Appendix G

Proofs for Section 3.4.2

Here we present the technical details for Section 3.4.2. First, we prove three lemmas that explore the relation between closed real intervals in terms of the lattice structure.

Prop. G.1. Given a continuous scalar $s \in S$, and $[x, y] \in I_S$, then $\downarrow(\perp,\ldots,[x, y],\ldots,\perp) = \bigcap\{\downarrow(\perp,\ldots,[z, z],\ldots,\perp) \mid x \leq z \leq y\}$.

Proof. $\downarrow(\perp,\ldots,[z, z],\ldots,\perp) = \{[u, v] \mid u \leq z \leq v\}$ so $\bigcap\{\downarrow(\perp,\ldots,[z, z],\ldots,\perp) \mid x \leq z \leq y\} = \{[u, v] \mid \forall z.\ (x \leq z \leq y \Rightarrow u \leq z \leq v)\} =$ $\{[u, v] \mid u \leq x \leq y \leq v\} = \downarrow(\perp,\ldots,[x, y],\ldots,\perp)$. ■

Prop. G.2. Given a continuous scalar $s \in S$, and a set $A \subseteq I_S \backslash \{\perp\}$ such that $\exists u'.\ \forall [u, v] \in A.\ u' \leq u$ and $\exists v'.\ \forall [u, v] \in A.\ v \leq v'$, then

$\downarrow(\perp,\ldots,[inf\{u \mid [u, v] \in A\}, sup\{v \mid [u, v] \in A\}],\ldots,\perp) =$

$\bigcap\{\downarrow(\perp,\ldots,[u, v],\ldots,\perp) \mid [u, v] \in A\}$.

Proof. Let $x = inf\{u \mid [u, v] \in A\}$ and $y = sup\{v \mid [u, v] \in A\}$. This *inf* and *sup* exist since the lower and upper bounds u' and v' exist. Then

$(\perp,\ldots,[a, b],\ldots,\perp) \in \downarrow(\perp,\ldots,[x, y],\ldots,\perp) \Leftrightarrow$

$a \leq x \leq y \leq b \Leftrightarrow$

$\forall [u, v] \in A.\ a \leq u \leq v \leq b \Leftrightarrow$

$\forall [u, v] \in A.\ (\perp,\ldots,[a, b],\ldots,\perp) \in \downarrow(\perp,\ldots,[u, v],\ldots,\perp) \Leftrightarrow$

$(\perp,\ldots,[a, b],\ldots,\perp) \in \bigcap\{\downarrow(\perp,\ldots,[u, v],\ldots,\perp) \mid [u, v] \in A\}$.

Thus $\downarrow(\perp,\ldots,[x, y],\ldots,\perp) = \bigcap\{\downarrow(\perp,\ldots,[u, v],\ldots,\perp) \mid [u, v] \in A\}$. ■

Prop. G.3. Given a display function $D{:}U \to V$, a continuous scalar $s \in S$, and $[x, y] \in I_s$, then $D(\downarrow(\perp,\ldots,[x, y],\ldots,\perp)) = \bigcap\{D(\downarrow(\perp,\ldots,[z, z],\ldots,\perp)) \mid x \le z \le y\}$.

Proof. $x \le w \le y \Rightarrow \bigwedge\{D(\downarrow(\perp,\ldots,[z, z],\ldots,\perp)) \mid x \le z \le y\} \le D(\downarrow(\perp,\ldots,[w, w],\ldots,\perp))$, so there is $A \in U$ such that $D(A) = \bigwedge\{D(\downarrow(\perp,\ldots,[z, z],\ldots,\perp)) \mid x \le z \le y\} = \bigcap\{D(\downarrow(\perp,\ldots,[z, z],\ldots,\perp)) \mid x \le z \le y\}$ (by Prop. C.8) and such that $x \le w \le y \Rightarrow A \le \downarrow(\perp,\ldots,[w, w],\ldots,\perp)$. Thus $A \le \bigwedge\{\downarrow(\perp,\ldots,[w, w],\ldots,\perp) \mid x \le w \le y\} = \bigcap\{\downarrow(\perp,\ldots,[w, w],\ldots,\perp) \mid x \le w \le y\} = \downarrow(\perp,\ldots,[x, y],\ldots,\perp)$ (by Prop. G.1).

On the other hand, $x \le z \le y \Rightarrow \downarrow(\perp,\ldots,[x, y],\ldots,\perp) \le \downarrow(\perp,\ldots,[z, z],\ldots,\perp) \Rightarrow D(\downarrow(\perp,\ldots,[x, y],\ldots,\perp)) \le D(\downarrow(\perp,\ldots,[z, z],\ldots,\perp))$, so $D(\downarrow(\perp,\ldots,[x, y],\ldots,\perp)) \le D(A)$ and thus $\downarrow(\perp,\ldots,[x, y],\ldots,\perp) \le A$. Therefore $\downarrow(\perp,\ldots,[x, y],\ldots,\perp) = A$ so $D(\downarrow(\perp,\ldots,[x, y],\ldots,\perp)) = D(A) = \bigcap\{D(\downarrow(\perp,\ldots,[z, z],\ldots,\perp)) \mid x \le z \le y\}$. ■

Now we define the values of display functions on embedded continuous scalar objects in terms of functions of real numbers.

Def. Given a display function $D{:}U \to V$ and a continuous scalar $s \in S$, by Prop. F.8 and Prop. F.11 there is a continuous $d \in DS$ such that values in U_s are mapped to values in V_d. Define functions $g_s{:}\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $h_s{:}\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by:
$\forall \downarrow(\perp,\ldots,[x, y],\ldots,\perp) \in U_s,\ D(\downarrow(\perp,\ldots,[x, y],\ldots,\perp)) = \downarrow(\perp,\ldots,[g_s(x, y), h_s(x, y)],\ldots,\perp) \in V_d$. Since $D(\{(\perp,\ldots,\perp)\}) = \{(\perp,\ldots,\perp)\}$ and D is injective, D maps intervals in I_s to intervals in I_d, so $g_s(x, y)$ and $h_s(x, y)$ are defined for all z. Also define functions $g'_s{:}\mathbf{R} \to \mathbf{R}$ and $h'_s{:}\mathbf{R} \to \mathbf{R}$ by $g'_s(z) = g_s(z, z)$ and $h'_s(z) = h_s(z, z)$.

In Prop. G.4 we show how the functions g_s and h_s can be defined in terms of the functions g'_s and h'_s.

Prop. G.4. Given a display function $D:U \to V$, a continuous scalar $s \in S$, and $[x, y] \in I_s$, then $g_s(x, y) = \mathit{inf}\{g'_s(z) \mid x \le z \le y\}$ and $h_s(x, y) = \mathit{sup}\{h'_s(z) \mid x \le z \le y\}$.

