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RESULTS CONCERNING PRODUCTS AND SUMS OF THE TERMS OF LINEAR RECURRENCES

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Abstract. Many papers have investigated perfect powers and polynomial values as terms of linear recursive sequences of rational integers. Many results show, under some restrictions, that if a term of a sequence is a perfect power or a polynomial value, then the exponent of the powers and the degree of the polynomials are bounded above. In this paper we show and prove some similar results where the terms are substituted by products and sums of the terms of sequences.

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1. Introduction

For a given positive integer $t \geq 1$ we define linear recursive sequences $G^{(i)} = \{G_n^{(i)}\}_{n=0}^{\infty}$ of order $t_i \geq 2$ $(i = 1, 2, \ldots, t)$ by the recursion formulae

$$G_n^{(i)} = A_1^{(i)} G_{n-1}^{(i)} + A_2^{(i)} G_{n-2}^{(i)} + \cdots + A_{t_i}^{(i)} G_{n-t_i}^{(i)},$$

where $A_1^{(i)}, \ldots, A_{t_i}^{(i)}$ and the initial values $G_0^{(i)}, \ldots, G_{t_i-1}^{(i)}$ are fixed rational integers such that $A_{t_i}^{(i)} \neq 0$ and the initial terms are not all zero for $1 \leq i \leq t$. The polynomial

$$g^{(i)}(x) = x^{t_i} - A_1^{(i)} x^{t_i-1} - \cdots - A_{t_i}^{(i)}$$

is called the characteristic polynomial of the sequence $G^{(i)}$ and we denote its distinct roots by $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_{k_i}^{(i)}$ and suppose that

$$|\alpha_1^{(i)}| \geq |\alpha_2^{(i)}| \geq \cdots \geq |\alpha_{k_i}^{(i)}|.$$

Denote the multiplicity of $\alpha_1^{(i)}, \ldots, \alpha_{k_i}^{(i)}$ by $m_1^{(i)}, \ldots, m_{k_i}^{(i)}$, respectively. Then, as it is well-known, the terms of the sequences can be expressed as

$$G_n^{(i)} = P_1^{(i)}(n)(\alpha_1^{(i)})^n + P_2^{(i)}(n)(\alpha_2^{(i)})^n + \cdots + P_{k_i}^{(i)}(n)(\alpha_{k_i}^{(i)})^n \tag{1}$$

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for any $n \geq 0$, where $P_j^{(i)}$ are polynomials of degree $m_j^{(i)} - 1$ and the coefficients of $P_j^{(i)}$ are algebraic numbers from the number field $\mathbf{Q}(\alpha_1^{(i)}, \dots, \alpha_{k_i}^{(i)})$. If $m_1^{(i)} = 1$ and $|\alpha_1^{(i)}| > |\alpha_j^{(i)}|$ $(j = 2, \dots, k_i)$ for some i, then $\alpha_1^{(i)}$ will be denote by α_i. In this case $|\alpha_i| > 1$, since $|A_{t_i}^{(i)}| \geq 1$, and by (1) we have

$$G_n^{(i)} = a_i \alpha_i^n + P_2^{(i)}(n)\left(\alpha_2^{(i)}\right)^n + \cdots + P_{k_i}^{(i)}(n)\left(\alpha_{k_i}^{(i)}\right)^n, \tag{2}$$

where $a_i \in \mathbf{Q}(\alpha_i, \alpha_2^{(i)}, \dots, \alpha_{k_i}^{(i)})$ and we suppose that $a_i \neq 0$. If $t = 1$ then we omit (i) in (2) and we write G_n instead of $G_n^{(1)}$.

In the following we need some notations. Let $p_1, \dots, p_r$ be given distinct prime numbers. In the results and theorems S will denote the set of integers defined by

$$S = \{\pm p_1^{e_1} \cdot p_2^{e_2} \cdots p_r^{e_r} : e_i \geq 0,\ 1 \leq i \leq r\}.$$

Furthermore $c_0, c_1, \dots, n_0, n_1, \dots$ will denote positive effectively computable constans depending only on t, the parameters of the sequences, the primes $p_1, \dots, p_r$ and the constans which are given in some of the mentioned results and theorems (δ, γ and K). We note that the constans can be exactly determined similary as in the papers [4] and [8].

