PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 96 (110) (2014), 233–238 DOI: 10.2298/PIM1410233S

ON A CONVERGENT PROCESS OF BERNSTEIN

László Szili and Péter Vértesi

Abstract. Bernstein in 1930 defined a convergent interpolation process based on the roots of the Chebyshev polynomials. We prove a similar statement for certain Jacobi roots.

1. Introduction. Preliminary results

1.1. In 1930, Bernstein [**1**] (cf. [**2**], too) defined the following convergent interpolatory process on the roots of

$$T_n(x) = \cos(n \arccos x) = \cos n\vartheta, \quad -1 \leqslant x \leqslant 1, \ \ 0 \leqslant \vartheta \leqslant \pi, \ \ n = 1, 2, \dots$$

(Chebyshev polynomials); the roots are

$$x_{kn} = \cos \vartheta_{kn} = \cos \frac{2k-1}{2n}\pi, \ \ k = 1, 2, \dots, n; \ \ n = 1, 2, \dots. \tag{1.1}$$

Let l, q be natural numbers; for simplicity we suppose that $n = 2lq$. We divide the nodes into q rows as follows.

$$\begin{matrix} x_{1n} & x_{2n} & \dots & x_{2l,n} \\ x_{2l+1,n} & x_{2l+2,n} & \dots & x_{4l,n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{2l(q-1)+1,n} & x_{2l(q-1)+2,n} & \dots & x_{2lq,n} \end{matrix}$$

If $f \in C$ (the set of continuous functions on $[-1, 1]$) and

$$\ell_{kn}(T, x) = \frac{T_n(x)}{T_n'(x_{kn})(x - x_{kn})}, \qquad k = 1, 2, \dots, n; \ \ n = 1, 2, \dots$$

are the Lagrange fundamental polynomials based on (1.1) we define the following interpolatory polynomials Q_{nl} if $l = 1, 2$ and 3.

$$\begin{aligned} Q_{n1}(f, x) \equiv Q_{n1}(f) = \big\{ f_1(\ell_1 + \ell_2) \big\} + \big\{ f_3(\ell_3 + \ell_4) \big\} \\ + \big\{ f_5(\ell_5 + \ell_6) \big\} + \dots + \big\{ f_{n-1}(\ell_{n-1} + \ell_n) \big\}, \end{aligned} \tag{1.2}$$

2010 *Mathematics Subject Classification:* 41A05.
Key words and phrases: interpolation, Bernstein process, Jacobi roots.
Dedicated to Professor Giuseppe Mastroianni on the occasion of his retirement.

$$
\begin{aligned}
(1.3)\qquad Q_{n2}(f,x) &\equiv Q_{n2}(f)\\
&= \{f_1(\ell_1+\ell_4)+f_2(\ell_2-\ell_4)+f_3(\ell_3+\ell_4)\}\\
&\quad+\{f_5(\ell_5+\ell_8)+f_6(\ell_6-\ell_8)+f_7(\ell_7+\ell_8)\}\\
&\quad+\{f_9(\ell_9+\ell_{12})+f_{10}(\ell_{10}-\ell_{12})+f_{11}(\ell_{11}+\ell_{12})\}+\dots\\
&\quad+\{f_{n-3}(\ell_{n-3}+\ell_n)+f_{n-2}(\ell_{n-2}-\ell_n)+f_{n-1}(\ell_{n-1}+\ell_n)\},
\end{aligned}
$$

$$
\begin{aligned}
(1.4)\qquad &Q_{n3}(f,x) \equiv Q_{n3}(f)\\
&= \{f_1(\ell_1+\ell_6)+f_2(\ell_2-\ell_6)+f_3(\ell_3+\ell_6)+f_4(\ell_4-\ell_6)+f_5(\ell_5+\ell_6)\}\\
&\quad+\{f_7(\ell_7+\ell_{12})+f_8(\ell_8-\ell_{12})+f_9(\ell_9+\ell_{12})\\
&\quad+f_{10}(\ell_{10}-\ell_{12})+f_{11}(\ell_{11}+\ell_{12})\}+\cdots\\
&\quad+\{f_{n-5}(\ell_{n-5}+\ell_n)+f_{n-4}(\ell_{n-4}-\ell_n)+f_{n-3}(\ell_{n-3}+\ell_n)\\
&\quad+f_{n-2}(\ell_{n-2}-\ell_n)+f_{n-1}(\ell_{n-1}+\ell_n)\}.
\end{aligned}
$$