Proof. By Prop. G.3, $D(\downarrow(\perp,...,[x, y],...,\perp)) =$
$\bigcap\{D(\downarrow(\perp,...,[z, z],...,\perp)) \mid x \le z \le y\} = \bigcap\{\downarrow(\perp,...,[g'_s(z), h'_s(z)],...,\perp) \mid x \le z \le y\}$. By Prop. F.8 this is $\downarrow(\perp,...,[a, b],...,\perp)$ for some $a, b \in \mathbf{R}$. Define
$A = \{[g'_s(z), h'_s(z)] \mid x \le z \le y\}$. Then $\forall[g'_s(z), h'_s(z)] \in A.\ a \le g'_s(z)$ and
$\forall[g'_s(z), h'_s(z)] \in A.\ h'_s(z) \le b$, and, by Prop. G.2,
$D(\downarrow(\perp,...,[x, y],...,\perp)) = \downarrow(\perp,...,[a, b],...,\perp) =$
$\downarrow(\perp,...,[\mathit{inf}\{g'_s(z) \mid x \le z \le y\}, \mathit{sup}\{h'_s(z) \mid x \le z \le y\}],...,\perp)$. ■

Next, we prove a two lemmas useful for studying the functions g_s and h_s.

Prop. G.5. Given a display function $D:U \to V$, a continuous scalar $s \in S$, and a finite set $A \subseteq I_s \backslash \{\perp\}$, then
$g_s(\mathit{inf}\{u \mid [u, v] \in A\}, \mathit{sup}\{v \mid [u, v] \in A\}) = \mathit{inf}\{g_s(u, v) \mid [u, v] \in A\}$ and
$h_s(\mathit{inf}\{u \mid [u, v] \in A\}, \mathit{sup}\{v \mid [u, v] \in A\}) = \mathit{sup}\{h_s(u, v) \mid [u, v] \in A\}$.

Proof. Since A is finite, $\mathit{inf}\{u \mid [u, v] \in A\}$ and $\mathit{sup}\{v \mid [u, v] \in A\}$ exist, so, by Prop. G.2, $\downarrow(\perp,...,[\mathit{inf}\{u \mid [u, v] \in A\}, \mathit{sup}\{v \mid [u, v] \in A\}],...,\perp) =$
$\bigcap\{\downarrow(\perp,...,[u, v],...,\perp) \mid [u, v] \in A\} = \bigwedge\{\downarrow(\perp,...,[u, v],...,\perp) \mid [u, v] \in A\}$. Let
$a = g_s(\mathit{inf}\{u \mid [u, v] \in A\}, \mathit{sup}\{v \mid [u, v] \in A\}$ and
$b = h_s(\mathit{inf}\{u \mid [u, v] \in A\}, \mathit{sup}\{v \mid [u, v] \in A\})$. Then
$\downarrow(\perp,...,[a, b],...,\perp) =$

$D(\downarrow(\perp,\ldots,[inf\{u \mid [u, v] \in A\}, sup\{v \mid [u, v] \in A\}],\ldots,\perp)) =$

$\bigwedge\{D(\downarrow(\perp,\ldots,[u, v],\ldots,\perp)) \mid [u, v] \in A\} =$

$\bigcap\{\downarrow(\perp,\ldots,[g_s(u, v), h_s(u, v)],\ldots,\perp) \mid [u, v] \in A\} =$ (by Prop. G.2)

$\downarrow(\perp,\ldots,[inf\{g_s(u, v) \mid [u, v] \in A\}, sup\{h_s(u, v) \mid [u, v] \in A\}],\ldots,\perp)$, so

$a = inf\{g_s(u, v) \mid [u, v] \in A\}$ and $b = sup\{h_s(u, v) \mid [u, v] \in A\}$. ■

Prop. G.6. Given a display function $D{:}U \rightarrow V$ and a continuous scalar $s \in S$, then $[a, b] \subset [x, y] \Leftrightarrow [g_s(a, b), h_s(a, b)] \subset [g_s(x, y), h_s(x, y)]$.

Proof. $[a, b] \subset [x, y] \Leftrightarrow \downarrow[a, b] > \downarrow[x, y] \Leftrightarrow$

$D(\downarrow(\perp,\ldots,[g_s(a, b), h_s(a, b)],\ldots,\perp)) > D(\downarrow(\perp,\ldots,[g_s(x, y), h_s(x, y)],\ldots,\perp)) \Leftrightarrow$

$[g_s(a, b), h_s(a, b)] \subset [g_s(x, y), h_s(x, y)]$. ■

Now we show that the overall behavior of a display function on a continuous scalar must fall into one of two categories.

Prop. G.7. Given a display function $D{:}U \rightarrow V$ and a continuous scalar $s \in S$, then either

(a) $\forall x, y, z \in \mathbf{R}.\ x < y < z$ implies that $g_s(x, z) = g_s(x, y)\ \&\ h_s(x, y) < h_s(x, z)$ and that $g_s(x, z) < g_s(y, z)\ \&\ h_s(y, z) = h_s(x, z)$,

or

(b) $\forall x, y, z \in \mathbf{R}.\ x < y < z$ implies that $g_s(x, z) < g_s(x, y)\ \&\ h_s(x, y) = h_s(x, z)$ and that $g_s(x, z) = g_s(y, z)\ \&\ h_s(y, z) < h_s(x, z)$.

Proof. Let $x < y < z$. Then, by Prop. G.5, $g_s(x, z) = \min\{g_s(x, y), g_s(y, z)\}$ and $h_s(x, z) = \max\{h_s(x, y), h_s(y, z)\}$. If $g_s(x, z) < g_s(x, y)$ and $h_s(x, y) < h_s(x, z)$ then

$g_s(y, z) = g_s(x, z)$ and $h_s(y, z) = h_s(x, z)$, so $[g_s(x, y), h_s(x, y)] \subset [g_s(y, z), h_s(y, z)]$ and by Prop. G.6, $[x, y] \subset [y, z]$, which is impossible. Thus either $g_s(x, y) = g_s(x, z)$ or $h_s(x, y) = h_s(x, z)$. However, both equalities cannot hold, since $\downarrow(\perp,\ldots,[g_s(x, y), h_s(x, y)],\ldots,\perp) = \downarrow(\perp,\ldots,[g_s(x, z), h_s(x, z)],\ldots,\perp) \Rightarrow$ $\downarrow(\perp,\ldots,[x, y],\ldots,\perp) = \downarrow(\perp,\ldots,[x, z],\ldots,\perp)$, which is impossible. Thus $g_s(x, z) = g_s(x, y)$ & $h_s(x, y) < h_s(x, z)$ or $g_s(x, z) < g_s(x, y)$ & $h_s(x, y) = h_s(x, z)$. A similar argument applies to the relation between $[y, z]$ and $[x, z]$, so $g_s(x, z) = g_s(y, z)$ & $h_s(y, z) < h_s(x, z)$ or $g_s(x, z) < g_s(y, z)$ & $h_s(y, z) = h_s(x, z)$.