Perfect powers and polynomial values among the terms of linear recurrences have been investigated for many years. For second order linear recurrences many particular results are known concerning perfect squares and higher powers in the sequences (see e.g. Cohn [2], Wylie [17], Mignotte and Pethő [9,11,12]). A general result was obtained by Shorey and Stewart [14] and Pethő [13]: Any non degenerate second order linear recursive sequence contains only finitely many perfect powers.

For general linear recurrences, which satisfy (2), Shorey and Stewart [14] proved that if $G_x \neq a\alpha^x$ and $G_x = dw^q$ for positive integers $w > 1, q > 1$ and a fixed integer $d \neq 0$, then $q < n_0$. In [3] we improved this result substituting d by integers $s \in S$, furthermore we showed, under some conditions, that $|sw^q - G_x| > e^{c_0 x}$ for all integers s, w and x with $s \in S$ and $x, q > n_1$. Similar results were obtain by Shorey and Stewart [15].

2. Results

If we replace G_x by the sums or products of the terms of linear recurrences $G^{(i)}$ we can obtain similar results as the above ones. E.g. Brindza, Liptai and Szalay [1] proved, under some conditions, that the equation

$$G_x^{(1)} G_y^{(2)} = w^q$$

can be satisfied only if q is bounded above. This result was extended by Szalay [16]. Now we present some other more general results. In the results we shall use the above notations and the following ones:

$$G_{x_1}^{(1)} \cdot G_{x_2}^{(2)} \cdots G_{x_t}^{(t)} = \Pi_{x_1,\ldots,x_t}$$

and

$$G_{x_1}^{(1)} + G_{x_2}^{(2)} + \cdots + G_{x_t}^{(t)} = \Sigma_{x_1,\ldots,x_t},$$

where $x_1, \ldots, x_t$ are positive integers.

Theorem 1. (Szalay [16]). *Let $G^{(i)}$ $(i = 1, \ldots, t)$ be linear recursive sequences defined in (2) and let $0 < \delta < 1$ be a real number. If $\Pi_{x_1,\ldots,x_t} \neq \Pi_{i=1}^t a_i \alpha_i^{x_i}$ and*

$$\Pi_{x_1,\ldots,x_t} = sw^q$$

with $w > 1$, $s \in S$ and $x_j > \delta \cdot \max(x_1, \ldots, x_t)$ for $1 \leq j \leq t$, then $q < n_2$.

Theorem 2. (Kiss and Mátyás [4]). *Let $G^{(i)}$ $(i = 1, \ldots, t)$ be linear recursive sequences defined in (2) and let $0 < \delta < 1$ be a fixed number. Then there is an effectively computable positive number c_1 such that if $sw^q \neq \Pi_{i=1}^t a_i \alpha_i^{x_i}$, then*

$$|sw^q - \Pi_{x_1,\ldots,x_t}| > e^{c_1 \cdot \max(x_1,\ldots,x_t)}$$

for any positive integer s, w, q, $x_1, \ldots,$ x_t satisfying the conditions $s \in S$, $w > 1$, $x_i > \delta \cdot \max(x_1, \ldots, x_t)$ and $\min(q, \max(x_1, \ldots, x_t)) > n_3$.

Theorem 3. (Kiss and Mátyás [5]). *Under the conditions of Theorem 2 concerning the sequences $G^{(i)}$ and integers $x_1, \ldots, x_t$, we have*

$$|s - \Pi_{x_1,\ldots,x_t}| > e^{c_2 \cdot \max(x_1,\ldots,x_t)}$$

for any $s \in S$ and $\max(x_1, \ldots, x_t) > n_4$.

