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ON RELATION BETWEEN STABILITY AND CORRECTNESS OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Consider the problem

$$dx(t) = dA(t)\cdot p(t)\cdot x(t) + df(t), \tag{1}$$

$$x(t_0) = c_0, \tag{2}$$

where $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R}_+ \to \mathbb{R}^n$ are, respectively, the real matrix- and vector-functions with locally bounded variation components, $p : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a matrix-function locally integrable with respect to A, $c_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}_+$.

Along with the problem (10), (2) let us consider the problem

$$dx(t) = d\widetilde{A}(t)\cdot \widetilde{p}(t)\cdot x(t) + d\widetilde{f}(t), \tag{3}$$

$$x(\widetilde{t_0}) = \widetilde{c_0}, \tag{4}$$

where $\widetilde{A} : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $\widetilde{f} : \mathbb{R}_+ \to \mathbb{R}^n$ are, respectively, real matrix– and vector-functions with locally bounded variation components, $\widetilde{p} : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a matrix-function locally integrable with respect to $\widetilde{A}$, $\widetilde{c_0} \in \mathbb{R}^n$ and $\widetilde{t_0} \in \mathbb{R}_+$.

Before passing to the statement of the basic results, we give some notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$ is the set of real numbers, $[a, b]$ and $]a, b[$ are, respectively, closed and open intervals; $\mathbb{R}_+ = [0, +\infty[$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $x = (x_{ij})_{ij=1}^{n,m}$ with the norm $\|x\| = \max_{j=1,\dots,m}\sum_{i=1}^n |x_{ij}|$.

$$\mathbb{R}_+^{n\times m} = \{(x_{ij})^{n,m} : x_{ij} \geq 0 \quad (i = 1,\dots,n;\ j = 1,\dots,m)\}.$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is a space of all real column n-vectors $x = (x_i)_{i=1}^n$.

If $x \in \mathbb{R}^{n\times n}$, then x^{-1} and $\det(x)$ are, respectively, the inverse to x matrix and the determinant of x; I_n is the identity $n\times n$ matrix;

$\overset{d}{\underset{c}{V}} = \sup\{\overset{b}{\underset{a}{V}}(x) : c < a < b < d\}$, where $\overset{b}{\underset{a}{x}}$ is the sum of total variations on a closed interval $[a, b]$ of components x_{ij} $(i = 1,\dots,n;\ j = 1,\dots,m)$ of the matrix-function $x :]c, d[\to \mathbb{R}^{n\times m}$; $v(x)(t) = (v(x_{ij})(t))_{i,j=1}^n$, where $v(x_{ij})(t) = (\overset{t}{\underset{-\infty}{V}} x_{ij})$ for $t \in]c, d[$ $(i = 1,\dots,n;\ j = 1,\dots,m)$[1];

$x(t-)$ and $x(t+)$ are the left and the right limits of the matrix-function $x :]c, d[\to \mathbb{R}^{n\times m}$ at the point $t \in]c, d[$, $d_1x(t) = x(t) - x(t-)$, $d_2x(t) = x(t+) - x(t)$.

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[1] x_{ij} as a constant outside $[a, b]$ is assumed to be continuous.

$BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n\times m})$ is the set of all matrix-functions $x : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ of bounded variations on every closed interval from $\mathbb{R}_+$.

If $g : \mathbb{R}_+ \to \mathbb{R}$ is a nondecreasing function, $x : \mathbb{R}_+ \to \mathbb{R}$ and $0 \le s < t < +\infty$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,dg_1(\tau) - \int_{]s,t[} x(\tau)\,dg_2(\tau) + \\ + \sum_{s<\tau\le t} x(\tau)\,d_1 g(\tau) - \sum_{s\le\tau<t} x(\tau)\,d_2 g(\tau),$$

where $g_j : \mathbb{R}_+ \to \mathbb{R}$ $(j = 1, 2)$ are continuous nondecreasing functions such that the function $g_1 - g_2$ is identically equal to the continuous part of g, and $\int_{]s,t[} dg_j(\tau)$ is the Lebesgue-Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function $g_j (j = 1, 2)$ (if $s = t$, then $\int_s^t x(\tau)dg(\tau) = 0$);

$L_{\text{loc}}(\mathbb{R}_+, \mathbb{R}; g)$ is the set of all functions $x : \mathbb{R}_+ \to \mathbb{R}$ $\mu(g)$-measurable (i.e, measurable with respect to the measures $\mu(g_1)$ and $\mu(g_2)$) and integrable on the closed interval $[0, b]$ for every $b \in \mathbb{R}_+$.

