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SEQUENCE OF DUALIZATIONS OF TOPOLOGICAL SPACES IS FINITE

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ABSTRACT. Problem 540 of J. D. Lawson and M. Mislove in Open Problems in Topology asks whether the process of taking duals terminate after finitely many steps with topologies that are duals of each other. The problem for T_1 spaces was already solved by G. E. Strecker in 1966. For certain topologies on hyperspaces (which are not necessarily T_1), the main question was in the positive answered by Bruce S. Burdick and his solution was presented on The First Turkish International Conference on Topology in Istanbul in 2000. In this paper we bring a complete and positive solution of the problem for all topological spaces. We show that for any topological space (X, τ) it follows $\tau^{dd} = \tau^{dddd}$. Further, we classify topological spaces with respect to the number of generated topologies by the process of taking duals.

0. DEFINITIONS AND NOTATION

Through the paper we mostly use the standard topological notions. The main source of definitions of some newer concepts is the paper [5] or the book [6] in which the reader can find many interesting connections to modern, computer science oriented topology. In this paper, all topological spaces are assumed without any separation axiom in general. Let (X, τ) be a topological space. Recall that the preorder of specialization is a reflexive and transitive binary relation on X defined by $x \leqslant y$ if and only if $x \in \operatorname{cl}\{y\}$. This relation is antisymmetric and hence a partial ordering if and only if X is a T_0 space. For any set $A \subseteq X$ we denote

$$\uparrow A = \{x : x \geqslant y \text{ for some } y \in A\}$$

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and

$$\downarrow A = \{x : x \leqslant y \text{ for some } y \in A\}.$$

In particular, for a singleton $x \in X$ it is clear that $\downarrow \{x\} = \operatorname{cl}\{x\}$. A set is said to be saturated in (X, τ) if it is the intersection of open sets. One can easily check that a set A is saturated in (X, τ) if and only if $A = \uparrow A$, that is, if and only if the set A is closed upward with respect to the preorder of specialization of (X, τ). By the dual topology τ^d for a topological space (X, τ) we mean the topology on X generated by taking the compact saturated sets of (X, τ) as a closed base. It is not very difficult to show that the dual operator switches the preorder of specialization, that is, $x \leqslant y$ in (X, τ) if and only if $x \geqslant y$ in (X, τ^d). Let ψ be a family of sets. We say that ψ has the finite intersection property, or briefly, that ψ has f.i.p., if

$$\text{for every } P_1, P_2, \ldots, P_k \in \psi \text{ it follows } P_1 \cap P_2 \cap \cdots \cap P_k \neq \varnothing.$$

For the definition of compactness, we do not assume any separation axiom. We say that a subset S of a topological space (X, τ) is compact if every open cover of S has a finite subcover. Let (X, τ) have a closed base Φ. It is well-known that $S \subseteq X$ is compact if and only if for every family $\zeta \subseteq \Phi$ such that the family $\{S\} \cup \zeta$ has f.i.p. it follows $S \cap (\bigcap \zeta) \neq \varnothing$, or equivalently, if and only if every filter base $\eta \subseteq \Phi$ such that every element of η meets S the filter base η has a cluster point in S.

1. Introduction

Problem 540 of J. D. Lawson and M. Mislove [5] in Open Problems in Topology (J. van Mill, G. M. Reed, eds., 1990) asks

- which topologies can arise as dual topologies and
- whether the process of taking duals terminate after finitely many steps with the topologies that are duals of each other.

For T_1 spaces, the solution of (2) follows simply from the fact that in T_1 spaces every set is saturated and hence the dual operator d coincide with the well-known compactness operator ρ of J. de Groot, G. E. Strecker and E. Wattel [3]. For more general spaces, the question (2) was partially answered by Bruce S. Burdick who found certain classes of (in general, non-T_1) spaces for which the process of taking duals of a topological space (X, τ) terminates by $\tau^{dd} = \tau^{dddd}$ — the lower Vietoris topology on any hyperspace, the Scott topology for reverse inclusion on any hyperspace, and the upper Vietoris topology on the hyperspace of a regular space. B. Burdick presented his paper on The First Turkish International Conference on Topology in Istanbul 2000 [1]. In this paper we will show that an analogous result, that is, $\tau^{dd} = \tau^{dddd}$ holds for arbitrary and general topological spaces without any restriction.