Theorem 4. (Kiss and Mátyás [6]). *Let $G^{(1)}$ and $G^{(i)}$ $(i = 2, \ldots, t)$ be linear recurrences defined by (2) and (1), respectively, and let $K > 1$ be a real number. Suppose that $|\alpha_1| \geq |\alpha_j^{(i)}|$ for $i = 2, \ldots, t$ and $j = 1, \ldots, k_i$. If*

$$|\Sigma_{x_1,\ldots,x_t}| \neq |a_1 \alpha_1^{x_1}|$$

and

$$\Sigma_{x_1,\ldots,x_t} = sw^q$$

for positive integers $w > 1$, $q, x_1, \ldots, x_t$ and $s \in S$ such that

$$x_1 > K \cdot \max(x_2, \ldots, x_t),$$

then $q < n_5$.

Theorem 5. (Mátyás [8]). *Under the conditions of Theorem 4 for the sequences* $G^{(i)}$ *and integers* $x_1, \dots, x_t$ *we have*

$$|sw^q - \Sigma_{x_1,\dots,x_t}| > e^{c_3 x_1}$$

for any $s \in S$ *and* $\min(x_1, q) > n_6$.

Theorem 6. (Kiss and Mátyás [5]). *Under the conditions of Theorem 4 for the sequences* $G^{(i)}$ *and integers* $x_1, \dots, x_t$ *we have*

$$|s - \Sigma_{x_1,\dots,x_t}| > e^{c_4 x_1}$$

for any $s \in S$ *and* $x_1 > n_7$.

Corollary 1. *Under the conditions implied by Theorem 2 and Theorem 4, Theorem 3 and Theorem 6 imply that the relations*

$$\Pi_{x_1,\dots,x_t} \in S \quad \text{and} \quad \Sigma_{x_1,\dots,x_t} \in S$$

hold only for finitely many positive integers $x_1, \dots, x_t$.

If we replace sw^q in Theorem 1, 2, 4 and 5 by a polynomial, we can obtain similar results. Nemes and Pethő [10] furthermore Kiss [7] proved, that if G is a linear recurrence defined by (2) and $F(y)$ is a polynomial satisfying some conditions, then the equation $G_x = F(y)$ implies that the degree of $F(y)$ is bounded above. Now we give some generalizations of this result.

Theorem 7. *Let* $G^{(i)}$ $(i = 1, \dots, t)$ *be linear recursive sequences defined by (2) and let* $0 < \delta < 1$ *be a fixed positive real number. Further let*

$$F(y) = by^q + b_k y^k + b_{k-1} y^{k-1} + \dots + b_0 \tag{3}$$

be a polynomial of integer coefficients with $b \neq 0$ *and* $k < \gamma q$, *where* $0 < \gamma < 1$. *If* $\gamma < c_6$ *and* $by^q \neq \prod_{i=1}^{t} a_i \alpha_i^{x_i}$, *then*

$$|F(y) - \Pi_{x_1,\dots,x_t}| > e^{c_5 \cdot \max(x_1,\dots,x_t)}$$

for any positive integers y, q, $x_1, \dots,$ x_t *satisfying the conditions* $y > 1$, $x_i > \delta \cdot \max(x_1, \dots, x_t)$, *and* $\min(q, \max(x_1, \dots, x_t)) > n_8$.

Theorem 8. *Let* $G^{(i)}$ $(i = 1, \dots, t)$ *be linear recurrences and* $x_1, \dots x_t$ *positive integers which satisfy the conditions of Theorem 4. Let* $F(y)$ *be a polynomial given in Theorem 7. Then*

$$|F(y) - \Sigma_{x_1,\dots,x_t}| > e^{c_7 x_1}$$

for any positive integers $y > 1$, $x_1, \ldots, x_t$ *with* $\min(q, x_1) > n_9$.

