Waveform Iterative Methods for Solving Initial Value Problems

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Overview

1. Motivation
2. Classical waveform techniques for solving initial value problems
3. Accelerating classical waveform techniques
4. Application: semiconductor device transient simulation
5. Experimental results
6. Conclusion

Everything you always wanted to know about waveform methods . . . .

1. What are they?
2. What are they good for?
3. Don't they converge too slowly?
4. Are they *really* useful for anything?
5. Do you have any experimental results?
6. What are the open questions?

Cluster-Based Computing with PVM

Advantages:
· Availability
· Cost
· Ease of use

Disadvantages
· Communication latency — O(10ms)

Motivation

If solving

$$Ax = b$$

takes longer in parallel than serially, can

$$\dot{x} + Ax = f$$

be solved faster in parallel than serially?

Solving Initial Value Problems

Consider the nonlinear initial value problem

$$F(x, \dot{x}, t) = 0$$
$$x(0) = x_0$$

Standard approach for numerical solution
1. Discretize in time (with implicit formula)
2. Solve nonlinear problem with Newton-Raphson
3. Solve linear system with Gaussian elimination

Disadvantages
· Parallelization = parallelization of matrix solution
· Bad when matrix solution does not parallelize well
· Bad when synchronization costs are high (as in a cluster of workstations)

Waveform Methods

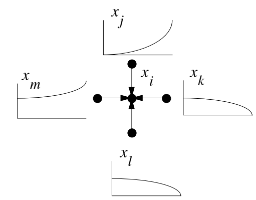
The equation $F(x, \dot{x}, t) = 0$ represents

$$F_1(x_1, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N, t) = 0$$
$$F_2(x_1, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N, t) = 0$$
$$\ldots$$
$$F_N(x_1, \ldots, x_N, \dot{x}_1, \ldots, \dot{x}_N, t) = 0$$

Basic idea:

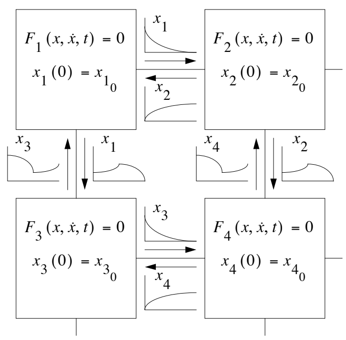
- Before discretization
- Map one equation per processor
- Each processor solves scalar initial value problem

Waveform Iteration



- Compute $x_i(t)$ for $t \in [0, T]$
- Hold $x_j(t)$, $x_k(t)$, $x_l(t)$, $x_m(t)$ fixed
- Compute $x_j(t)$, etc.

Data to Processor Mapping



Waveform Relaxation

Gauss-Jacobi for $Ax = b$

Iterate: For $k = 0, 1, \ldots$ solve
For $i = 0, 1, \ldots$ solve

$$a_{ii} x_i^{k+1} = b_i - \sum_{j \neq i} a_{ij} x_j^k$$

Jacobi WR for $\dot{x} + Ax = f$

Iterate: For $k = 0, 1, \ldots$ solve
For $i = 0, 1, \ldots$ solve

$$\dot{x}_i^{k+1} + a_{ii} x_i^{k+1} = f_i - \sum_{j \neq i} a_{ij} x_j^k$$

Methods for Nonlinear Problems

Jacobi WR for solving $\dot{x} + F(x) = 0$

Iterate: For $k = 0, 1, \ldots$ solve
For $i = 0, 1, \ldots$ solve

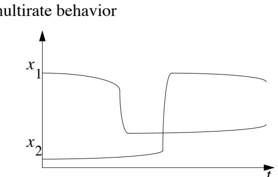
$$\dot{x}_i^{k+1} + F_i(x_i^k, \ldots, x_i^{k+1}, \ldots, x_N^k) = 0$$
$$x_i^{k+1}(0) = x_{i_0}$$

Jacobi WRN

- Take only one Newton step
- Use timepoint value from previous waveform iterate as initial guess for Newton

