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A DISPROOF OF A CONJECTURE OF ROBERTSON AND GENERALIZATIONS

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In [**1**, pp. 264, 274–280, Section 4], we established the Taylor expansion

$$d(z) := \left\{ \frac{1}{2xz} \left[\left(\frac{1+z}{1-z} \right)^x - 1 \right] \right\}^y = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} d_{nj}(y) x^j \tag{1}$$

for $|z| < 1$, where x and y are arbitrary complex numbers, $1^x = 1^y = 1$, and the coefficients $d_{nj}(y)$ are found explicitly. In particular, the coefficients $d_{nj}(y)$ are equal to zero if n and j are of different parity. We proved that the coefficients $d_{nj}(y)$, $0 \le j \le n$, in (1) are nonnegative for $n = 1, \dots, [y]+1$ if $y > 0$ is not a positive integer ($[y]$ denotes the greatest integer less than y), and for all $n = 1, 2, \dots$ if y is a positive integer ($y = 1, 2, \dots$). Therefore, we conjectured that the coefficients $d_{nj}(y)$, $0 \le j \le n$, in (1) are nonnegative for the "tail-end" as well, i.e. for $n > [y]+1$ if $y \ge y_0$ is not a positive integer where y_0 is a certain number of the interval $0 < y_0 \le 1/2$. In particular, for $y = 1/2$ this conjecture is due to Robertson [**2**, pp. 8, 20–21, 176]. In [**3**] Robertson conjecture has been disproved. At the author's request, 1) Staffan Wrigge and Arne Fransén of National Defence Research Institute, Systems Analysis Department (FOA 1), P. O. Box 27322, S-10254, Stockholm, Sweden, 2) Karl Dilcher of Dalhousie University, Department of Mathematics, Statistics and Computing Science, Halifax, Nova Scotia, Canada B3H 3J5 and 3) Pierre Barrucand of Université Pierre et Marie-Curie (Paris VI) Institut de mathématiques pures et appliquées, 4, place Jussieu, 75252 Paris Cedex 05, France, computed independently the coefficients $d_{nj}(1/2)$, $0 \le j \le n$, for 1) $9 \le n \le 15$, 2) $8 \le n \le 20$ and 3) $0 \le n \le 29$, respectively, and found that the first negative coefficient occurs for $n = 13$ and $j = 13$. In addition, the author called the attention of A. A. Jagers (University of Twente, Department of Mathematics, Enschede, The Netherlands) to the Robertson conjecture, who in his turn suggested it to F. W. Steutel of Eindhoven University of Technology, Department of Mathematics and Computing Science, P. O. Box 513, 5600 MB Eindhoven, The Netherlands. In

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a private communication to the author, Steutel [**4**] also disproved the Robertson conjecture. Now in this paper we shall show that our general conjecture is false for any rational (but not integer) $y > 0$ as well. For this we need two lemmas.

Lemma 1 (Steutel [**4**] and [**5**, p. 137]). *Let*

$$P(z) := \sum_{n=0}^{\infty} p_n z^n \tag{2}$$

be a (possibly formal) power series generating a strictly logarithmically convex sequence of positive numbers p_n, $n = 0, 1, 2, \ldots$, *i.e.*

$$p_n^2 < p_{n+1} p_{n-1}, \qquad n = 1, 2, \ldots, \tag{3}$$

and let

$$\big(P(z)\big)^y := \sum_{n=0}^{\infty} P_n(y) z^n, \qquad y > 0, \quad P_0(y) > 0. \tag{4}$$

Then

$$P_n(y) > 0, \qquad n = 0, 1, 2, \ldots\ . \tag{5}$$

Corollary. *Under the conditions of Lemma* 1, *if* $y < 0$ *in* (4), *then there exists at least one subscript* $n > 0$ *such that* $P_n(y) < 0$.

Proof. The Corollary follows from (5) and the identities

$$\sum_{k=0}^{n} P_k(y) P_{n-k}(-y) \equiv 0, \qquad n = 1, 2, \ldots, \quad y < 0, \quad P_0(y) > 0,$$

resulting from the multiplication of the series (4) and the series obtained from (4) after substituting $-y$ for y.