Corollary 2. *From Theorem 7 and 8 it follows, that if the sequences* $G^{(i)}$, *the integers* $x_1, \ldots, x_t$ *and the polynomial* $F(y)$ *satisfy the conditions of Theorem 7 and Theorem 8, then the equations*

$$\Pi_{x_1,\ldots,x_t} = F(y)$$

and

$$\Sigma_{x_1,\ldots,x_t} = F(y)$$

imply the inequalities $q < n_{10}$ *and* $q < n_{11}$, *respectively.*

3. Proofs

The proofs of the Theorems 1–6 can be found in the papers mentioned in the theorems. The proofs are based upon Baker-type estimations of linear forms of logarithms of algebraic numbers, using the explicit form of the terms of the sequences.

Proof of Theorem 7. Let $G^{(i)}$ and $F(y)$ be linear recurrences given in the theorem and let $y, q, x_1, \ldots, x_t$ be positive integers such that $y, q > 1$, $k < \gamma q$ and $x_i > \delta \cdot \max(x_1, \ldots, x_t)$ for $i = 1, \ldots, t$ Denote by x the maximum values of $x_1, \ldots, x_t$, i.e.

$$x = \max(x_1, \ldots, x_t).$$

Suppose that

$$|F(y) - \Pi_{x_1,\ldots,x_t}| < e^{cx} \tag{4}$$

for some $c > 0$. Then by (2) and (3), using that $\delta x < x_i \leq x$ and $k < \gamma q$

$$\left| by^q(1+\varepsilon_1) - \left(\prod_{i=1}^{t} a_i \alpha_i^{x_i}\right)(1+\varepsilon_2)\right| < e^{cx} \tag{5}$$

follows, where

$$|\varepsilon_1| < e^{-c_8 q} \quad \text{and} \quad |\varepsilon_2| < e^{-c_9 x}$$

if $q, x > n_{12}$. By (5), using that $x_i > \delta x$, we obtain the inequalities

$$\left| \frac{by^q}{\prod_{i=1}^{t} a_i \alpha_i^{x_i}} - \frac{1+\varepsilon_2}{1+\varepsilon_1} \right| < \frac{e^x}{\left|\prod_{i=1}^{t} a_i \alpha_i^{x_i}\right|} \cdot \frac{1}{|1+\varepsilon_1|} < \frac{e^{cx}}{e^{c_{10}x}} < e^{-c_{11}x}$$

if $c < c_{10}$. From these it follows that

$$(6)\qquad 1-\varepsilon < \left| \frac{by^q}{\prod_{i=1}^{t} a_i \alpha_i^{x_i}} \right| < 1+\varepsilon,$$

where $0 \le \varepsilon < c_{12} \cdot \max(|\varepsilon_1|, |\varepsilon_2|, e^{-c_{11}x})$. By (6) we get the inequality

$$|by^q| < (1+\varepsilon) \left| \prod_{i=1}^{t} a_i \alpha_i^{x_i} \right| < e^{c_{13}x}$$

and so

$$(7)\qquad q \cdot \log y < c_{14} x.$$

Using (7), by Theorem 2 we have

$$|F(y) - \Pi_{x_1,\dots,x_t}| \ge \big||by^q - \Pi_{x_1,\dots,x_t}| - |d_k y^k + \dots + b_0|\big| \ge$$
$$|e^{c_{15}x} - y^{c_{16}k}| = |e^{c_{15}x} - e^{c_{16}k \cdot \log y}| >$$
$$|e^{c_{15}x} - e^{c_{16}\gamma q \cdot \log y}| > |e^{c_{15}x} - e^{c_{17}\gamma x}| > e^{c_{18}x}$$

if $c_{15} > c_{17}\gamma$, i.e. if $\gamma < c_{15}/c_{17}$. It contradicts to (4) if $c < c_{18}$, which proves the theorem with $c_5 = c_{18}$, $c_6 = c_{15}/c_{17}$ and $n_8 = \max(n_{12}, n_{13})$, where n_{13} is implied by Theorem 2.

Proof of Theorem 8. The theorem can be proved similary as Theorem 7 using the result of Theorem 5.

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