Since $g_s(x, z) = \min\{g_s(x, y), g_s(y, z)\}$ and $h_s(x, z) = \max\{h_s(x, y), h_s(y, z)\}$, if $g_s(x, z) = g_s(x, y)$ then $h_s(x, y) < h_s(x, z)$ so $h_s(x, z) = h_s(y, z)$, and if $g_s(x, z) = g_s(y, z)$ then $h_s(y, z) < h_s(x, z)$ so $h_s(x, z) = h_s(x, y)$. Thus, for all $x, y, z \in \mathbf{R}$, $x < y < z$ implies that

(c) $g_s(x, z) = g_s(x, y)$ & $h_s(x, y) < h_s(x, z)$ and $g_s(x, z) < g_s(y, z)$ & $h_s(y, z) = h_s(x, z)$,

or

(d) $g_s(x, z) < g_s(x, y)$ & $h_s(x, y) = h_s(x, z)$ and $g_s(x, z) = g_s(y, z)$ & $h_s(y, z) < h_s(x, z)$.

We need to show that either (c) is true for all $x < y < z$, or that (d) is true for all $x < y < z$.

Now let $x < y < z < w$. Apply (c) and (d) to $x < y < z$ and $x < z < w$, but assume that (c) applies in one case and that (d) applies in the other case. That is, assume that $g_s(x, w) = g_s(x, z) < g_s(x, y)$ and $h_s(x, y) = h_s(x, z) < h_s(x, w)$, or that $g_s(x, w) < g_s(x, z) = g_s(x, y)$ and $h_s(x, y) < h_s(x, z) = h_s(x, w)$. Under both of these assumptions, $g_s(x, w) < g_s(x, y)$ and $h_s(x, y) < h_s(x, w)$, which is impossible (applying the result of the previous paragraph to $x < y < w$). Thus either (c) applies to both $x < y < z$ and $x < z < w$, or (d) applies to both $x < y < z$ and $x < z < w$.

Similarly, apply (c) and (d) to $x < y < w$ and $y < z < w$, but assume that (c) applies in one case and that (d) applies in the other case. That is, assume that $g_s(x, w) < g_s(y, w) = g_s(z, w)$ and $h_s(z, w) < h_s(y, w) = h_s(x, w)$, or that

$g_s(x, w) = g_s(y, w) < g_s(z, w)$ and $h_s(z, w) = h_s(y, w) < h_s(x, w)$. Under both of these assumptions, $g_s(x, w) < g_s(z, w)$ and $h_s(z, w) < h_s(x, w)$, which is impossible (applying the result of the previous paragraph to $x < z < w$). Thus either (c) applies to both $x < y < w$ and $y < z < w$, or (d) applies to both $x < y < w$ and $y < z < w$.

Now let $x < y < z < x' < y' < z'$. The results of the last two paragraphs can be applied to show that (c) and (d) are applied consistently to the following chain of triples:

$x < y < z$

$x < y < x'$

$x < x' < y'$

$x < y' < z'$

$y < y' < z'$

$z < y' < z'$

$x' < y' < z'$.

Thus either (c) applies to both $x < y < z$ and $x' < y' < z'$, or (d) applies to both $x < y < z$ and $x' < y' < z'$.

Given any two triples $x < y < z$ and $x' < y' < z'$, pick $x'' < y'' < z''$ with $z < x''$ and $z' < x''$. Then $x < y < z < x'' < y'' < z''$ and $x' < y' < z' < x'' < y'' < z''$ so either (c) or (d) applies uniformly to the triples $x < y < z$, $x'' < y'' < z''$ and $x' < y' < z'$. Thus either (c) or (d) applies uniformly to all triples, proving the proposition. ■

Next we define names for the two categories established in Prop. G.7.

Def. Given a display function $D:U \rightarrow V$ and a continuous scalar $s \in S$, by Prop. G.7, either (a) or (b) is applies to all triples $x < y < z$. If (a) applies, say that D is *increasing* on s, and if (b) applies, say that D is *decreasing* on s.

Prop. G.8 is useful for showing how the categories established in Prop. G.7 apply to the functions g'_s and h'_s.

Prop. G.8. Given a display function $D:U \to V$, a continuous scalar $s \in S$, $z \in \mathbf{R}$, and a set $A \subseteq I_s \backslash \{\perp\}$ such that $[z, z] = \bigcap A$, then
$g'_s(a) = sup\{g_s(a, b) \mid [a, b] \in A\}$ and
$h'_s(a) = inf\{h_s(a, b) \mid [a, b] \in A\}$.

Proof.
$\downarrow(\perp,...,[z, z],...,\perp) = \{(\perp,...,[u, v],...,\perp) \mid u \leq z \leq v\} =$
$\{(\perp,...,[u, v],...,\perp) \mid \exists[a, b] \in A.\ u \leq a \leq b \leq v\} =$
$\bigcup\{\downarrow(\perp,...,[a, b],...,\perp) \mid [a, b] \in A\}$. This union of closed sets is closed (since it equals $\downarrow(\perp,...,[z, z],...,\perp)$), so, by Prop. C.8,
$\downarrow(\perp,...,[z, z],...,\perp) = \bigvee\{\downarrow(\perp,...,[a, b],...,\perp) \mid [a, b] \in A\}$. Then, by Prop. B.3,
$D(\downarrow(\perp,...,[z, z],...,\perp)) = \bigvee\{D(\downarrow(\perp,...,[a, b],...,\perp)) \mid [a, b] \in A\} =$
$\bigvee\{\downarrow(\perp,...,[g_s(a, b), h_s(a, b)],...,\perp) \mid [a, b] \in A\}$. Therefore
$\downarrow(\perp,...,[g'_s(a), h'_s(a)],...,\perp) = \bigvee\{\downarrow(\perp,...,[g_s(a, b), h_s(a, b)],...,\perp) \mid [a, b] \in A\}$. Thus
$\forall[a, b] \in A.\ \downarrow(\perp,...,[g_s(a, b), h_s(a, b)],...,\perp) \leq \downarrow(\perp,...,[g'_s(a), h'_s(a)],...,\perp)$, so
$\forall[a, b] \in A.\ g_s(a, b) \leq g'_s(a) \leq h'_s(a) \leq h_s(a, b)$. Therefore
$sup\{g_s(a, b) \mid [a, b] \in A\} \leq g'_s(a)$ and $h'_s(a) \leq inf\{h_s(a, b) \mid [a, b] \in A\}$.