Lemma 2. *Let*

$$B(t) := \left(\frac{e^t - 1}{t}\right)^y := \sum_{n=0}^{\infty} B_n(y) t^n, \qquad |t| < 2\pi, \quad 1^y = 1. \tag{6}$$

Then:

(i) *for any rational (but not integer)* $y > 0$, *and* (ii) *for any rational number* $y < 0$, *there exists at least one subscript* $n > 0$ *such that* $B_n(y) < 0$;

(iii) *for any irrational number* $y > 0$, *there exists at least one subscript* $n > 0$ *such that either* $B_n(y) < 0$ *or* $B_n(-y) < 0$.

Proof. (i) Let $p \geq 1$ and $q \geq 2$ be integers such that p is not a multiple of q. Then for $y = p/q$ from (6) we obtain

$$\left(\frac{e^t - 1}{t}\right)^p = \left(\sum_{n=0}^{\infty} B_n t^n\right)^q, \qquad B_n := B_n(p/q), \quad |t| < 2\pi, \tag{7}$$

where $B_0 = 1$ and all B_n, $n = 1, 2, \dots$, are real numbers. We have

$$\left(\frac{e^t - 1}{t}\right)^p = p! \sum_{n=0}^{\infty} \frac{S(n+p,p)}{(n+p)!} t^n \tag{8}$$

where

$$S(n+p,p) = \frac{1}{p!} \sum_{\nu=1}^{p} (-1)^{p-\nu} \binom{p}{\nu} \nu^{n+p}, \qquad n \geq 0, \tag{9}$$

are the Stirling numbers of the second kind which are positive integers (see, for example, in [**6**, pp. 313 and 310, formulas (21) and (5)–(6), respectively], or in [**7**, Chapter V]), and

$$\left(\sum_{n=0}^{\infty} B_n t^n\right)^q = \sum_{n=0}^{\infty} t^n \sum B_{k_1} B_{k_2} \cdot \ldots \cdot B_{k_q} \tag{10}$$

where the inner sum is taken over all nonnegative integers $k_1, k_2, \dots, k_q$, satisfying

$$k_1 + k_2 + \cdots + k_q = n, \qquad n \geq 0. \tag{11}$$

From (7), (8), (10) and (11) it follows that

$$\frac{p! S(n+p,p)}{(n+p)!} = \sum B_{k_1} B_{k_2} \cdot \ldots \cdot B_{k_q}, \quad n \geq 0. \tag{12}$$

(From (12) and (11) for $n = 0$ we obtain again $B_0 = 1$, and for $n = 1$ we obtain $B_1 = p/2q$ since $S(p+1,p) = p(p+1)/2$ (see, for example, in [**7**, p. 227]). Of course, these values of B_0 and B_1 follow directly from (7).) From (12) and (11) for $n \geq 2$ we obtain

$$B_n = \frac{1}{q}\left(\frac{p! S(n+p,p)}{(n+p)!} - \sum B_{k_1} B_{k_2} \cdot \ldots \cdot B_{k_q}\right) \tag{13}$$

where the sum is taken over all nonnegative integers $k_1, k_2, \dots, k_q$ satisfying simultaneously the inequalities $0 \leq k_j \leq n-1$, $j = 1, 2, \dots, q$, and the equation (11). Now if we assume that all B_n, $n = 0, 1, 2, \dots$, are nonnegative, then from (13) it follows that

$$0 \leq B_n \leq \frac{p! S(n+p,p)}{q(n+p)!}, \qquad n \geq 2. \tag{14}$$

Further, with the help of (9) we obtain that

$$\begin{aligned} \lim_{n\to\infty} \left(\frac{p! S(n+p,p)}{q(n+p)!}\right)^{1/n} &= \lim_{n\to\infty} \frac{S(n+p+1,p)}{(n+p+1) S(n+p,p)} \\ &= \lim_{n\to\infty} \frac{p}{n+p+1} \cdot \frac{\displaystyle\sum_{\nu=1}^{p} (-1)^{p-\nu} \binom{p}{\nu} \left(\frac{\nu}{p}\right)^{n+p+1}}{\displaystyle\sum_{\nu=1}^{p} (-1)^{p-\nu} \binom{p}{\nu} \left(\frac{\nu}{p}\right)^{n+p}} = 0. \end{aligned} \tag{15}$$

Therefore, from (15) and (14) we conclude that the series (6) for $y = p/q$ has an infinite radius of convergence; but this is a contradiction as $t = 2m\pi i$, $m = \pm 1, \pm 2, \dots$, are branch points of the function (6) for $y = p/q$. Hence, for $y = p/q$, not all coefficients $B_n(p/q)$, $n = 0, 1, 2, \dots$, in the series (6) are nonnegative.