The definitions for $l \geqslant 4$ are analogous:

$$
\begin{aligned}
(1.5)\quad &Q_{nl}(f,x) \equiv Q_{nl}(f)\\
&= \{f_1(\ell_1+\ell_{2l})+f_2(\ell_2-\ell_{2l})+\cdots+f_{2l-1}(\ell_{2l-1}+\ell_{2l})\}+\\
&+\{f_{2l+1}(\ell_{2l+1}+\ell_{4l})+f_{2l+2}(\ell_{2l+2}-\ell_{4l})+\cdots+f_{4l-1}(\ell_{4l-1}+\ell_{4l})\}+\cdots\\
&+\{f_{n-(2l-1)}(\ell_{n-(2l-1)}+\ell_n)+\cdots+f_{n-1}(\ell_{n-1}+\ell_n)\}.
\end{aligned}
$$

You may consult with [**1**] or [**2**] (above $f_k = f(x_{kn})$ and $\ell_k \equiv \ell_{kn}(T,x)$; moreover q is large enough).

If $N = n + r$, $n = 2lq$, $0 < r < 2l$, the definition of Q_{Nl} is as follows (cf. [**1**] or [**2**])

$$Q_{Nl}(f) := Q_{nl}(f) + \sum_{k=n+1}^{N} f_k \ell_k.$$

1.2. By the above definitions we have with $e_0(x) \equiv 1$

$$Q_{nl}(e_0,x) \equiv \sum_{k=1}^{n} \ell_{kn}(T,x) \equiv 1, \tag{1.6}$$

$$Q_{nl}(f,x_{kn}) = f(x_{kn}) \quad \text{if} \quad k \neq 2l,\ 4l,\ \dots,\ 2lq, \tag{1.7}$$

i.e. Q_{nl} interpolates at $n - q = 2lq - q$ nodes. This number is “very close" to n if the (fixed) l is large enough while q (and n, too) tends to infinity, i.e., for large l our Q_{nl} is “very close" to the Lagrange interpolation L_n. However, Q_{nl} converges for every $f \in C$, when $n \to \infty$ (cf. Proposition 1.1 and Theorem 2.1), which generally does not hold for L_n.

Later we use that (1.6) and (1.7) hold true for*arbitrary* point system.

1.3. In [**1**] Bernstein proved

Proposition 1.1. *Let l be a fixed positive integer and $f \in C$. Then*
$$\lim_{n\to\infty} \|f(x) - Q_{nl}(f,x)\| = 0.$$

Above, $\|g(x)\| = \max_{|x|\leqslant 1} |g(x)|$, $g \in C$. Actually, he proved for $N = n + r$, too; the case when $N = n + r$ demands only small technical changes in the proof.

1.4. The Bernstein process and its generalizations were exhaustively investigated by Kis (sometimes with coauthors). For more details we suggest the papers [**6, 7, 8**] and references therein.

2. The Bernstein process for Jacobi abscissas

2.1. The aim of this note is to prove a statement similar to Proposition 1.1 for Jacobi roots. Let the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ be defined by
$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\right\} \qquad (\alpha,\beta > -1).$$
For the roots $x_{kn}^{(\alpha,\beta)} = \cos\vartheta_{kn}^{(\alpha,\beta)}$, $0 < \vartheta_{kn}^{(\alpha,\beta)} < \pi$, of $P_n^{(\alpha,\beta)}(x)$ we have
$$-1 < x_{nn}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \cdots < x_{1n}^{(\alpha,\beta)} < 1.$$
Let
$$\ell_{kn}^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)\prime}(x_{kn})(x-x_{kn})}.$$

For a fixed positive integer l, we define $Q_{nl}^{(\alpha,\beta)}(f,x)$ according to (1.2)–(1.5); now ℓ_k and f_k stand for $\ell_{kn}^{(\alpha,\beta)}(x)$ and $f(x_{kn}^{(\alpha,\beta)})$, respectively. As we noticed we have the properties analogous to (1.6) and (1.7) for $Q_{nl}^{(\alpha,\beta)}(f,x)$, too.

2.2. We prove (compare with Vértesi [**3**] dealing with Lagrange interpolation)

Theorem 2.1. *Let l be a fixed positive integer, $n = 2lq$ $(q = 1, 2, \dots)$ and $f \in C$. Then*
$$\lim_{n\to\infty} \|f(x) - Q_{nl}^{(\alpha,\beta)}(f,x)\| = 0 \tag{2.1}$$
for any processes $Q_{nl}^{(\alpha,\beta)}$ supposing $-1 < \alpha,\beta < 0.5$.