Now assume that $sup\{g_s(a, b) \mid [a, b] \in A\} < g'_s(a)$ and pick u such that $sup\{g_s(a, b) \mid [a, b] \in A\} < u < g'_s(a)$. Then for all $[a, b] \in A$, $g_s(a, b) < u$ so
$\downarrow(\perp,...,[g_s(a, b), h_s(a, b)],...,\perp) \leq \downarrow(\perp,...,[u, h'_s(a)],...,\perp)$. Therefore
$\bigvee\{\downarrow(\perp,...,[g_s(a, b), h_s(a, b)],...,\perp) \mid [a, b] \in A\} \leq$
$\downarrow(\perp,...,[u, h'_s(a)],...,\perp) < \downarrow(\perp,...,[g'_s(a), h'_s(a)],...,\perp)$,

which contradicts

$\bigvee\{\downarrow(\perp,\ldots,[g_s(a, b), h_s(a, b)],\ldots,\perp) \mid [a, b] \in A\} = \downarrow(\perp,\ldots,[g'_s(a), h'_s(a)],\ldots,\perp)$. Thus $g'_s(a) = sup\{g_s(a, b) \mid [a, b] \in A\}$. A similar argument shows that $h'_s(a) = inf\{h_s(a, b) \mid [a, b] \in A\}$. ■

Now we show how the categories of behavior established in Prop. G.7 apply to the functions g'_s and h'_s.

Prop. G.9. Given a display function $D{:}U \rightarrow V$, a continuous scalar $s \in S$, and $z < z'$, if D is increasing on s then $g'_s(z) < g'_s(z')$ and $h'_s(z) < h'_s(z')$, and if D is decreasing on s then $g'_s(z) > g'_s(z')$ and $h'_s(z) > h'_s(z')$.

Proof. First assume that D is increasing on s. Then, by Prop. G.8, $g'_s(z) = sup\{g_s(z, x) \mid z < x\}$. By Prop. G.7, $\forall x > z.\ \forall y > z.\ g_s(z, x) = g_s(z, y)$, so $\forall x > z.\ g'_s(z) = g_s(z, x)$. Similarly, $\forall x > z'.\ g'_s(z') = g_s(z', x)$. Pick $x > z' > z$. Then, by Prop. G.7, $g'_s(z) = g_s(z, x) < g_s(z', x) = g'_s(z')$.

By Prop. G.8, $h'_s(z) = inf\{h_s(x, z) \mid x < z\}$. By Prop. G.7, $\forall x < z.\ \forall y < z.\ h_s(x, z) = h_s(y, z)$, so $\forall x < z.\ h'_s(z) = h_s(x, z)$. Similarly, $\forall x < z'.\ h'_s(z') = h_s(x, z')$. Pick $x < z < z'$. Then, by Prop. G.7, $h'_s(z) = h_s(x, z) < h_s(x, z') = h'_s(z')$.

Next assume that D is decreasing on s. Then, by Prop. G.8, $g'_s(z) = sup\{g_s(x, z) \mid x < z\}$. By Prop. G.7, $\forall x < z.\ \forall y < z.\ g_s(x, z) = g_s(y, z)$, so $\forall x < z.\ g'_s(z) = g_s(x, z)$. Similarly, $\forall x < z'.\ g'_s(z') = g_s(x, z')$. Pick $x < z < z'$. Then, by Prop. G.7, $g'_s(z) = g_s(x, z) > g_s(x, z') = g'_s(z')$.

By Prop. G.8, $h'_s(z) = inf\{h_s(z, x) \mid z < x\}$. By Prop. G.7, $\forall x > z.\ \forall y > z.\ h_s(z, x) = h_s(z, y)$, so $\forall x > z.\ h'_s(z) = h_s(z, x)$. Similarly,

$\forall x > z'.\ h'_s(z') = h_s(z', x)$. Pick $x > z' > z$. Then, by Prop. G.7,

$h'_s(z) = h_s(z, x) > h_s(z', x) = h'_s(z')$. ■

Next we show that the functions g'_s and h'_s must be continuous functions of real variables. The key idea is that g'_s and h'_s are either increasing or decreasing, so if they are discontinuous there must be a gap in their values, which contradicts Prop. B.2.

Prop. G.10. Given a display function $D{:}U \to V$ and a continuous scalar $s \in S$, the functions g'_s and h'_s are continuous (in the topological sense).

Proof. Assume that D is increasing on s. Then, by Prop. G.9, g'_s and h'_s are monotone increasing. Now assume that g'_s is discontinuous at z. Then

(a) $\exists \varepsilon > 0.\ \forall \delta > 0.\ \exists w.$

$z - \delta < w < z \ \&\ g'_s(w) \leq g'_s(z) - \varepsilon$ or

$z < w < z + \delta \ \&\ g'_s(z) + \varepsilon \leq g'_s(w)$

Fix ε satisfying (5). If

(b) $\exists w_-.\ (w_- < z \ \&\ g'_s(z) - \varepsilon < g'_s(w_-))$

then

(c) $\forall w.\ w_- < w < z \Rightarrow g'_s(z) - \varepsilon < g'_s(w) < g'_s(z)$

and if

(d) $\exists w_+.\ (z < w_+ \ \&\ g'_s(w_+) < g'_s(z) + \varepsilon)$

then

(e) $\forall w.\ z < w < w_+ \Rightarrow g'_s(z) < g'_s(w) < g'_s(z) + \varepsilon$.

Now, ((c) & (e)) contradicts (a), so ($\neg$(b) or $\neg$(d)).

$\neg\text{(b)} \equiv \forall w.\ w < z \Rightarrow g'_s(w) \leq g'_s(z) - \varepsilon$

and

$\neg$(d) $\equiv \forall w.\ z < w \Rightarrow g'_s(z) + \varepsilon \leq g'_s(w)$.

In the $\neg$(b) case, since $z \leq w \Rightarrow g'_s(z) \leq g'_s(w)$, there is no $w \in \mathbf{R}$ such that $g'_s(z) - \varepsilon < g'_s(w) < g'_s(z)$. Now, $[g'_s(z), h'_s(z)] \subset [g'_s(z) - \varepsilon/2, h'_s(z)]$ so $\downarrow(\perp,...,[g'_s(z) - \varepsilon/2, h'_s(z)],...,\perp) \leq \downarrow(\perp,...,[g'_s(z), h'_s(z)],...,\perp)$. Thus, by Prop. B.2, there is $u \in U$ such that $D(u) = \downarrow(\perp,...,[g'_s(z) - \varepsilon/2, h'_s(z)],...,\perp)$, and by Prop. F.9 and Prop. F.10, $u \in U_s$. Let $u = \downarrow(\perp,...,[a, b],...,\perp)$. Then, by Prop. G.4, $g'_s(z) - \varepsilon/2 = g_s(a, b) = inf\{g'_s(w) \mid a \leq w \leq b\}$. However, since there is no w such that $g'_s(z) - \varepsilon < g'_s(w) < g'_s(z)$, this is impossible. Thus g'_s cannot be discontinuous at z.