2. Main Results

Through this paper, let Φ denote a certain closed topology base of a topological space (X, τ). Let Φ^d be the collection of all compact saturated sets in (X, τ) and hence the closed topology base of (X, τ^d). For $n = 2, 3 \dots$ let Φ^{d^n} be the collection of all compact saturated sets in $(X, \tau^{d^{n-1}})$ and hence the closed topology base of (X, τ^{d^n}). The following lemma is due to Bruce S. Burdick [1]. We repeat it because of completeness.

Lemma 2.1 (B. S. Burdick, 2000))**.** *Let C be compact in (X, τ) and $P \in \Phi^{dd}$. Then $C \cap P$ is compact in (X, τ).*

Proof. Let $\zeta \subseteq \Phi$ be a filter base such that its every element meets $C \cap P$. Let

$$\eta = \{\uparrow (C \cap D) : D \in \zeta\}.$$

Then $\eta \subseteq \Phi^d$ and $\{P\} \cup \eta$ has f.i.p. Since $P \in \Phi^{dd}$ it follows that there exists some

$$x \in P \cap (\bigcap \eta) \neq \varnothing.$$

Hence, $x \in P \cap \uparrow (C \cap D)$ for every $D \in \zeta$, which implies that there is some $t \in C \cap D$ (depending on the choice of D) such that $x \geqslant t$. Then

$$\downarrow \{x\} \cap C \cap D \neq \varnothing \quad \text{for every } D \in \zeta,$$

so the collection

$$\{C\} \cup (\{\downarrow \{x\}\} \cup \zeta)$$

has f.i.p. Since C is compact and $\downarrow \{x\}$ as well as every element of ζ is closed, there is some

$$y \in C \cap \downarrow \{x\} \cap (\bigcap \zeta) \subseteq C \cap P \cap (\bigcap \zeta).$$

Hence, y is a cluster point of the closed filter base ζ in $C \cap P$. $\square$

Lemma 2.2. *Let $C \in \Phi^d$, $M \in \psi \subseteq \Phi^{dd}$ and let $\{C\} \cup \psi$ has f.i.p. Then there exist $\xi(M) \in M$ such that*

$$\{C\} \cup \psi \cup \{\downarrow \{\xi(M)\}\}$$

has f.i.p.

Proof. Let

$$\varphi = \{\uparrow (C \cap P_1 \cap P_2 \cap \dots \cap P_k) : P_1, P_2, \dots, P_k \in \psi\}.$$

From Lemma 2.1 it follows that $\varphi \subseteq \Phi^d$ and $\{M\} \cup \varphi$ has f.i.p. Hence, since $M \in \Phi^{dd}$ is compact in (X, τ^d), there exist

$$\xi(M) \in M \cap (\bigcap \varphi).$$

Then for every $P_1, P_2, \dots, P_k \in \psi$ it follows that

$$\xi(M) \in \uparrow (C \cap P_1 \cap P_2 \cap \dots \cap P_k)$$

so there exist $t \in C \cap P_1 \cap P_2 \cap \cdots \cap P_k$ (depending on the choice of $P_1, P_2, \dots, P_k$) with $\xi(M) \geqslant t$. Then $t \in \downarrow \{\xi(M)\}$ which implies that

$$\downarrow \{\xi(M)\} \cap C \cap P_1 \cap P_2 \cap \cdots \cap P_k \neq \varnothing.$$

It follows that

$$\{C\} \cup \psi \cup \{\downarrow \{\xi(M)\}\}$$

has f.i.p. □

Lemma 2.3. *Let $C \in \Phi^d$, $\psi \subseteq \Phi^{dd}$ and let $\{C\} \cup \psi$ has f.i.p. Then for every $M \in \psi$ there exist $\xi(M) \in M$ such that*

$$\{C\} \cup \{\downarrow \{\xi(M)\} : M \in \psi\}$$

has f.i.p.