A matrix-function t_0 is said to be nondecreasing if each of its components is such.

If $G = (g_{ik})_{i,k=1}^{\ell,n} : \mathbb{R}_+ \to \mathbb{R}^{\ell\times n}$ is a nondecreasing matrix-functions, then $L(\mathbb{R}_+, \mathbb{R}^{n\times n} : G)$ is the set of all matrix-functions $x = (x_{kj})_{k,j=1}^{n,m} : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ such that $x_{kj} \in L(\mathbb{R}_+, \mathbb{R}; g_{ik})$ $(i = 1, \dots, \ell, ; k = 1, \dots, n; j = 1, \dots, m)$;

$$\int_s^t dG(\tau)\cdot x(\tau) = \Big(\sum_{k=1}^n \int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\Big)_{i,j=1}^{\ell,m} \quad \text{for} \ \ 0 \le s \le t < +\infty.$$

If $G_j : \mathbb{R}_+ \to \mathbb{R}^{\ell\times n}$ $(j = 1, 2)$ are nondecreasing matrix-functions, $G \equiv G_1 - G_2$ and $x : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot x(\tau) = \int_s^t dG_1(\tau)\cdot x(\tau) - \int_s^t dG_2(\tau)\cdot x(\tau) \ \ \text{for} \ \ 0 \le s \le t < +\infty;$$

$$L(\mathbb{R}_+, \mathbb{R}^{n\times m}; G) = \bigcap_{j=1}^{2} L(\mathbb{R}_+, \mathbb{R}^{n\times m}; G_j).$$

Under a solution of the system (1) is understood a vector-function $x \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ such that

$$x(t) - x(s) = \int_s^t dA(\tau)\cdot p(\tau)\cdot x(\tau) + f(t) - f(s) \ \ \text{for} \ \ 0 \le s \le t < +\infty.$$

We will assume that $f \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, $A \in BV_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ and $p \in L_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{n\times n}, A)$ are such that

$$\det(I_n + (-1)^j \, djA(t)\cdot p(t)) \ne 0 \ \ \text{for} \ \ t \in \mathbb{R}_+ \ \ (j = 1, 2). \tag{5}$$

Then the problem (1), (2) has a unique solution (see [1]).

Definition 1. The problem (1), (2) is said to be *correct* if for every arbitrarily small $\varepsilon > 0$ and arbitrarily large $\rho > 0$ there exists $\delta > 0$ such that for any $\widetilde{t}_0 \in \mathbb{R}_+$, $\widetilde{c}_0 \in \mathbb{R}^n$,

$\widetilde{A} \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n}$, $\widetilde{f} \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^n)$ and $\widetilde{p} \in L_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n}, A)$ satisfying the conditions

$$|t_0 - \widetilde{t}_0| < \delta, \quad \|c_0 - \widetilde{c}_0\| < \delta, \tag{6}$$
$$\|M(t) - \widetilde{M}(t)\| < \delta \quad \|f(t) - \widetilde{f}(t)\| < \delta, \overset{+\infty}{\underset{0}{V}}(M - \widetilde{M}) < \rho$$

and

$$\det(I_n + (-1)^j \, dj\widetilde{A}(t) \cdot \widetilde{p}(t)) \neq 0 \;\text{ for }\; t \in \mathbb{R}_+ \;\; (j = 1, 2) \tag{7}$$

with

$$M(t) = \int_0^t dA(\tau) \cdot p(\tau), \quad \widetilde{M}(t) = \int_0^t d\widetilde{A}(\tau) \cdot \widetilde{p}(\tau), \tag{8}$$

the inequality

$$\|x(t) - y(t)\| < \varepsilon \;\text{ for }\; t \in \mathbb{R}_+ \tag{9}$$

holds, where x and y are the solutions of the problems (1), (2) and (3), (4), respectively.