In the $\neg$(d) case, since $w \leq z \Rightarrow g'_s(w) \leq g'_s(z)$, there is no $w \in \mathbf{R}$ such that $g'_s(z) < g'_s(w) < g'_s(z) + \varepsilon$, and furthermore, $z < z' \Rightarrow g'_s(z) < g'_s(z')$, so there is z' such that $g'_s(z) + \varepsilon \leq g'_s(z')$. Now, $[g'_s(z'), h'_s(z')] \subset [g'_s(z) + \varepsilon/2, h'_s(z')]$ so $\downarrow(\perp,...,[g'_s(z) + \varepsilon/2, h'_s(z')],...,\perp) \leq \downarrow(\perp,...,[g'_s(z'), h'_s(z')],...,\perp)$. Thus, by Prop. B.2, there is $u \in U$ such that $D(u) = \downarrow(\perp,...,[g'_s(z) + \varepsilon/2, h'_s(z')],...,\perp)$, and by Prop. F.9 and Prop. F.10, $u \in U_s$. Let $u = \downarrow(\perp,...,[a, b],...,\perp)$. Then, by Prop. G.4, $g'_s(z) + \varepsilon/2 = g_s(a, b) = inf\{g'_s(w) \mid a \leq w \leq b\}$. However, since there is no w such that $g'_s(z) < g'_s(w) < g'_s(z) + \varepsilon$, this is impossible. Thus g'_s cannot be discontinuous at z.

The proof that h'_s is continuous, and the proofs that g'_s and h'_s are continuous when D is decreasing on s, are virtually identical to this. ■

Prop. G.11 completes the list of conditions on the functions g'_s and h'_s that will allow us to define necessary and sufficient conditions for display functions.

Prop. G.11. Given a display function $D:U \rightarrow V$ and a continuous scalar $s \in S$, then g'_s has no lower bound and h'_s has no upper bound. Furthermore, $\forall z \in \mathbf{R}.\ g'_s(z) \leq h'_s(z)$.

Proof. If $\exists a. \forall z. g'_s(z) > a$ then,

$D(\downarrow(\perp,\ldots,[0, 0],\ldots,\perp)) = \downarrow(\perp,\ldots,[g'_s(0), h'_s(0)],\ldots,\perp) \geq \downarrow(\perp,\ldots,[a\text{-}1, h'_s(0)],\ldots,\perp)$

[since $a\text{-}1 < a \leq g'_s(0)$], so there must be $u \in U$ such that

$D(u) = \downarrow(\perp,\ldots,[a\text{-}1, h'_s(0)],\ldots,\perp)$. By Prop. F.9 and Prop. F.10, $u \in U_s$. However, by Prop. G.4, there is no $[x, y] \in I_s$ such that

$D(\downarrow(\perp,\ldots,[x, y],\ldots,\perp)) = \downarrow(\perp,\ldots,[a\text{-}1, h'_s(0)],\ldots,\perp)$. Thus g'_s has no lower bound. The proof that h'_s has no upper bound is virtually identical.

If $g'_s(z) > h'_s(z)$ then $[g'_s(z), h'_s(z)] \notin I_s$, which is impossible, so $\forall z \in \mathbf{R}.\ g'_s(z) \leq h'_s(z)$. ■

The results of this appendix can be summarized in the following definition.

Def. A pair of functions $g'_s{:}\mathbf{R} \to \mathbf{R}$ and $h'_s{:}\mathbf{R} \to \mathbf{R}$ are called a *continuous display pair* if:

(a) g'_s has no lower bound and h'_s has no upper bound,

(b) $\forall z \in \mathbf{R}.\ g'_s(z) \leq h'_s(z)$, and

(c) g'_s and h'_s are continuous,

(d) either g'_s and h'_s are increasing:

$\forall z, z' \in \mathbf{R}.\ z < z' \Rightarrow g'_s(z) < g'_s(z')\ \&\ h'_s(z) < h'_s(z')$,

or g'_s and h'_s are decreasing:

$\forall z, z' \in \mathbf{R}.\ z < z' \Rightarrow g'_s(z) > g'_s(z')\ \&\ h'_s(z) > h'_s(z')$.

Appendix H

Proofs for Section 3.4.3

Here we present the technical details for Section 3.4.3.

Def. Given a finite set S of scalars, a finite set DS of display scalars, $X = \mathbf{X}\{I_s \mid s \in S\}$, $Y = \mathbf{X}\{I_d \mid d \in DS\}$, $U = CL(\mathrm{X})$, and $V = CL(\mathrm{Y})$, then a function $D{:}U \rightarrow V$ is a *scalar mapping function* if:

(a) there is a function $MAP_D{:}\ S \rightarrow POWER(DS)$ such that

$\forall s, s' \in S.\ MAP_D(s) \cap MAP_D(s') = \phi$,

(b) for all continuous $s \in S$, $MAP_D(s)$ contains a single continuous $d \in DS$,

(c) for all discrete $s \in S$, all $d \in MAP_D(s)$ are discrete,

(d) $D(\phi) = \phi$ and $D(\{(\perp,\ldots,\perp)\}) = \{(\perp,\ldots,\perp)\}$,

(e) for all continuous $s \in S$, g'_s and h'_s are a continuous display pair,

for all $[u, v] \in I_s$, $g_s(u, v) = inf\{g'_s(z) \mid u \leq z \leq v\}$ and

$h_s(u, v) = sup\{h'_s(z) \mid u \leq z \leq v\}$,

and, given $\{d\} = MAP_D(s)$, then for all $[u, v] \in I_s \backslash \{\perp\}$,

$D(\downarrow(\perp,\ldots,[u, v],\ldots,\perp)) = \downarrow(\perp,\ldots,[g_s(u, v), h_s(u, v)],\ldots,\perp) \in V_d$,

(f) for all discrete $s \in S$, for all $a \in I_s \backslash \{\perp\}$,

$D(\downarrow(\perp,\ldots,a,\ldots,\perp)) = \mathrm{b} \in V_d$ for some $d \in MAP_D(s)$, where $b \neq \{(\perp,\ldots,\perp)\}$,

and, for all $a, a' \in I_s \backslash \{\perp\}$, $a \neq a' \Rightarrow D(\downarrow(\perp,\ldots,a,\ldots,\perp)) \neq D(\downarrow(\perp,\ldots,a',\ldots,\perp))$

(g) for all $x \in X$, $D(\downarrow x) = \downarrow \bigvee \{y \mid \exists s \in S.\ x_s \neq \perp\ \&\ \downarrow y = D(\downarrow(\perp,\ldots,x_s,\ldots,\perp))\}$,

where x_s represents tuple components of x, and using the values for D defined in (e) and (f),

(h) for all $u \in U$, $D(u) = \bigvee\{D(\downarrow x) \mid x \in u\}$, using the values for D defined in (g).