Our statement follows from the next more informative pointwise estimations (compare with the result in Vértesi [**4**] on Lagrange interpolation).

Theorem 2.2. *Let l be fixed natural number. Then for arbitrary fixed $\alpha,\beta > -1$ and $f \in C$*
$$\left|Q_{nl}^{(\alpha,\beta)}(f,x) - f(x)\right| = O(1)\sum_{i=1}^{n}\omega\left(f;\frac{\sqrt{1-x^2}}{n}i + \frac{i^2}{n^2}\right)\frac{1}{i^\gamma} \tag{2.2}$$
uniformly in n and $x \in [-1,1]$, where $\gamma = \min(2; 1.5-\alpha; 1.5-\beta)$. $\big($*$\omega(f;t)$ is the modulus of continuity of $f(x)$.*$\big)$

2.3. It is easy to get (2.1) using Theorem 2.2. Indeed, let

$$\varepsilon_n = \begin{cases} \frac{1}{n}\log n & \text{if } -1 < \alpha, \beta \leqslant -0.5 \\ n^{\delta - 0.5} & \text{if } \max(\alpha, \beta) =: \delta > -0.5. \end{cases}$$

We have by (2.2)

$$\|Q_{nl}^{(\alpha,\beta)}(f,x) - f(x)\| = O(1)\omega(f;\varepsilon_n)$$

if $f \in C$, whence we obtain (2.1).

2.4. Another consequence of Theorem 2.2 is the following

Corollary 2.1. *If $-1 < \alpha, \beta \leqslant -0.5$ and $\omega(f;t) \sim t^\varrho$ $(0 < \varrho < 0.5)$ then for $f \in C$*

$$\left|Q_{nl}^{(\alpha,\beta)}(f,x) - f(x)\right| = O(1)\Big[\Big(\frac{1}{n}\sqrt{1-x^2}\Big)^\varrho + \frac{1}{n^{2\varrho}}\Big]$$

uniformly for n and $|x| \leqslant 1$.

This formula of Timan type can be obtained by simple calculation.

Other estimations showing the connections between the parameters $\gamma \in (0,2]$ and $\varrho \in (0,1]$ are as follows

$$\begin{aligned} &\left|Q_{nl}^{(\alpha,\beta)}(f,x) - f(x)\right| \\ &= O(1)\begin{cases} \Big(\frac{1}{n}\sqrt{1-x^2}\Big)^\varrho + \begin{cases} n^{-2\varrho} & \text{if } 0 < \varrho < \frac{1}{2}(\gamma-1), \\ n^{-2\varrho}\log n & \text{if } \varrho = \frac{1}{2}(\gamma-1), \\ n^{-\gamma+1} & \text{if } \frac{1}{2}(\gamma-1) < \varrho < \gamma - 1; \end{cases} \\ \Big(\frac{1}{n}\sqrt{1-x^2}\Big)^\varrho \log n + \frac{1}{n^{\gamma-1}} \quad \text{if } \varrho = \gamma - 1, \\ \frac{\big(\sqrt{1-x^2}\big)^\varrho}{n^{\gamma-1}} + \frac{1}{n^{\gamma-1}} \quad \text{if } \gamma - 1 < \varrho \leqslant 1 \end{cases} \end{aligned}$$

uniformly for n and $|x| \leqslant 1$. These formulae can be obtained by simple calculation.

2.5. It is interesting to compare (2.2) to

$$\left|H_n^{(\alpha,\beta)}(f;x) - f(x)\right| = O(1)\sum_{i=1}^n \omega\Big(f; \frac{1}{n}\sqrt{1-x^2}\, i + \frac{i^2}{n^2}\Big) i^{2\eta-1} \qquad (x \in [-1,1])$$

where $H_n^{(\alpha,\beta)}(f;x)$ is the Hermite–Fejér interpolatory polynomial of degree $\leqslant 2n-1$ defined by $H_n^{(\alpha,\beta)}\big(f;x_{kn}^{(\alpha,\beta)}\big) = f\big(x_{kn}^{(\alpha,\beta)}\big)$, $H_n^{(\alpha,\beta)\prime}\big(f;x_{kn}^{(\alpha,\beta)}\big) = 0$ $(k = 1,2,\dots,n)$, $f \in C$ and $\eta = \max(-0.5, \alpha, \beta)$ (see [**5**, 2.1]).