Advantages

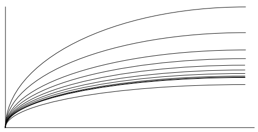
- Avoid directly solving large matrices
- Exploit multirate behavior



- Parallelizable
- Synchronize only once per WR iteration (independent of number of timesteps)
- Communicate in large packets

Disadvantages

- Slow convergence



Acceleration schemes

- Waveform SOR
- Waveform Krylov-subspace techniques

Operator Equation Formulation

Jacobi WR for $\dot{x} + Ax = f$ (compactly)

Iterate: For $k = 0, 1, \ldots$ solve

$$\dot{x}^{k+1} + Mx^{k+1} = Nx^{k+1} + f$$

where $A = M - N$

Solving for x^{k+1}

$$x^{k+1} = \int_0^t \Phi_M(t,s)\, N x^k(s)\, ds$$

$$+ \Phi_M(t,0)\, x(0) + \int_0^t \Phi_M(t,s)\, f(s)\, ds$$

Operator Equation Formulation

Rewrite compactly as:

$$x^{k+1} = \boldsymbol{K} x^k + \psi$$

Solution x satisfies

$$(I - \boldsymbol{K})\, x = \psi$$

Properties of $\boldsymbol{K}$

- Compact
- Not self adjoint

Waveform GMRES

1. *Start:* Set $r^0 = \psi - (I - \boldsymbol{K})\, x^0$, $v^1 = r^0 / \| r^0 \|$
2. *Iterate:* For $k = 1, 2, \ldots$, until satisfied do:

$$h_{j,k} = \langle (I - \boldsymbol{K})\, v^k, v^j \rangle, j = 1, \ldots, k$$

$$\hat{v}^{k+1} = (I - \boldsymbol{K})\, v^k - \sum_{j=1}^{k} h_{j,k} v^j$$

$$h_{k+1,k} = \| \hat{v}^{k+1} \|$$

$$v^{k+1} = \hat{v}^{k+1} / h_{k+1,k}$$

3. *Form approximate solution:*

$x^k = x^0 + V^k y^k$ where y^k minimizes $\| \beta e_1 - \bar{H}^k y^k \|$

Fundamental Operations

- Operator-Waveform product
  To calculate $w = (I - \boldsymbol{K})\, p$
  1. Calculate $\boldsymbol{K} p$ with one step of normal WR, i.e., solve

$$\dot{y} + My = Np$$
$$y(0) = 0$$

  2. Set $w = p - y$

- Inner product
  Use L_2 inner product

$$\langle x, y \rangle = \sum_{i=1}^{N} \int_0^T x_i(t)\, y_i(t)\, dt$$

  Integration performed in parallel followed by global sum

Nonlinear Systems

Apply Newton to $\dot{x} + F(x) = 0$

Iterate: For $m = 0, 1, \ldots$, solve

$$\dot{x}^{m+1} + J_F(x^m)\, x^{m+1} = J_F(x^m)\, x^m - F(x^m)$$

$$x^{m+1} = x_0$$

with WGMRES

Device Transient Simulation

- Governing equations

$$\frac{kT}{q}\nabla\cdot(\varepsilon\nabla u) + q(p-n+N_D-N_A) = 0$$

$$\nabla\cdot J_n - q\left(\frac{\partial n}{\partial t}+R\right) = 0$$

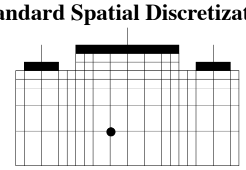
$$\nabla\cdot J_p + q\left(\frac{\partial p}{\partial t}+R\right) = 0$$