Proof. Let $\psi = \{M_\alpha : \alpha < \mu\}$ where μ is an ordinal number. From Lemma 2.2 it follows that there exist $\xi(M_0) \in M_0$ such that

$$\{C\} \cup \psi \cup \{\downarrow \{\xi(M_0)\}\}$$

has f.i.p. Suppose that for some $\beta < \mu$ and every $\alpha < \beta$ there exist $\xi(M_\alpha) \in M_\alpha$ such that, in the notation

$$\chi_\alpha = \psi \cup \{\downarrow \{\xi(M_\gamma)\} : \gamma \leq \alpha\},$$

the family $\{C\} \cup \chi_\alpha$ has f.i.p. Let $\chi = \bigcup_{\alpha<\beta} \chi_\alpha$. Obviously, the family $\{C\} \cup \chi$ has f.i.p. and $M_\beta \in \psi \subseteq \chi \subseteq \Phi^{dd}$. Then, by Lemma 2.2 there exist $\xi(M_\beta) \in M_\beta$ such that

$$\{C\} \cup \chi \cup \{\downarrow \{\xi(M_\beta)\}\}$$

has f.i.p. But

$$\chi_\beta = \chi \cup \{\downarrow \{\xi(M_\beta)\}\}$$

which implies that the family $\{C\} \cup \chi_\beta$ has f.i.p. By induction, we have defined $\xi(M_\beta) \in M_\beta$ for every $\beta < \mu$. Obviously, the family

$$\{C\} \cup (\bigcup_{\beta<\mu} \chi_\beta) = \{C\} \cup \psi \cup \{\downarrow \{\xi(M_\beta)\} : \beta < \mu\}$$

has f.i.p which implies that also its subfamily

$$\{C\} \cup \{\downarrow \{\xi(M)\} : M \in \psi\}$$

has f.i.p. □

Theorem 2.4. *For any topological space (X, τ) it follows $\Phi^d \subseteq \Phi^{ddd}$.*

Proof. Let $C \in \Phi^d$ and let $\psi \subseteq \Phi^{dd}$ be a family such that $\{C\} \cup \psi$ has f.i.p. From Lemma 2.3. it follows that for every $M \in \psi$ there exist $\xi(M) \in M$ such that

$$\{C\} \cup \{\downarrow \{\xi(M)\} : M \in \psi\}$$

has f.i.p. But C is compact in (X, τ) and for every $M \in \psi$, the sets

$$\downarrow \{\xi(M)\} = \mathrm{cl}\,\{\xi(M)\} \subseteq M$$

are closed. Hence

$$\varnothing \neq C \cap (\bigcap \{\downarrow \{\xi(M)\} : M \in \psi\}) \subseteq C \cap (\bigcap \psi).$$

It follows that C is compact in (X, τ^{dd}). But C is saturated in (X, τ) which is the same as C is saturated in (X, τ^{dd}). Hence $C \in \Phi^{ddd}$. □

Corollary 2.5. *For any topological space (X, τ) it follows $\Phi^{dd} = \Phi^{dddd}$.*

Proof. From Theorem 2.4 it follows that $\Phi^{d} \subseteq \Phi^{ddd}$ which implies that $\Phi^{dd} \supseteq \Phi^{dddd}$. Applying Theorem 2.4 to the space (X, τ^{d}) we obtain that $\Phi^{dd} \subseteq \Phi^{dddd}$. □

Corollary 2.6. *For any topological space (X, τ) it follows $\tau^{dd} = \tau^{dddd}$.*

3. Some Classification Notes

Now, let e be an identity operator on the class of all topological spaces, i.e. $e(\tau) = \tau$ for a topology τ of a space (X, τ). Then e, d, dd and ddd form, with the composition operation $\circ$, a commutative monoid having the unit element e and the following multiplicative table:

$\circ$	e	d	dd	ddd
e	e	d	dd	ddd
d	d	dd	ddd	dd
dd	dd	ddd	dd	ddd
ddd	ddd	dd	ddd	dd

Table 1.

