With some restrictions, P. Kiss and F. Mátyás [7] proved an additive result in this theme, namely, if $\Sigma_{x_1,x_2,\dots,x_m} = sw^q$ for positive integers $w > 1, x_1, x_2, \dots, x_m, q$ and there is a dominant sequence among the sequences $G^{(i)}$, then q is bounded above.

P. Kiss investigated the difference between perfect powers and products or sums of terms of linear recurrences. Such a result is proved in [3] for the sequence $G^{(1)}$ which has the form of (4). Namely, under some restrictions, $\left|sw^q - G^{(1)}_x\right| > e^{cx}$ for all integers $w > 1, x, q$ and s, if x and $q > n_1$, where c and n_1 are effectively computable positive numbers. Using some conditions, P. Kiss and F. Mátyás [6] generalized this result substituting $G^{(1)}_x$ by $\prod_{i=1}^{m} G^{(i)}_{x_i}$, where the sequences $G^{(i)}$ have the form of (4).

F. Mátyás [8] proved a similar result in additive case.

2. Results and proofs

Using the notations mentioned above, we shall prove the following theorems.

Theorem 1. *Let $G^{(i)}$ $(i = 1, 2, \dots, m;\ m \geq 2)$ be linear recursive sequences of integers defined by (1). Suppose that $G^{(1)}$ is the dominant recurrence among the sequences $G^{(i)}$ and $\alpha \notin \mathbf{Z}$. Let $K > 1$ and $0 < \delta_1 < 1$ be real constants, $F(x)$ and $\Sigma_{x_1,x_2,\dots,x_m}$ are defined by (5) and (6) with the condition $p < \delta_1 q$. If*

$$x_1 > K \max_{2 \leq i \leq m} (x_i)$$

then the equation

$$\Sigma_{x_1,x_2,\dots,x_m} = F(x), \tag{9}$$

in positive integers $x \geq 2, x_1 > x_2, \dots, x_m, q$ implies that $q < q_1$, where q_1 is an effectively computable number depending on $K, \delta_1, F(x), m$ and the sequences $G^{(i)}$.

Theorem 2. *Let $G^{(i)}$ $(i = 1, 2, \dots, m;\ m \geq 2)$ be linear recursive sequences of integers defined by (1). Suppose that $|\alpha^{(i)}_1| > |\alpha^{(i)}_j|$ for $1 \leq i \leq m$ and $2 \leq j \leq t_i$,* *moreover $\alpha_1^{(i)}$-s are not integers. Let $0 < \gamma < 1$ and $0 < \delta_2 < 1$ be real constants, $F(x)$ and $\mathcal{G}_{x_1,x_2,\dots,x_m}$ are defined by (5) and (7) with the condition $p < \delta_2 q$. If $x_i > \gamma \max(x_1,\dots,x_m)$ for $i = 1,\dots,m$ then the equation*

$$\mathcal{G}_{x_1,x_2,\dots,x_m} = F(x), \tag{10}$$

in positive integers $x \geq 2, x_1 > x_2,\dots,x_t,q$ implies that $q < q_2$, where q_2 is an effectively computable number depending on $\gamma, \delta_2, F(x), m$ and the sequences $G^{(i)}$.

Remark. P. Kiss in [5] proved similar results with other conditions.

In what follows we need the following auxiliary results.

Lemma 1. *Let $\omega_1, \omega_2, \dots, \omega_n$ $(\omega_i \neq 0$ or $1)$ be algebraic numbers with heights at most $M_1, M_2, \dots, M_n \geq 4$, respectively. If $b_1, b_2, \dots, b_n$ are non-zero integers with $\max(|b_1|, |b_2|, \dots, |b_{n-1}|) \leq B$ and $|b_n| \leq B'$, $B' \geq 3$, furthermore*

$$\Lambda = |b_1 \log \omega_1 + b_2 \log \omega_2 + \dots + b_n \log \omega_n| \neq 0,$$

where the logarithms are assumed to have their principal values, then there exists an effectively computable positive constant C, depending only on n, $M_1, \dots, M_{n-1}$ and the degree of the field $\mathbf{Q}(\omega_1, \dots \omega_n)$ such that

$$\Lambda > \exp\left(-C \log B' \log M_n - B/B'\right).$$

Lemma 1. is a result of A. Baker (see Theorem 1. in [1] with $\delta = 1/B'$).