Definition 2. The problem (1), (2) is said to be *weakly correct* if for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\widetilde{t}_0 \in \mathbb{R}^n$ and $\widetilde{c}_0 \in \mathbb{R}^n$, $\widetilde{A} \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, $\widetilde{f} \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^n)$ and $\widetilde{p} \in L_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n}, A)$ satisfying the conditions (6), (7) and

$$\overset{+\infty}{\underset{0}{V}}(M - \widetilde{M}) < \delta, \quad \overset{+\infty}{\underset{0}{V}}(f - \widetilde{f}) < \delta,$$

where the matrix-functions M and $\widetilde{M}$ are defined by (8), the inequality (9) holds, where x and y are the solutions of the problems (1), (2) and (3), (4), respectively.

Definition 3. Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\lim_{t\to+\infty} \xi(t) = +\infty$. A solution x of the system (1) is said to be *ξ-exponentially asymptotically stable* if there exists a positive number η such that for every $\varepsilon > 0$ there exists $\delta = dl(\varepsilon) > 0$ such that an arbitrary solution y of the system (1) the satisfying the inequality

$$\|x(t_0) - y(t_0)\| < \delta$$

for some $t_0 \in \mathbb{R}_+$, admits the estimate

$$\|x(t) - y(t)\| < \varepsilon \exp(-\eta(\xi(t) - \xi(t_0))) \;\text{ for }\; t \geq t_0$$

The uniform stability of the solution x is defined just in the same way as for systems of ordinary differential equations (see, e.g., [2] or [3]).

Definition 4. The system (1) is said to be *uniformly stable (ξ-exponentially asymptotically stable)* if every solution of that system is uniformly stable (ξ-exponentially asymptotically stable).

Definition 5. A pair (A, p) of matrix-functions $A \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n}$ and $p \in L_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n}, A)$ satisfying the condition (5) is said to be *uniformly stable (ξ-exponentially asymptotically stable)* if the system

$$dx(t) = dA(t) \cdot p(t) \cdot x(t)$$

is uniformly stable (ξ-exponentially asymptotically stable).

Theorem 1. *Let* $A \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, $f \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^n)$, $p \in L_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n}, A)$, *and let the condition* (5) *hold. Moreover, let the pair* (A,p) *be* ξ*-exponentially asymptotically stable and the conditions*

$$\limsup_{t\to+\infty} \overset{\nu(\xi)(t)}{\underset{t}{V}}(B) < +\infty,$$

and

$$\lim_{t\to+\infty} \overset{\nu(\xi)(t)}{\underset{t}{V}}(\widetilde{B}) = 0$$

hold, where

$$\nu(\xi)(t) = \sup\{\tau \geq t : \xi(\tau) \leq \xi(t)+1\},$$

$$B(A,p)(t) = \int_0^t dA(\tau)\cdot p(\tau) + \sum_{0\leq\tau<t} d_1A(\tau)\cdot p(\tau)(I_n - d_1A(\tau)\cdot p(\tau))^{-1}\cdot d_1A(\tau)\cdot(\tau) -$$

$$- \sum_{0\leq\tau<t} d_2A(\tau)\cdot p(\tau)(I_n + d_2A(\tau)\cdot p(\tau))^{-1}\cdot d_2A(\tau)\cdot(\tau),$$

$$\widetilde{B}(A,p,f)(t) = f(t) + \sum_{0<\tau\leq t} d_1A(\tau)\cdot p(\tau)(I_n - d_1A(\tau)\cdot p(\tau))^{-1}\cdot d_1f(\tau) -$$

$$- \sum_{0\leq\tau<t} d_2A(\tau)\cdot p(\tau)(I_n + d_2A(\tau)\cdot p(\tau))^{-1}\cdot d_2f(\tau).$$

Then the problem (1), (2) *is correct.*

Theorem 2. *Let* $A \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, $f \in BV_{\rm loc}(\mathbb{R}_+, \mathbb{R}^n)$, $p \in L_{\rm loc}(\mathbb{R}_+, \mathbb{R}^{n\times n}, A)$, *and let the condition* (5) *hold. Let, moreover, the pair* (A,p) *be uniformly stable. Then the problem* (1), (2) *is weakly correct.*

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