This definition contains a variety of expressions for the value of D on various subsets of U. The next proposition shows that these expressions are consistent where the subsets of U overlap. This involves showing that D is monotone.

Prop. H.1. In the definition of scalar mapping functions, the values defined for D in (d), (e), (f), (g) and (h) are consistent. Furthermore, D is monotone.

Proof. (e), (f), (g) and (h) do not apply to ϕ and thus do not conflict with the definition of $D(\phi)$ in (d). (e) and (f) do not apply to $\{(\bot,\ldots,\bot)\}$ and thus do not conflict with the definition of $D(\{(\bot,\ldots,\bot)\})$ in (d). The definition of $D(\{(\bot,\ldots,\bot)\})$ in (d) is consistent with (g) and (h) if the *sup* of an empty set of objects is defined as $(\bot,\ldots,\bot)$. (e) and (f) apply to disjoint sets and thus do not conflict. For all $s \in S$, (g) applies to objects $x \in U_s$, but defines $D(\downarrow x)$ as the *sup* of the singleton set containing the value of $D(\downarrow x)$ defined by (e) or (f), and is thus consistent with that value. (h) applies to objects $x \in U_s$, and is consistent with (e) and (f) if it is consistent with (g) on these objects. Thus we need to show the consistency of (g) and (h).

If $u = \downarrow y$ then (h) defines $D(\downarrow y) = \bigvee\{D(\downarrow x) \mid x \in \downarrow y\} = \bigvee\{D(\downarrow x) \mid x \leq y\}$. To Show consistency with (g), it is necessary to show that $x \leq y \Rightarrow D(\downarrow x) \leq D(\downarrow y)$ for the definition of D in (d), (e), (f) and (g) (that is, that D is monotone). Clearly D in (d) is monotone, in itself and in relation to D in (e), (f) and (g). If $s \in S$ is discrete, then for all $a, a' \in I_s\backslash\{\bot\}$, $a \neq a' \Rightarrow \neg(a \leq a')$, so D in (f) is monotone by default. If $s \in S$ is continuous then for all $[u, v], [u', v'] \in I_s\backslash\{\bot\}$,

$\downarrow(\bot,\ldots,[u, v],\ldots,\bot) \leq \downarrow(\bot,\ldots,[u', v'],\ldots,\bot) \Rightarrow$

$[u', v'] \subseteq [u, v] \Rightarrow$

$[inf\{g'_s(z) \mid u' \leq z \leq v'\}, sup\{h'_s(z) \mid u' \leq z \leq v'\}] \subseteq$

$[inf\{g'_s(z) \mid u \leq z \leq v\}, sup\{h'_s(z) \mid u \leq z \leq v\}] \Rightarrow$

$D(\downarrow(\perp,\ldots,[u, v],\ldots,\perp)) \leq D(\downarrow(\perp,\ldots,[u', v'],\ldots,\perp))$.

Thus D in (e) is monotone. For all $x, x' \in X$,

$x \leq x' \Rightarrow$

$\forall s \in S. x_s \leq x_s' \Rightarrow$ (since D in (e) and (f) is monotone)

$\forall s \in S. D(\downarrow(\perp,\ldots,x_s,\ldots,\perp)) \leq D(\downarrow(\perp,\ldots,x'_s,\ldots,\perp)) \Rightarrow$

$D(\downarrow x) \leq D(\downarrow x')$.

Thus D in (g) is monotone, so D is consistent in (g) and (h).

All that remains is to show that D in (h) is monotone. For all $u, u' \in U$, $u \leq u' \Rightarrow u \subseteq u'$ so $\bigvee\{D(\downarrow x) \mid x \in u\} \leq \bigvee\{D(\downarrow x) \mid x \in u'\}$. Thus D is monotone. ■

As we will show in Prop. H.5, the values of a scalar mapping function D can be decomposed into the values of an auxiliary function D' from X to Y. Now we define this auxiliary function, show that it is an order embedding, and prove two lemmas that will be useful in the proof of Prop. H.5.

Def. Given a scalar mapping function $D:U \rightarrow V$, define $D':X \rightarrow Y$ by $D'(x) = \bigvee\{(\perp,\ldots,a_d,\ldots,\perp) \mid s \in S \;\&\; x_s \neq \perp \;\&\; D(\downarrow(\perp,\ldots,x_s,\ldots,\perp)) = \downarrow(\perp,\ldots,a_d,\ldots,\perp)\}$.

Prop. H.2. Given a scalar mapping function $D:U \rightarrow V$, D' is an order embedding.

Proof. Given $x, x' \in X$, $x \leq x' \Leftrightarrow \forall s \in S. x_s \leq x'_s$. Let $D'((\perp,\ldots,x_s,\ldots,\perp)) = (\perp,\ldots,a_d,\ldots,\perp)$ and $D'((\perp,\ldots,x'_s,\ldots,\perp)) = (\perp,\ldots,a'_d,\ldots,\perp)$ where $d \in MAP_D(s)$. Note that $x_s \leq x'_s \Rightarrow (\perp,\ldots,x_s,\ldots,\perp) \leq (\perp,\ldots,x'_s,\ldots,\perp) \Rightarrow$ $(\perp,\ldots,a_d,\ldots,\perp) \leq (\perp,\ldots,a'_d,\ldots,\perp)$ (since a D is monotone) so a_d and a'_d are in the same I_d.

For all $s \in S$, $x_s \le x'_s \Leftrightarrow (\bot,...,x_s,...,\bot) \le (\bot,...,x'_s,...,\bot) \Leftrightarrow$ $\downarrow(\bot,...,a_d,...,\bot) = D(\downarrow(\bot,...,x_s,...,\bot)) \le D(\downarrow(\bot,...,x'_s,...,\bot)) = \downarrow(\bot,...,a'_d,...,\bot) \Leftrightarrow$ $(\bot,...,a_d,...,\bot) \le (\bot,...,a'_d,...,\bot) \Leftrightarrow a_d \le a'_d$. Thus $(\forall s \in S.\ x_s \le x'_s) \Leftrightarrow (\forall d \in DS.\ a_d \le a'_d)$. Since $\forall s, s' \in S.\ MAP_D(s) \cap MAP_D(s') = \phi$, $c = D'(x) \Rightarrow (\forall d \in DS.\ c_d \ne \bot \Rightarrow \exists s \in S.\ D(\downarrow(\bot,...,x_s,...,\bot)) = \downarrow(\bot,...,c_d,...,\bot))$ (that is, $c_d = a_d$), and thus $(\forall d \in DS.\ a_d \le a'_d) \Leftrightarrow D'(x) \le D'(x')$. Therefore, by a chain of logical equivalences, $x \le x' \Leftrightarrow D'(x) \le D'(x')$. ■

Prop. H.3. Let $D{:}U \rightarrow V$ be a scalar mapping function. Then, for all $u \in U$, $x \in u$ and $b \le D'(x) = a$, there is $y \le x$ such that $b = D'(y)$.