Then it is natural to introduce the following classes G_1, $G_2 = G_{2_a} \cup G_{2_b}$, $G_3 = G_{3_a} \cup G_{3_b} \cup G_{3_c}$ and G_4 of topological spaces, where:

$$G_1 = \left\{(X, \tau) : \tau^{d} = \tau\right\},$$

$$G_{2_a} = \left\{(X, \tau) : \tau^{dd} = \tau\right\}, \ G_{2_b} = \left\{(X, \tau) : \tau^{dd} = \tau^{d}\right\},$$

$$G_{3_a} = \left\{(X, \tau) : \tau^{ddd} = \tau\right\},$$
$$G_{3_b} = \left\{(X, \tau) : \tau^{ddd} = \tau^{d}\right\},$$
$$G_{3_c} = \left\{(X, \tau) : \tau^{ddd} = \tau^{d}\right\},$$

$$G_4 = \left\{(X, \tau) : \tau^{dddd} = \tau^{dd}\right\} = \mathsf{Top}.$$

Obviously, each class G_n consists exactly of those topological spaces such that the process of taking duals generates at most $n = 1, 2, 3, 4$ different topologies while the class G_4 contains all topological spaces. Hence, we suggest the following terminology:

Definition 3.1. We say that a topological space (X, τ) is *n-generative* if $n = 1, 2, 3, 4$ is the least number such that (X, τ) belongs to the class G_n.

Propostion 3.2. *The classes G_n satisfy the following relationships:*

(i) $G_1 = G_{3_a} = G_{2_a} \cap G_{2_b} = G_{2_a} \cap G_{3_c}$
(ii) $G_{2_b} = G_{3_b} \cap G_{3_c}$
(iii) $G_3 = G_{3_b} \cup G_{3_c}$

The proof is an easy consequence of the definitions of the classes G_n and the identity $dd = dddd$ of Corollary 2.6. We leave it to the reader. Hence, the relationships between the classes G_n can be described by the following diagram:

$$\begin{tikzcd}
 & G_4 \arrow[dl, dash] \arrow[dr, dash] & \\
G_{3_b} \arrow[d, dash] \arrow[drr, dash] & & G_{3_c} \arrow[d, dash] \\
G_{2_a} \arrow[dr, dash] & & G_{2_b} \arrow[dl, dash] \\
 & G_1 &
\end{tikzcd}$$

Diagram 1.

Since for T_1 spaces the dual operator d coincide with the compactness operator ρ of J. de Groot, G. E. Strecker and E. Wattel [3] we immediately obtain that $G_1 \subsetneq G_{2_a} \subsetneq G_{3_b} \subsetneq G_4$, $G_1 \subsetneq G_{2_b} \subsetneq G_{3_b}$ and $G_{3_c} \subsetneq G_4$ (the paper [2] is a good source of proper examples of such spaces, see also the section 4). Hence it remains to determine whether the class G_{2_b} is a proper subclass of the class G_{3_c}.

Lemma 3.3. *Let $\tau = \tau^d$ for a topological space (X, τ). Then for every subset $A \subseteq X$ it follows that $\uparrow A = \downarrow A$.*

Proof. Let $x \in X$. Then $\downarrow \{x\} = \mathrm{cl}_\tau \{x\}$ and $\uparrow \{x\} = \mathrm{cl}_{\tau^d} \{x\}$. Since $\tau = \tau^d$ it follows that $\uparrow \{x\} = \downarrow \{x\}$. Then

$$\begin{aligned} \uparrow A &= \bigcup_{x \in A} \uparrow \{x\} \\ &= \bigcup_{x \in A} \downarrow \{x\} \\ &= \downarrow A. \end{aligned}$$

$\square$

Lemma 3.4. *Let $\uparrow A = \downarrow A$ for every subset $A \subseteq X$. If $S \in \Phi^{dd}$ and $H \in \Phi$ then $S \cap H \in \Phi^{dd}$.*

