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A SURVEY OF J-SPACES

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ABSTRACT. This note is a survey of J-spaces.

1. Basic concepts

A space X is a *J-space* if, whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is compact. A space X is a *strong J-space* if every compact $K \subset X$ is contained in a compact $L \subset X$ with $X \backslash L$ connected. [As in [4], all maps are continuous and all spaces are Hausdorff.]

1.1. Every strong J-space X is a J-space. The two concepts coincide when X is locally connected, but in general (even for closed subsets of $\mathbb{R}^2$) they do not.

2. Examples

2.1. A topological linear space X is a (strong) J-space if and only if $X \neq \mathbb{R}$.

2.2. If X and Y are connected and non-compact, then $X \times Y$ is a strong J-space.[1]

2.3. Let Y be a compact manifold with boundary B, and let $A \subset B$. Then $Y \backslash A$ is a (strong) J-space if and only if A is connected.

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[1] This was proved in [5].

3. Characterizations by closed maps

A map $f : X \to Y$ is called *boundary-perfect* if f is closed and $\operatorname{bdry} f^{-1}(y)$ is compact for every $y \in Y$. It follows from [3] that every closed map $f : X \to Y$ from a paracompact space X to a q-space Y is boundary-perfect.[2]

3.1. A space X is a J-space if and only if every boundary-perfect map $f : X \to Y$ onto a non-compact space Y is perfect.

3.2. If X is a J-space, then every boundary-perfect map $f : X \to Y$ has at most one non-compact fiber. The converse holds if X is locally compact.

3.3. Let X be paracompact and locally compact. Then the following are equivalent.

(a) X is a J-space.
(b) Every closed map $f : X \to Y$ onto a non-compact, locally compact space Y is perfect.
(c) Every closed map $f : X \to Y$ onto a locally compact space Y has at most one non-compact fiber.

3.4. Let X be metrizable. Then the following are equivalent

(a) X is a J-space.
(b) Every closed map $f : X \to Y$ onto a non-compact, metrizable space Y is perfect.

4. Characterization by compactifications

Call a set $A \subset Y$ a *boundary set* for Y if $\operatorname{Int} A = \emptyset$ and, whenever $U \supset A$ is open in Y and $\{W_1, W_2\}$ is a disjoint, relatively open cover of $U \backslash A$, then no $y \in A$ lies in $\overline{W}_1 \cap \overline{W}_2$. Call a set $A \subset Y$ a *strong boundary set* for Y if $\operatorname{Int} A = \emptyset$ and, whenever $U \supset A$ is open in Y, then every $y \in A$ has an open neighborhood $V \subset U$ with $V \backslash A$ connected.

It is easy to see that, if Y is a manifold with boundary B, then every $A \subset B$ is a strong boundary set for Y. And it follows from the proof of [1, Lemma 4] (or from [2, Proposition 3.5]) that, if Y is completely regular, then $\beta X \backslash X$ is a boundary set for βX.

4.1. Let Y be a compactification of X, and suppose either that X is locally compact or that Y is metrizable. Then the following are equivalent.

(a) X is a (strong) J-space.
(b) $Y \backslash X$ is connected and a (strong) boundary set for Y.

5. Preservation

5.1. J-spaces are preserved by boundary-perfect images. (False for strong J-spaces, even with perfect images.)

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[2] q-spaces (see [3]) include all locally compact and all metrizable spaces.

5.2. J-spaces and strong J-spaces are preserved by monotone, perfect pre-images.

5.3. If X_1, X_2 are connected, then $X_1 \times X_2$ is a (strong) J-space if and only if either X_1, X_2 are both (strong) J-spaces or both are non-compact.

5.4. Let $\{X_1, X_2\}$ be a closed cover of X with $X_1 \cap X_2$ compact. Then X is a (strong) J-space if and only if X_1, X_2 are both (strong) J-spaces and X_1 or X_2 is compact.

5.5. If X is a (strong) J-space, so is every component of X. (False for J-spaces).

References

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