Proof. For all $d \in DS$, $b_d \ne \bot$ implies that $\exists s \in S.\ D'((\bot,...,x_s,...,\bot)) = (\bot,...,a_d,...,\bot)$ and $b_d \le a_d$. For discrete s, $b_d \le a_d\ \&\ b_d \ne \bot \Rightarrow b_d = a_d$. Thus $D'((\bot,...,x_s,...,\bot)) = (\bot,...,b_d,...,\bot)$. Let $y_s = x_s$.

For continuous s, let $a_d = [inf\{g'_s(z) \mid u \le z \le v\}, sup\{h'_s(z) \mid u \le z \le v\}]$ where $x_s = [u, v]$. There are $e, f \in \mathbf{R}$ such that $b_d = [e, f]$ where $e \le inf\{g'_s(z) \mid u \le z \le v\} \le sup\{h'_s(z) \mid u \le z \le v\} \le f$. Since g'_s is continuous and has no lower bound, $\exists u'.\ g'_s(u') = e$, and since h'_s is continuous and has no upper bound, $\exists v'.\ h'_s(v') = f$. Now g'_s and h'_s are either increasing or decreasing.

If g'_s and h'_s are increasing then $u' \le u$ and $v \le v'$, so $e = inf\{g'_s(z) \mid u' \le z \le v'\}$ [since $u' \le z \Rightarrow g'_s(u') \le g'_s(z)$] and $f = sup\{h'_s(z) \mid u' \le z \le v'\}$ [since $z \le v' \Rightarrow h'_s(z) \le h'_s(v')$]. Then $b_d = [e, f] = [inf\{g'_s(z) \mid u' \le z \le v'\}, sup\{h'_s(z) \mid u' \le z \le v'\}]$ and $D'((\bot,...,[u', v'],...,\bot)) = (\bot,...,b_d,...,\bot)$. Let $y_s = [u', v']$.

If g'_s and h'_s are decreasing then $v' \leq u$ and $v \leq u'$, so $e = inf\{g'_s(z) \mid v' \leq z \leq u'\}$ [since $z \leq u' \Rightarrow g'_s(u') \leq g'_s(z)$] and $f = sup\{h'_s(z) \mid v' \leq z \leq u'\}$ [since $v' \leq z \Rightarrow h'_s(z) \leq h'_s(v')$]. Then $b_d = [e, f] = [inf\{g'_s(z) \mid v' \leq z \leq u'\}, sup\{h'_s(z) \mid v' \leq z \leq u'\}]$ and $D'((\perp,...,[v', u'],...,\perp)) = (\perp,...,b_d,...,\perp)$. Let $y_s = [v', u']$.

Thus for all $d \in DS$ such that $b_d \neq \perp$, there is $y_s \leq x_s$ such that $D'((\perp,...,y_s,...,\perp)) = (\perp,...,b_d,...,\perp)$. For any $s \in S$ such that y_s is not determined by any b_d, set $y_s = \perp$. Then $D'(y) = b$. ■

Prop. H.4. Given a scalar mapping function $D{:}U \rightarrow V$, and a directed set $M \subseteq X$, $D'(\bigvee M) = \bigvee D'(M)$.

Proof. Given a directed set $M \subseteq X$, let $x = \bigvee M$ and y $= D'(x)$. Since D' is an order embedding, $D'(M)$ is directed so $z = \bigvee D'(M)$ exists. Also, $\forall m \in M.\ m \leq x$, so $\forall m \in M.\ D'(m) \leq y$ and thus $z \leq y$. For all $d \in DS$, if $y_d \neq \perp$ then there is $s \in S$ such that $\downarrow(\perp,...,y_d,...,\perp) = D(\downarrow(\perp,...,x_s,...,\perp))$, and so $(\perp,...,y_d,...,\perp) = D'((\perp,...,x_s,...,\perp))$. Since *sups* are taken componentwise in X, $x_s = \bigvee\{m_s \mid m \in M\}$.

If s is discrete, then $\exists m \in M.\ x_s = m_s$ so $(\perp,...,y_d,...,\perp) = D'((\perp,...,m_s,...,\perp)) \leq D'(m) \leq z$, and thus $y_d \leq z_d$. Since $z \leq y$, and thus $z_d \leq y_d$, this gives $y_d = z_d$.

If s is continuous, then $x_s = [u, v]$ and $m_s = [u_m, v_m]$ are real intervals (we adopt the convention that $u_m = -\infty$ and $v_m = \infty$ for $m_s = \perp$). Then $[u, v]$ is the intersection of the $[u_m, v_m]$, for all $m \in M$, so $u = sup\{u_m \mid m \in M\}$ and $v = inf\{v_m \mid m \in M\}$ and thus $y_d = [a, b] = [inf\{g'_s(z) \mid u \leq z \leq v\}, sup\{h'_s(z) \mid u \leq z \leq v\}]$. Also let $z_d = [e, f]$. Then, since $MAP_D(s)$ contains only d,
$e = sup\{inf\{g'_s(z) \mid u_m \leq z \leq v_m\} \mid m \in M\}$ and

$f = inf\{sup\{h'_s(z) \mid u_m \leq z \leq v_m\} \mid m \in M\}$.

If g'_s and h'_s are increasing then, since they are continuous,

$a = inf\{g'_s(z) \mid sup\{u_m \mid m \in M\} \leq z \leq inf\{v_m \mid m \in M\}\} = g'_s(sup\{u_m \mid m \in M\}) =$

$sup\{g'_s(u_m) \mid m \in M\} = sup\{inf\{g'_s(z) \mid u_m \leq z \leq v_m\} \mid m \in M\} = e$ and

$b = sup\{h'_s(z) \mid sup\{u_m \mid m \in M\} \leq z \leq inf\{v_m \mid m \in M\}\} = h'_s(inf\{v_m \mid m \in M\}) =$

$inf\{h'_s(v_m) \mid m \in M\} = inf\{sup\{h'_s(z) \mid u_m \leq z \leq v_m\} \mid m \in M\} = f$.