For the sake of brevity we introduce the following abbreviations. For non-negative integers $x_1, x_2, \dots, x_m$ let

$$\varepsilon_j^{(i)} = \frac{p_j^{(i)}(x_i)}{a} \frac{\left(\alpha_j^{(i)}\right)^{x_i}}{\alpha^{x_1}}, \quad \varepsilon_1 = \sum_{j=2}^{t_1} \varepsilon_j^{(1)}, \quad \varepsilon_2 = \sum_{i=2}^{m} \sum_{j=1}^{t_i} \varepsilon_j^{(i)} \tag{11}$$

and $\varepsilon = \varepsilon_1 + \varepsilon_2$. Using (2), (4) and (6)

$$\Sigma_{x_1,x_2,\dots,x_m} = a\alpha^{x_1} + \sum_{j=2}^{t_1} p_j^{(1)}(x_1) \left(\alpha_j^{(1)}\right)^{x_1} + \sum_{i=2}^{m} \sum_{j=1}^{t_i} p_j^{(i)}(x_i) \left(\alpha_j^{(i)}\right)^{x_i}$$

and by (11) we have

$$\Sigma_{x_1,x_2,\dots,x_m} = a\alpha^{x_1}(1 + \varepsilon_1 + \varepsilon_2) = a\alpha^{x_1}(1 + \varepsilon). \tag{12}$$

Let

$$\varepsilon_3 = \left(\frac{d_p}{d} \left(\frac{1}{x}\right)^{q-p}\right) \left(1 + \frac{d_{p-1}}{d_p} \left(\frac{1}{x}\right) + \dots + \frac{d_0}{d_p} \left(\frac{1}{x}\right)^p\right). \tag{13}$$

So (5) can be written in the form

$$F(x) = dx^q(1+\varepsilon_3). \tag{14}$$

The following three lemmas are due to F. Mátyás [8], where n_1, n_2, n_3 means effectively computable constants.

Lemma 2. *Let $G^{(1)}$ be the dominant sequence among the recurrences $G^{(i)}$ $(1 \leq i \leq m)$ defined by (1). Then there are effectively computable positive constants c_1 and n_1 depending only on the sequence $G^{(1)}$ such that*

$$|\varepsilon_1| < e^{-c_1 x_1}$$

for any $n_1 < x_1$.

Lemma 3. *Let $G^{(1)}$ be the dominant sequence among the recurrences $G^{(i)}$ $(1 \leq i \leq m)$ defined by (1), $1 < K \in \mathbf{R}$ and $x_1 > K \max\limits_{2 \leq i \leq m}(x_i)$. Then there are effectively computable positive constants c_2 and n_2 depending only on K and the sequences $G^{(i)}$ such that*

$$|\varepsilon_2| < e^{-c_2 x_1}$$

for any $n_2 < x_1$.

Lemma 4. *Suppose that the conditions of Lemma 2 and Lemma 3 hold. Then there exist effectively computable positive constants c_3, c_4, n_3 depending only on K and the sequences $G^{(i)}$ such that*

$$e^{c_3 x_1} < |\Sigma_{x_1, x_2, \ldots, x_m}| < e^{c_4 x_1}$$

for any integer $x_1 > n_3$.

The following lemma is due to P. Kiss and F. Mátyás [6].

Lemma 5. *Let γ be a real number with $0 < \gamma < 1$ and let $\mathcal{G}_{x_1,\ldots,x_m}$ be an integer defined by (7), where $x_1, \ldots, x_m$ are positive integers satisfying the condition $x_i > \gamma \max(x_1, \ldots, x_t)$ and $|\alpha_1^{(i)}| > |\alpha_j^{(i)}|$ for $1 \leq i \leq m$ and $2 \leq j \leq t_i$. Then there are effectively computable positive constants c_5 and n_4, depending only on the sequences $G^{(i)}$ and γ, such that*

$$\mathcal{G}_{x_1,\ldots,x_m} = \left(\prod_{i=1}^{m} a_i \alpha_i^{x_i}\right)(1+\varepsilon_4), \tag{15}$$

where

$$|\varepsilon_4| < e^{-c_5 \cdot \max(x_1,\ldots,x_m)}$$

for any $\max(x_1, \ldots, x_m) > n_4$.