(ii) Let $p \geq 1$ and $q \geq 1$ be integers. Then for $y = -p/q$ from (6) we obtain the identity

$$\left(\sum_{n=0}^{\infty} B_n t^n\right)^q \left(\frac{e^t - 1}{t}\right)^p \equiv 1, \qquad B_n := B_n(-p/q), \tag{16}$$

having in mind (10) and (8). If all $B_n \geq 0$ for $n = 0, 1, 2, \dots$, then the coefficients of t^n for $n = 1, 2, \dots$ in the expansion of the left-hand side of (16) will be positive but not equal to zero according to the right-hand side of (16). Hence, for $y = p/q$, not all coefficients $B_n(-p/q)$, $n = 0, 1, 2, \dots$, in the series (6) are nonnegative.

(iii) Let $y > 0$ be an irrational number. Then from (6) we deduce the identities

$$\sum_{k=0}^{n} B_k(y) B_{n-k}(-y) \equiv 0, \qquad n = 1, 2, \dots . \tag{17}$$

From (17) it follows that not all coefficients $B_n(y)$ and $B_n(-y)$ for $n = 1, 2, \dots$, are nonnegative.

This completes the proof of Lemma 2.

In particular, for $y = 1/q$, where $q \geq 2$ is an integer, Lemma 2 is due to Steutel [**4**].

An open problem is whether for any irrational number $y > 0$ there always exists at least one subscript $n > 0$ such that $B_n(y) < 0$; the same question holds for $B_n(-y)$.

Theorem. *For any rational (but not integer) $y > 0$, not all coefficients $d_{nj}(y)$ in (1) are nonnegative.*

Remark. Evidently, for any integer $y > 0$, all coefficients $d_{nj}(y)$ in (1) are positive.

Proof. Let

$$t := x \log \frac{1+z}{1-z}, \tag{18}$$

$$A(z) := \frac{1}{2z} \log \frac{1+z}{1-z} = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n+1}, \qquad |z| < 1, \tag{19}$$

$$\big(A(z)\big)^y := \sum_{n=0}^{\infty} A_n(y) z^{2n}, \qquad |z| < 1,\ y > 0,\ A_0(y) = 1, \tag{20}$$

where according to (2)–(5) in Steutel's Lemma 1 we shall have

$$A_n(y) > 0, \qquad n = 0, 1, 2, \dots \quad (y > 0). \tag{21}$$

With the help of (18)–(20) and (6) the equation (1) takes the form

$$d(z) = \big(A(z)\big)^y B(t) = \sum_{j=0}^{\infty} B_j(y)(2xz)^j \big(A(z)\big)^{y+j}$$

$$= \sum_{n=0}^{\infty} z^{2n} \sum_{j=0}^{n} A_{n-j}(y+2j)B_{2j}(y)(2x)^{2j} \tag{22}$$

$$+ \sum_{n=0}^{\infty} z^{2n+1} \sum_{j=0}^{n} A_{n-j}(y+2j+1)B_{2j+1}(y)(2x)^{2j+1}$$

for $|z| < 1$. From (1) and (22) we obtain the following formulas for the nonvanishing coefficients