Proof. Take $\zeta \subseteq \Phi^d$ such that the family $\{S \cap H\} \cup \zeta$ has f.i.p. We put

$$\eta = \{\uparrow (H \cap D) : D \in \zeta\}.$$

It follows that $\eta \in \Phi^d$ and the family $\{S\} \cup \eta$ has f.i.p. Since $S \in \Phi^{dd}$, we have

$$S \cap (\bigcap \eta) \neq \varnothing.$$

It follows that there exist some $x \in S \cap (\bigcap \eta)$. Then $x \in \uparrow (H \cap D)$ for every $D \in \zeta$. Hence, there exist $t \in H \cap D$ (depending on the choose of D) such that $x \geqslant t$. Obviously, $x \in D$ for every $D \in \zeta \subseteq \Phi^d$ since the elements of Φ^d are saturated and hence closed upward with respect to the specialization preorder. Since H is a closed set, it follows that

$$H = \downarrow H = \uparrow H$$

which implies that $x \in H$. Therefore, we have

$$x \in S \cap H \cap (\bigcap \zeta)$$

and so $S \cap H \in \Phi^{dd}$. □

Propostion 3.5. *The equality $G_{2_b} = G_{3_c}$ holds.*

Proof. Let (X, τ) be a space that belongs to the class G_{3_c}. Then $\tau^{dd} = \tau^{ddd}$. Applying Lemma 3.3 to the space (X, τ^{dd}) we obtain that for any subset $A \subseteq X$ it follows that $\uparrow A = \downarrow A$. We will show that (X, τ) belongs to the class G_{3_b}. It follows from Theorem 2.4 that $\Phi^d \subseteq \Phi^{ddd}$. Conversely, let $S \in \Phi^{ddd}$. It means that S is compact saturated in (X, τ^{dd}) which is the same as in (X, τ^{ddd}). Hence $S \in \Phi^{dddd} = \Phi^{dd}$ by Corollary 2.5. We will show that $S \in \Phi^d$. Let $\zeta \subseteq \Phi$ be a family such that $\{S\} \cup \zeta$ has f.i.p. We put

$$\eta = \{S \cap H : H \in \zeta\}\,.$$

It follows from Lemma 3.4 that $\eta \subseteq \Phi^{dd}$ and, obviously, the family $\{S\} \cup \eta$ has f.i.p. Then

$$S \cap (\bigcap \zeta) = S \cap (\bigcap \eta) \neq \varnothing,$$

which implies that $S \in \Phi^d$. Then $\Phi^d = \Phi^{ddd}$ and so $\tau^d = \tau^{ddd}$. Hence, the space (X, τ) belongs to the class G_{3_b}. Now, we have that $G_{3_c} \subseteq G_{3_b}$ and $G_{2_b} = G_{3_b} \cap G_{3_c}$ by Proposition 3.2. Together it gives that $G_{2_b} = G_{3_c}$. □

Corollary 3.6. *The equality $G_3 = G_{3_b}$ holds.*

Now, we can slightly improve Diagram 1:

$$\begin{tikzcd}
& G_4 \arrow[d, dash] & \\
& G_3 \arrow[dl, dash] \arrow[dr, dash] & \\
G_{2_a} \arrow[dr, dash] & & G_{2_b} \arrow[dl, dash] \\
& G_1 &
\end{tikzcd}$$

Diagram 2.

The two following simple results give a partial answer to the question which topologies can arise by the process of taking duals.

