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ON THE EXISTENCE OF PROPER AND VANISHING AT INFINITY SOLUTIONS OF ODD ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Consider a nonlinear ordinary differential equation

$$u^{(n)}+\sum_{k=1}^{n-1}p_k(t)u^{(k)}=f(t,u,u',\dots,u^{(n-1)}) \tag{1}$$

on $[a,+\infty[$, where $n\geq 2$, $0<a<+\infty$ each of the functions $p_k:[a,+\infty[\to\mathbb{R}$, $k\in\{1<\dots<n-1\}$ are locally absolutely continuous together with its derivatives up to order $k-1$, inclusive, and the function $f:[a,+\infty[\times\mathbb{R}^n\to\mathbb{R}$ is locally summable in the first argument and satisfies the local Lipschitz condition in the last n variables.

As is well-known, even in the case where $p_k(t)\equiv 0$, the general theory of the Cauchy problem does not answer the question on the existence of a global nontrivial solution of the equation (1) if the increase order with respect to at least one phase variable is greater than 1.

In 1986, I. Kiguradze [1] investigated a certain boundary value problem for (1) and obtained sufficient conditions for the existence of proper solutions in the above-mentioned case. The same question was considered by the author in [2] for equation (1). In the present paper we complement the results of [2], in the case, where n is odd.

Throughout this work, the use will be made of the following notation:

$\mathbb{R}$ is the set of real numbers;

$\mathbb{R}_+=[0,+\infty[$;

$\mathbb{R}^n$ is the n-dimensional real Euclidean space;

$\mu^k(k=1,2,\dots;\ k=2i,\ 2i+1,\dots)$ are the real constants defined by the recurrence relation

$$\mu_0^{i+1}=\frac{1}{2};\ \ \mu_i^{2i}=1;\ \ \mu_{i+1}^k=\mu_{i+1}^{k-1}+\mu_i^{k-2}\ \ (i=0,1,\dots;\ k=2i+3,\dots).$$

The solution u of the equation (1) is said to be *proper*, if it is defined on $[t_0,+\infty[\subset[a,+\infty[$ and does not equal identically to zero in any neighborhood of $+\infty$.

We say that the proper solution u of the equation (1) *vanishes at infinity*, if

$$\lim_{t\to+\infty}u(t)=0.$$

Let n_0 be the entire part of the number $\frac{n}{2}$ and $\varphi_i:\mathbb{R}^n\to\mathbb{R}$ be continuous functions satisfying

$$0<\sum_{i=0}^{n_0-1}\Big|\varphi_i(x_0,x_1,\dots,x_{n-1})\Big|\leq c\Big(1+\sum_{k=n_0}^{n-1}|x_k|\Big)^{-\lambda} \tag{2}$$

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on $\mathbb{R}^n$, where $c \geq 0$ and $\lambda \in [0,1]$.

Below, unless otherwise specified, the function f is assumed to satisfy the conditions

$$\begin{gathered}(-1)^{n_0} f(t, x_0, x_1, \dots, x_{n-1}) \operatorname{sgn} x_0 \geq -\sum_{i=0}^{n_0-1} \alpha_i(t)|x_0|, \\ \Big| f(t, x_0, x_1, \dots, x_{n-1}) \Big| \leq h(t, |x_0|, |x_1|, \dots, |x_{n_0-1}|)\end{gathered} \tag{3}$$

on $[a,+\infty[\times\mathbb{R}^n$, where the functions $\alpha_i : [a,+\infty[\to \mathbb{R}$ $(i = 0, \dots, n_0 - 1)$ are locally summable, while the function $h : [a,+\infty[\to \mathbb{R}_+^{n_0} \to \mathbb{R}_+$ is locally summable in the first argument, nondecreasing in the last n_0 arguments and for any $\rho_0 > 0$ satisfies the condition

$$\limsup_{\substack{t\to a_k \\ \rho\to+\infty}} \frac{1}{\rho^2}\bigg(\int_a^t h(\tau, \rho_0, \rho, \dots, \rho)\, d\tau\bigg)^{1-\lambda} < +\infty. \tag{4}$$

Theorem 1. *Let there exist the constant $\delta > 0$ such that the inequalities*

$$\begin{gathered}\sum_{k=2i}^{n-1} (-1)^{n_0+k-i} \mu_i^k \Big[p_k(t)\Big]^{(k-2i)} \leq 0 \quad (i = 1, \dots, n_0 - 1), \\ p_{n-1}(t) \leq -\delta \ \textit{ and } \ \sum_{k=1}^{n-1} (-1)^{n_0+k-1} \mu_0^k \Big[p_k(t)\Big]^{(k)} - \sum_{i=0}^{n_0-1} \alpha_i(t) \geq \delta\end{gathered}$$

hold on $[a,+\infty[$. Then there exists a proper, vanishing at infinity, solution u of the equation (1) *satisfying the initial conditions*

$$u^{(i)}(a) = \varphi_i(u(a), u'(a), \dots, u^{(n-1)}(a)) \quad (i = 0, \dots, n_0 - 1). \tag{5}$$

The theorem below deals with the case, where $h(x_0, x_1, \dots, x_{n-1}) \equiv h_0(t, x_0 x)$, $\alpha_i(t) \equiv 0$ $(i = 1, \dots, n_0 - 1)$ $\lambda = 0$ i.e., when the conditions (2) and (3) are of the form

$$\begin{gathered}(-1)^{n_0} f(t, x_0, \dots, x_{n-1}) \operatorname{sgn} x_0 \geq -\alpha_0(t)|x_0|, \\ 0 < \sum_{i=0}^{n_0-1} \Big|\varphi_i(x_0, \dots, x_{n-1})\Big| \leq c.\end{gathered}$$

The condition (4) in this case is fulfilled automatically.

Theorem 2. *Let there exist a continuous solution $\delta : [a,+\infty[\to [0,+\infty[$ such that $\delta(a) > 0$ and the inequalities*

$$\begin{gathered}\sum_{k=2i}^{n-1} (-1)^{n_0+k-i} \mu_i^k p_k^{(k-2i)}(t) \leq 0 \quad (i = 1, \dots, n_0 - 1), \\ p_{n-1}(t) \leq 0, \quad \sum_{k=1}^{n-1} (-1)^{n_0+k-1} \mu_0^k p_k^{(k)}(t) \geq \alpha_0(t) + \delta(t)\end{gathered}$$

hold on $[a,+\infty[$. Then there exists at least one proper solution of the equation (1) *satisfying the initial conditions* (5).

In a special case where $p_k(t) \equiv 0$ for $k \in \{1, \dots, n-3, n-1\}$ and $p_{n-2}(t) \equiv 1$, Theorem 2 implies one result from [3].

References

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