$$d_{2n,2j}(y) = 2^{2j} A_{n-j}(y+2j)B_{2j}(y), \tag{23}$$

$$d_{2n+1,2j+1}(y) = 2^{2j+1} A_{n-j}(y+2j+1)B_{2j+1}(y) \tag{24}$$

for $0 \le j \le n$, $n = 0, 1, 2, \ldots$. It is clear from (21), (23) and (24) that the signs of $d_{2n,2j}(y)$ and $d_{2n+1,2j+1}(y)$ are determined by those of $B_{2j}(y)$ and $B_{2j+1}(y)$, respectively. Therefore, for any rational (but not integer) $y > 0$, according to our Lemma 2, there exists at least one integer $j > 0$ such that either $B_{2j}(y) < 0$ or $B_{2j+1}(y) < 0$, i.e. either $d_{2n,2j}(y) < 0$ or $d_{2n+1,2j+1}(y) < 0$ for all integers $n \ge j$, respectively. (For example, in [**3**] we have shown that $d_{13,13}(1/2) < 0$, i.e. $B_{13}(1/2) < 0$. Hence, $d_{2n+1,13}(1/2) < 0$ for all integers $n \ge 6$.)

This completes the proof of the theorem.

In particular, for $y = 1/q$, where $q \ge 2$ is an integer, the theorem is due to Steutel [**4**].

From the Theorem proved it follows that our general conjecture for any rational (but not integer) $y > 0$ as well as the special conjecture of Robertson for $y = 1/2$ for the coefficients in (1) are false. But our general conjecture is open for the irrational numbers $y > 0$.

Application. For $j = n$ and $y = 1/q$ from (23) and (24) we obtain the formula

$$d_{nn}(1/2) = 2^n B_n, \qquad 0, 1, 2, \ldots, \tag{25}$$

where the numbers B_n are generated by the expansion (6) for $y = 1/2$ and $B_n := B_n(1/2)$. From (13) for $p = 1$ and $q = 2$ we obtain the recurrence relation

$$B_n = \frac{1}{2}\left(\frac{1}{(n+1)!} - \sum_{k=1}^{n-1} B_k B_{n-k}\right), \qquad n \ge 2,\ B_0 = 1,\ B_1 = 1/4, \tag{26}$$

for the calculation of the numbers B_n. Thus from (26) we obtain successively

(27)

$$B_2 = \frac{5}{2^5 \cdot 3}, \quad B_3 = \frac{1}{2^7}, \quad B_4 = \frac{79}{2^{11} \cdot 3^2 \cdot 5}, \quad B_5 = \frac{3}{2^{13} \cdot 5},$$
$$B_6 = \frac{71}{2^{16} \cdot 3^3 \cdot 7}, \quad B_7 = \frac{113}{2^{18} \cdot 3^3 \cdot 5 \cdot 7}, \quad B_8 = \frac{3053}{2^{23} \cdot 3^4 \cdot 5^2 \cdot 7},$$

$$B_9 = \frac{1}{2^{25}\cdot 3^3\cdot 5^2}, \quad B_{10} = \frac{17}{2^{28}\cdot 3^2\cdot 5\cdot 7\cdot 11},$$
$$B_{11} = \frac{19}{2^{30}\cdot 3^2\cdot 7\cdot 11}, \quad B_{12} = \frac{935917}{2^{34}\cdot 3^4\cdot 5^3\cdot 7^2\cdot 11\cdot 13},$$
$$B_{13} = -\frac{20287103}{2^{36}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 13} < 0, \quad B_{14} = -\frac{2452337}{2^{39}\cdot 3^7\cdot 5^2\cdot 7^2\cdot 11\cdot 13} < 0!$$

The last two equations in (27), having in mind (25), show that the Robertson conjecture is false. More general, with the help of (27) from (23) and (24) we conclude that

$$\begin{aligned} d_{2n,2j}(1/2) &> 0, \qquad 0 \le j \le n,\ n = 0,1,2,3,4,5,6,\\ d_{2n+1,2j+1}(1/2) &> 0, \qquad 0 \le j \le n,\ n = 0,1,2,3,4,5, \end{aligned}$$

and

$$\begin{aligned} d_{2n,2j}(1/2) &> 0, \qquad 0 \le j \le 6,\ n \ge 7,\\ d_{2n+1,2j+1}(1/2) &> 0, \qquad 0 \le j \le 5,\ n \ge 6, \end{aligned}$$

but

$$\begin{aligned} d_{2n+1,13}(1/2) &< 0, \qquad n \ge 6,\\ d_{2n,14}(1/2) &< 0, \qquad n \ge 7, \end{aligned}$$

respectively. Again from the last two inequalities it follows that the Robertson conjecture is false.

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