Corollary 3.7. *Let (X, τ) be a topological space such that τ arise as a dual topology. Then (X, τ) belongs to the class G_3 and does not belong to the class $G_{2_b} \setminus G_1$.*

Proof. If $\tau = \sigma^d$ where σ is a topology on X, then
$$\tau^d = \sigma^{dd} = \sigma^{dddd} = \tau^{ddd}.$$
Hence (X, τ) belongs to the class G_3. Suppose that (X, τ) belongs to the class G_{2_b}. Then
$$\sigma^{dd} = \tau^d = \tau^{dd} = \sigma^{ddd}$$
which implies that (X, σ) belongs to the class G_{3_c}. By Proposition 3.5 it follows that $G_{3_c} = G_{2_b}$ and so
$$\tau = \sigma^d = \sigma^{dd} = \tau^d.$$
Hence, (X, τ) belongs to the class G_1. $\square$

Corollary 3.8. *Let (X, τ) belongs to the class G_2. Then τ arise as a dual topology.*

Question 3.9. Is it true that all spaces of G_3 arise as duals? Or conversely, does there exist a space (X, τ) belonging to the class G_3 such that there is no topology σ with $\sigma^d = \tau$?

4. Examples

The following examples are due to J. de Groot, H. Herrlich, G. E. Strecker, E. Wattel (Examples from 4.1 to 4.4, see [2]) and B. S. Burdick (Example 4.5, [1]) and we repeat them because of completeness. In the most cases, their listed properties follow directly from the fact that for T_1 spaces the dual operator d and the compactness operator ρ of [3] coincide.

Example 4.1. Every compact T_2 space belongs to the class G_1.

Example 4.2. Every non-compact Hausdorff k-space belongs to the class G_{2_a} but does not belong to the class G_1.

Example 4.3. Let $W = \omega_1$ be the first uncountable ordinal with its natural order topology, $\mathbb{N}$ be the discrete space of all natural numbers, $Y = W \times \mathbb{N}$ be their product space. Let $a \notin Y$, $b \notin Y$, $a \neq b$. We set $X = Y \cup \{a, b\}$ and define the closed subbase Φ for the topology τ of the space (X, τ): A subset $F \subseteq X$ belongs to Φ if and only if fulfills at least one of the following three conditions:

(i) F is a compact subset of Y.
(ii) There are $\alpha \in W$, $n \in \mathbb{N}$ with $F = \{(\beta, n) : \beta \geq \alpha\} \cup \{a\}$
(iii) There are $\alpha \in W$, $\beta \in W$, $n \in \mathbb{N}$ with
$$F = \{(\gamma, m) : \alpha < \gamma \leq \beta, n < m\} \cup \{b\}$$

Then (X, τ) belongs to the class G_{2_b} but does not belong to the class G_1.

Example 4.4. Let $X = \mathbb{N} \cup \{z\}$ for some $z \in \beta\mathbb{N} \setminus \mathbb{N}$ with the topology τ induced from the Čech-Stone compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Then (X, τ) belongs to the class G_3 but does not belong to the class G_2.

Example 4.5. Let (X, τ) be the first uncountable ordinal $X = \omega_1$ with the topology

$$\tau = \{[0, \alpha) \setminus F : 0 \leq \alpha \leq \omega_1, \text{F is finite}\}.$$

Then (X, τ) belongs to the class G_4 but does not belong to the class G_3 (since τ^d is cocountable, τ^{dd} is cofinite and τ^{ddd} is discrete).

References

[1] Bruce S. Burdick, *A note on iterated duals of certain topological spaces*, Theoret. Comput. Sci. **275** (2002), no. 1–2, 69–77.

[2] J. de Groot, H. Herrlich, G. E. Strecker, and E. Wattel, *Compactness as a operator*, Compositio Math. **21** (1969), 349–375. MR 41 #4490

[3] J. de Groot, G. E. Strecker, and E. Wattel, *The compactness operator in general topology*, General Topology and its Relations to Modern Analysis and Algebra, II (Proc. Second Prague Topological Sympos., 1966), Academia, Prague, 1967, pp. 161–163. MR 38 #657

[4] Martin Maria Kovár Kovár, *The solution to Problem 540*, To appear in Theoret. Comput. Sci. B, 2002.

[5] Jimmie D. Lawson and Michael Mislove, *Problems in domain theory and topology*, Open problems in topology, North-Holland, Amsterdam, 1990, pp. 349–372. MR 1 078 658

[6] Steven Vickers, *Topology via logic*, Cambridge University Press, Cambridge, 1989. MR **90j:**03110

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