Remark. In general $\alpha_1^{(i)}$ is named the dominant root of the i-th sequence, if $\left|\alpha_1^{(i)}\right| > \left|\alpha_j^{(i)}\right|$ for $2 \le j \le t_i$.

Proof of Theorem 1. In the proof $c_6, c_7, \dots$ denote effectively computable constants, which depend on $K, \delta_1, F(x)$ and the sequences $G^{(i)}$. Suppose that (9) holds with the conditions given in the Theorem 1. and x_1 is sufficiently large. Using (9), (14) and Lemma 4. we have

$$|dx^q(1+\varepsilon_3)| = |F(x)| = |\Sigma_{x_1,x_2,\dots,x_m}| < e^{c_6 x_1}. \tag{16}$$

Taking the logarithms of the both side we get

$$|\log|d| + q\log x + \log|1+\varepsilon_3|| < c_6 x_1$$

that is

$$q \log x < c_7 x_1. \tag{17}$$

Now, using (11) and (13), the equation (9) can be written in the form

$$\left|\frac{a\alpha^{x_1}}{dx^q}\right| = |1+\varepsilon_3|\,|1+\varepsilon|^{-1}. \tag{18}$$

We distinguish two cases. First we suppose that

$$a\alpha^{x_1} = dx^q.$$

Let $\alpha' \neq \alpha$ be any conjugate of α and let φ be an automorphism of $\overline{\mathbf{Q}}$ with $\varphi(\alpha) = \alpha'$. Moreover,

$$\varphi(a)\,(\varphi(\alpha))^{x_1} = \varphi(dx^q).$$

Thus

$$\frac{a}{\varphi(a)} = \left(\frac{\alpha'}{\alpha}\right)^{x_1}.$$

whence x_1 is bounded, which implies that q is bounded. Now we can suppose that $\frac{a\alpha^{x_1}}{dx^q} \neq 1$. Put

$$L_1 = \left|\log\left|\frac{a\alpha^{x_1}}{dx^q}\right|\right| = \left|\log|a| + x_1\log|\alpha| - q\log x - \log d\right|$$

and employ Lemma 1. with $M_4 = x, B' = q$ and $B = x_1$. We have

$$L_1 > \exp(-c_8 \log q \log x - \frac{x_1}{q}). \tag{19}$$

Using (9), (11), (12), (13), (14) and (17) we have

$$c_9 x^q < dx^q(1+\varepsilon_3) = a\alpha^{x_1}(1+\varepsilon_1+\varepsilon_2) < c_{10}x^q,$$

that is

$$c_{11}x^q < \alpha^{x_1} < c_{12}x^q.$$

Using (13), the previous inequalities and the condition $p < \delta_1 q$ we have

$$|\varepsilon_3| < \left(\frac{1}{x}\right)^{c_{13}(q-p)} < \left(\frac{1}{x}\right)^{c_{13}q(1-\delta_1)} < \exp(-c_{14}x_1). \tag{20}$$

Recalling that $|log(1+x)| \leq x$ and $|log(1-x)| \leq 2x$ for $0 \leq x < \frac{1}{2}$ and using (20), Lemma 2. and Lemma 3. we find that

$$\left|\log|1+\varepsilon_3|\,|1+\varepsilon|^{-1}\right| < \exp(-c_{15}x_1)$$

Using (17), (18), (19) and (20) we have the following inequalities

$$c_{15}x_1 < c_8 \log q \log x + \frac{x_1}{q} < c_8 \log q \frac{c_7 x_1}{q} + \frac{x_1}{q} < c_{16}x_1 \frac{\log q}{q}.$$

This implies

$$\frac{c_{15}}{c_{16}} < \frac{\log q}{q}.$$

The previous inequality can be satisfied by only finitely many q and this completes the proof.