3. Proof of Theorem 2.2

We apply the main idea from [**4**]. Let $x = \cos\vartheta$, $x \in [-1,1]$, $\vartheta \in [0,\pi]$ and define the index $j = j(n)$ by $\min_{1\leqslant k\leqslant n}\big|x - x_{kn}^{(\alpha,\beta)}\big| = \big|x - x_{jn}^{(\alpha,\beta)}\big|$.

3.1. First let $l = 1$. By (1.5) and (1.6) we can write

$$\begin{aligned}(3.1)\qquad Q_{nl}^{(\alpha,\beta)}(f,x) - f(x) &= \sum_{k=1}^{q} \Big\{ \Big(f\big(x_{2k-1}^{(\alpha,\beta)}\big) - f(x) \Big) \Big(\ell_{2k-1}^{(\alpha,\beta)}(x) + \ell_{2k}^{(\alpha,\beta)}(x) \Big) \Big\} \\ &= \sum_{k=1}^{cq} \cdots + \sum_{k>cq} \cdots = \sum\nolimits_I + \sum\nolimits_{II}.\end{aligned}$$

Now we use Lemma 4.1 of [**3**], which says the following: Let $-1 < \alpha, \beta$ and $\varepsilon, \eta > 0$ be fixed. If $k \geqslant M$, $\vartheta_{kn}^{(\alpha,\beta)} \leqslant \pi - \varepsilon$, then for any $x \in [-1+\eta, 1]$ we have

$$(3.2)\qquad \big|\ell_{kn}^{(\alpha,\beta)}(x) + \ell_{k+1,n}^{(\alpha,\beta)}(x)\big| = O(1)\big|\ell_{kn}^{(\alpha,\beta)}(x)\big| \Big[\frac{1}{k} + \frac{k}{(k+j)(|k-j|+1)}\Big]$$

uniformly in x and k.

We note that instead of $\tilde{\ell}_{kn}^{(\alpha,\beta)}(x)$ of [**3**] one can write $\ell_{kn}^{(\alpha,\beta)}(x)$. Moreover (3.2) obviously holds true if $1 \leqslant k \leqslant M$ (maybe with another $O(1)$).

From (3.1) with obvious short notations we have

$$\begin{aligned}(3.3)\qquad \sum\nolimits_I = O(1) &\sum_{k=1}^{cq} |f(x_{2k-1}) - f(x)| \\ &\times \Big(|\ell_{2k-1}(x)| \Big[\frac{1}{2k-1} + \frac{2k-1}{(2k-1+j)(|2k-1-j|+1)}\Big]\Big)\end{aligned}$$

if $\alpha, \beta > -1$ and $\varepsilon, \eta > 0$ are fixed.

By (3.3) we get as in [**3**]: If $\gamma = \min(2; 1.5-\alpha; 1.5-\beta)$, then

$$(3.4)\qquad \sum_{k=1}^{n-1} |f(x_{2k-1}) - f(x)|\, |\ell_{2k-1}(x) + \ell_{2k}(x)| = O(1) \sum_{i=1}^{n} \omega\Big(f; \frac{\sin\vartheta}{n} i + \frac{i^2}{n^2}\Big) \frac{1}{i^\gamma}$$

uniformly in $x \in [-1, 1]$; see [**3**, 4.10], where $\sum |f - f_k|\, |\ell_k k^{-1}|$ (which by (3.2), is analogous to $\sum |f - f_k|\, |\ell_k + \ell_{k+1}|$) is estimated.

Let us remark that getting (3.4) we have to define $J = \big[\vartheta_{j-1,n}^{(\alpha,\beta)}, \vartheta_{j+1,n}^{(\alpha,\beta)}\big)$ and for $r = 1, 2, \dots$

$$I_r = \big[\vartheta_{j-2^r,n}^{(\alpha,\beta)}, \vartheta_{j-2^{r-1},n}^{(\alpha,\beta)}\big), \quad K_r = \big[\vartheta_{j+2^{r-1},n}^{(\alpha,\beta)}, \vartheta_{j+2^r,n}^{(\alpha,\beta)}\big)$$

instead of the definition (4.2) of [**4**].

From the above formulas we obtain our theorem for $l = 1$.