- Drift-Diffusion approximation

$$J_n = -qD_n(n\nabla u - \nabla n)$$

$$J_p = -qD_p(p\nabla u + \nabla p)$$

Standard Spatial Discretization



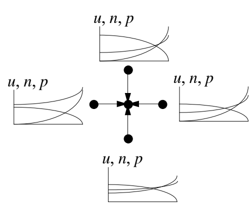
Each mesh node contributes three equations

$$\frac{\varepsilon kT}{q}\sum_j\left\{\frac{d_{ij}}{L_{ij}}(u_i-u_j)\right\} - qA_i(p_i-n_i+N_D-N_A) = 0$$

$$qD_n\sum_j\left\{\frac{d_{ij}}{L_{ij}}[n_jB(u_j-u_i) - n_iB(u_i-u_j)]\right\} - qA_i\left(\frac{dn_i}{dt}+R_i\right) = 0$$

$$qD_p\sum_j\left\{\frac{d_{ij}}{L_{ij}}[p_iB(u_j-u_i) - p_jB(u_i-u_j)]\right\} + qA_i\left(\frac{dp_i}{dt}+R_i\right) = 0$$

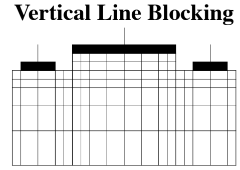
Waveform Relaxation Step



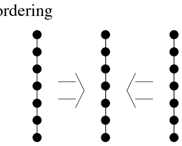
At each grid node i solve 3×3 system for

$$\begin{pmatrix} u_i^{k+1}(t) \\ n_i^{k+1}(t) \\ p_i^{k+1}(t) \end{pmatrix} \text{ using } \begin{pmatrix} u_j^{k}(t) \\ n_j^{k}(t) \\ p_j^{k}(t) \end{pmatrix}$$

Vertical Line Blocking

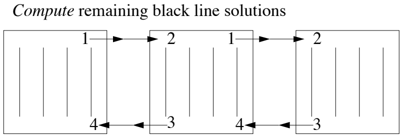


- For each vertical mesh line
  Solve for u, n, p for nodes in line
- Use red/black ordering

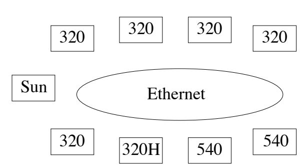


Parallel Implementation

- Manager divides mesh and sends lines to worker programs
- Worker WR iteration:
  1. *Send* black line solutions
     Compute internal red line solutions
  2. *Receive* black line solutions
     Compute remaining red line solutions
  3. *Send* red line solutions
     Compute internal black line solutions
  4. *Receive* red line solutions
     Compute remaining black line solutions

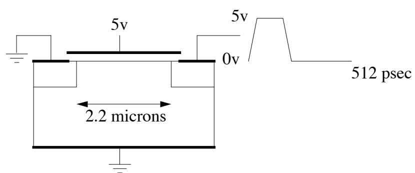


Experimental Setup



- Workstation cluster
  IBM RS/6000 as workers (8)
  Sun SparcStation as manager (1)
- Standard Ethernet (10Mbit/sec)
- PVM 2.4
- 1,2,4 machine results on model 320 machines

Experimental Device Model



- N-channel MOSFET
- 19×31 mesh
- First order BDF
- 256 fixed timesteps

Experimental Results

Method	**#Procs**	**Wall clock sec**
WRN	1	8230.23
	2	4469.91
	4	2712.58
	8	1571.92
WN/WGMRES	1	*
	2	*
	4	925.6
	8	504.5
Point (direct)	1	2462.48
Point (GMRES)	1	1221.98
	2	6931.86

Future Work

- Multirate implementation
- PVM 3.1
- Other parallel machines
- Parallelize in time (timepoint pipelining)
- Theoretical convergence estimates
- Other acceleration techniques
- Other applications

Conclusion

- Accelerated waveform methods are a very effective parallel approach for initial value problems
- Waveform methods are especially attractive for loosely coupled processors (e.g., workstation cluster with PVM)
- For semiconductor device transient simulation:
  - Fastest serial method did not parallelize well (it slowed down on two processors)
  - WGMRES on 8 workstations more than twice as fast as fastest serial method for device transient simulation