Proof of Theorem 2. Similarly the previous proof, c_i-s denote effectively computable positive constants, which depend on $\gamma, \delta_2, F(x)$ and the sequences $G^{(i)}$. Suppose that (10) holds with the conditions given in Theorem 2. Let $x_1, \ldots, x_t$ be positive integers and let $x_0 = \max(x_1, \ldots, x_t)$. We suppose that α_s is the dominant root of the sequence which belongs to x_0. Using Lemma 5. we have

$$e^{c_{17}x_0} < \mathcal{G}_{x_1,\ldots,x_m} = F(x) < e^{c_{18}x_0} \tag{21}$$

if $x_0 > n_4$. So by (10) and (21) we get

$$|dx^q(1+\varepsilon_3)| = |F(x)| = |\mathcal{G}_{x_1,x_2,\ldots,x_m}| < e^{c_{18}x_0}. \tag{22}$$

Taking the logarithms of the both side we get

$$|\log|d| + q\log x + \log|1+\varepsilon_3|| < c_{18}x_0$$

that is

$$q \log x < c_{19} x_0. \tag{23}$$

The equation (10) can be written in the form

$$\frac{\prod_{i=1}^{m} a_i \alpha_i^{x_i}}{dx^q} = (1+\varepsilon_3)(1+\varepsilon_4)^{-1}. \tag{24}$$

We distinguish two cases. First we suppose that

$$\prod_{i=1}^{m} a_i \alpha_i^{x_i} = dx^q.$$

Let $\alpha_s' \neq \alpha_s$ be any conjugate of α_s and let φ be an automorphism of $\overline{\mathbf{Q}}$ with $\varphi(\alpha) = \alpha'$. Moreover,

$$\varphi\left(\prod_{i=1}^{m} a_i \alpha_i^{x_i}\right) = \varphi\left(dx^q\right)$$

that is

$$\prod_{i=1}^{m} a_i \alpha_i^{x_i} = \varphi\left(\prod_{i=1}^{m} a_i \alpha_i^{x_i}\right).$$

Since α dominant root, $\varphi(\alpha_i) \leq \alpha_i$ $i = 1, 2, \ldots, m$ we have

$$\left(\frac{\alpha_s}{\varphi(\alpha_s)}\right)^{x_0} \leq \frac{\varphi\left(\prod_{i=1}^{m} a_i\right)}{\prod_{i=1}^{m} a_i},$$

whence x_0 is bounded, which implies that q is bounded. Now we can suppose that $\prod_{i=1}^{t} a_i \alpha_i^{x_i} \neq dx^q$. Put

$$L_2 = \left|\log\left|\frac{\prod_{i=1}^{m} a_i \alpha_i^{x_i}}{dx^q}\right|\right| = \left|\sum_{i=1}^{m} \log|a_i| + \sum_{i=1}^{m} x_i \log|\alpha_i| - \log d - q \log x\right|$$

and employ Lemma 1. with $M_{2t+2} = x, B' = q$ and $B = x_0$. We have

$$L_2 > \exp(-c_{20} \log q \log x - \frac{x_0}{q}). \tag{25}$$

Using (15) and Lemma 5. we have

$$c_{20}x^q < dx^q(1+\varepsilon_3) = \prod_{i=1}^{m} a_i\alpha_i^{x_i}(1+\varepsilon_4) < c_{21}x^q$$

that is

$$\alpha_s^{x_0} < c_{21}x^q.$$

Using (13), the previous inequality and the condition $p < \delta_2 q$ we have

$$|\varepsilon_3| < \left(\frac{1}{x}\right)^{c_{22}(q-p)} < \left(\frac{1}{x}\right)^{c_{22}q(1-\delta_2)} < \exp(-c_{23}x_0). \tag{26}$$

Recalling that $|log(1+x)| \le x$ and $|log(1-x)| \le 2x$ for $0 \le x < \frac{1}{2}$ and using (26) and Lemma 5. we find that

$$\left|\log|1+\varepsilon_3|\,|1+\varepsilon_4|^{-1}\right| < \exp(-c_{24}x_0). \tag{27}$$

Using (23), (24), (25), and (27) we have the following inequalities

$$c_{24}x_0 < c_{20}\log q\log x + \frac{x_0}{q} < c_{20}\log q\frac{c_{19}x_0}{q} + \frac{x_0}{q} < c_{25}x_0\frac{\log q}{q}.$$

This implies

$$\frac{c_{24}}{c_{25}} < \frac{\log q}{q}.$$

The previous inequality can be satisfied by only finitely many q and this completes the proof.

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