3.2. Now let $l = 2$. By (1.3) and (1.6) we get

$$\begin{aligned}(3.5)\qquad Q_{n2}(f) - f = &\big\{(f_1 - f)(\ell_1 + \ell_4) + (f_2 - f)(\ell_2 - \ell_4) + (f_3 - f)(\ell_3 + \ell_4)\big\}_1 \\ &+ \big\{(f_5 - f)(\ell_5 + \ell_8) + (f_6 - f)(\ell_6 - \ell_8) + (f_7 - f)(\ell_7 + \ell_8)\big\}_2 + \cdots.\end{aligned}$$

In $\{\cdots\}_1$,

$$|\ell_1+\ell_4| = |(\ell_1+\ell_2)-(\ell_2+\ell_3)+(\ell_3+\ell_4)| \leqslant |\ell_1+\ell_2|+|\ell_2+\ell_3|+|\ell_3+\ell_4|$$
$$\leqslant c\sum_{k=1}^{3}\bigl|\ell_k(x)\bigr|\Bigl[\frac{1}{k}+\frac{k}{|k+j|\,(|k-j|+1)}\Bigr]$$
$$\leqslant c\Bigl\{\bigl|\ell_s(x)\bigr|\Bigl[\frac{1}{s}+\frac{s}{|s+j|\,(|s-j|+1)}\Bigr]\Bigr\}_{s=k=1}.$$

Here we used (3.2) and that $\bigl|\ell_k(x)\cdot\ell_{k\pm m}^{-1}\bigr|\sim 1$ for any k whenever $0\leqslant m\leqslant C$.

Similar considerations are valid for the second term in $\{\cdots\}_1$ by $\ell_2-\ell_4=(\ell_2+\ell_3)-(\ell_3+\ell_4)$.

Taking into account that $|f_k-f|\leqslant c\omega\bigl(\frac{\sin\vartheta}{n}i+\frac{i^2}{n^2}\bigr)$ whenever $i=|k-j|+m$ (see [**4**, (4.4)]; $0\leqslant m\leqslant C$), we get that

$$|\{\cdots\}_1|\leqslant c\omega\Bigl(\frac{\sin\vartheta}{n}i+\frac{i^2}{n^2}\Bigr)\Bigl(\bigl|\ell_s(x)\bigr|\Bigl[\frac{1}{s}+\frac{s}{|s+j|\,(|s-j|+1)}\Bigr]\Bigr)_{s=1}.$$

Using this last estimation and similar ones for $\{\cdots\}_2$, $\{\cdots\}_3$, ..., we can get (3.4).

If $l>2$, the argument is similar. We may omit the further details. □

Acknowledgement. The authors thank the referee for the careful work. Some of the referee's advices are in the paper.

References

1. S. Bernstein, *Sur une modification de la formule d'interpolation de Lagrange* (French), Communications Kharkow **5** (1932), 49–57. [Collected Works. Vol. II. The constructive theory of functions [1931–1953] (Russian), Izdat. Akad. Nauk SSSR, Moscow, 1954, 130–140.]
2. ______, *Sur une formule d'interpolation* (French), C. R.**191** (1930), 635–637. [Collected Works. Vol. I. The constructive theory of functions [1905–1930] (Russian), Izdat. Akad. Nauk SSSR, Moscow, 1952, 520–522.]
3. P. Vértesi, *One-sided convergence conditions for Lagrange interpolation based on the Jacobi-roots*, Acta Sci. Math. **45** (1983), 419–428.
4. ______, *Lagrange interpolation for continuous functions of bounded variation*, Acta Math. Acad. Sci. Hung. **35** (1980), 23–31.
5. ______, *Notes on the Hermite–Fejér interpolation based on the Jacobi abscissas*, Acta Math. Acad. Sci. Hung. **24** (1973), 233–239.
6. O. Kis, Ho Tho Kau, *Investigation of a certain interpolation process. III*, (Russian) Acta Math. Acad. Sci. Hung. **28** (1976), 157–176.
7. O. Kis, J. Szabados, *On some de la Vallé-Poussin type discrete linear operators*, Acta Math. Hung. **47** (1986), 239–260.
8. T. Hermann, O. Kis, P. Vértesi, *On certain rational interpolatory operators*, Acta Math. Hung. **49** (1987), 29–49.

Department of Numerical Analysis, Loránd Eötvös University
Budapest, Hungary
`szili@caesar.elte.hu`

Alfréd Rényi Mathematical Institute of the Hungarian Academy of Sciences
Budapest, Hungary
`veter@renyi.hu`