If g'_s and h'_s are decreasing then, since they are continuous,

$a = inf\{g'_s(z) \mid sup\{u_m \mid m \in M\} \leq z \leq inf\{v_m \mid m \in M\}\} = g'_s(inf\{v_m \mid m \in M\}) =$

$sup\{g'_s(v_m) \mid m \in M\} = sup\{inf\{g'_s(z) \mid u_m \leq z \leq v_m\} \mid m \in M\} = e$ and

$b = sup\{h'_s(z) \mid sup\{u_m \mid m \in M\} \leq z \leq inf\{v_m \mid m \in M\}\} = h'_s(sup\{u_m \mid m \in M\}) =$

$inf\{h'_s(u_m) \mid m \in M\} = inf\{sup\{h'_s(z) \mid u_m \leq z \leq v_m\} \mid m \in M\} = f$.

In either case, $y_d = [a, b] = [e, f] = z_d$.

Thus $y_d = z_d$ for all $d \in DS$ such that $y_d \neq \perp$. However, we also have $z \leq y$ so $z_d = \perp$ whenever $y_d = \perp$, so $y_d = z_d$ for all $d \in DS$ and thus $y = z$. ■

Now we show how a scalar mapping function D can be defined in terms of the auxiliary function D'.

Prop. H.5. Given a scalar mapping function $D{:}U \to V$, for all $u \in U$, $D(u) = \{D'(x) \mid x \in u\}$.

Proof. First, we show that for all $u \in U$, u is closed $\Rightarrow \{D'(x) \mid x \in u\}$ is closed. Assume $x \in u$ and $b \leq D'(x)$. Then, by Prop. H.3, $\exists y \leq x.\ b = D'(y)$. Further, $y \leq x \Rightarrow y \in u$ so $b \in \{D'(x) \mid x \in u\}$. Now assume $N \subseteq \{D'(x) \mid x \in u\}$ and N is directed. Then there is $M \subseteq u$ such that $N = D'(M)$, and, since D' is an order embedding, M is directed. Thus $\bigvee M \in u$ and, by Prop. H.4, $\bigvee N = D'(\bigvee M) \in \{D'(x) \mid x \in u\}$. Thus

$\{D'(x) \mid x \in u\}$ is closed.

Second, we show that for all $x \in X$, $D(\downarrow x) = \{D'(y) \mid y \leq x\}$. By (g) in the definition of scalar mapping functions, $\forall y \in X. \exists b \in Y. D(\downarrow y) = \downarrow b$. Furthermore, comparing (g) with the definition of D', $\forall y \in X. D(\downarrow y) = \downarrow b \Leftrightarrow D'(y) = b$. Then, given $D(\downarrow x) = \downarrow a$, $b \leq a \Leftrightarrow \downarrow b \leq \downarrow a \Leftrightarrow \exists y \leq x. D(\downarrow y) = \downarrow b \Leftrightarrow \exists y \leq x. D'(y) = b$. Thus $D(\downarrow x) = \downarrow a = \{b \mid b \leq a\} = \{D'(y) \mid y \leq x\}$.

By Prop. C.8, $\bigvee\{D(\downarrow x) \mid x \in u\}$ is the smallest closed set containing $\bigcup\{D(\downarrow x) \mid x \in u\}$. However, $\bigcup\{D(\downarrow x) \mid x \in u\} = \bigcup\{\{D'(y) \mid y \leq x\} \mid x \in u\} = \{D'(x) \mid x \in u\}$, which is closed, so $\bigvee\{D(\downarrow x) \mid x \in u\} = \bigcup\{D(\downarrow x) \mid x \in u\}$. Thus, for all $u \in U$, $D(u) = \bigvee\{D(\downarrow x) \mid x \in u\} = \{D'(x) \mid x \in u\}$. ■

The next two propositions show that a scalar mapping function satisfies the conditions of a display function.

Prop. H.6. A scalar mapping function $D:U \to V$ is an order embedding (and thus injective).

Proof. By Prop. H.5, for all $u \in U$, $D(u) = \{D'(x) \mid x \in u\}$. Members of U are ordered by set inclusion, so $u \leq u' \Rightarrow u \subseteq u' \Rightarrow D(u) = \{D'(x) \mid x \in u\} \subseteq \{D'(x) \mid x \in u'\} = D(u') \Rightarrow D(u) \leq D(u')$. By Prop. H.2, D' is an order embedding, and thus injective, so $u = \{(D')^{-1}(x) \mid x \in D(u)\}$. Therefore $D(u) \leq D(u') \Rightarrow D(u) \subseteq D(u') \Rightarrow$ $u = \{(D')^{-1}(x) \mid x \in D(u)\} \subseteq \{(D')^{-1}(x) \mid x \in D(u')\} = u' \Rightarrow u \leq u'$. Thus D is an order embedding. ■

Prop. H.7. A scalar mapping function $D{:}U \rightarrow V$ is a surjective function onto $\downarrow D(X)$.

Proof. Assume that $v' < v = D(X)$. We need to show that there is $u' \in U$ such that $v' = D(u')$. As we saw in the proof of Prop. H.6, if there is such a u', then $u' = \{(D')^{-1}(x) \mid x \in v'\}$. Thus let $u' = \{(D')^{-1}(x) \mid x \in v'\}$, and we will show that this is a closed set, and thus a member of U.

Assume that $y \in u'$ and $b \leq y$. Then $D'(b) \leq D'(y)$, and since $D'(y) \in v'$ and v' is closed, $D'(b) \in v'$ so $b \in u'$. Now assume that $N \subseteq u'$ and N is directed. Then $M = D'(N) \subseteq v'$ is directed (since D' is an order embedding), so $\bigvee M \in v'$ and $(D')^{-1}(\bigvee M) \in u'$. By Prop. H.4, $\bigvee M = D'(\bigvee N)$ so $\bigvee N = (D')^{-1}(\bigvee M) \in u'$. Thus u' is closed. ■

The results of the last three sections show that display functions are completely characterized as scalar mapping functions. This is summarized by the following theorem.

Theorem H.8. $D{:}U \rightarrow V$ is a display function if and only if it is a scalar mapping function.

Proof. If $D{:}U \rightarrow V$ is a display function then Theorem F.14 shows that D satisfies conditions (a), (b), (c) and (f) of the definition of scalar mapping functions. Theorem F.14, along with Props. G.4, G.9, G.10 and G.11 show that D satisfies condition (e). Prop. F.2 shows that D satisfies condition (d). Prop. F.12 shows that D satisfies condition (g), and the proof of Prop. F.13 shows that D satisfies condition (h). Thus D is a scalar mapping function.

If $D{:}U \rightarrow V$ is a scalar mapping function then Props. H.6 and H.7 show that D is a display function. ■