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CHARACTERIZING CONTINUITY BY PRESERVING COMPACTNESS AND CONNECTEDNESS

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Abstract. Let us call a function f from a space X into a space Y *preserving* if the image of every compact subspace of X is compact in Y and the image of every connected subspace of X is connected in Y. By elementary theorems a continuous function is always preserving. Evelyn R. McMillan [6] proved in 1970 that if X is Hausdorff, locally connected and Frèchet, Y is Hausdorff, then the converse is also true: any preserving function $f : X \to Y$ is continuous. The main result of this paper is that if X is any product of connected linearly ordered spaces (e.g. if $X = \mathbb{R}^\kappa$) and $f : X \to Y$ is a preserving function into a regular space Y, then f is continuous.

Let us call a function f from a space X into a space Y *preserving* if the image of every compact subspace of X is compact in Y and the image of every connected subspace of X is connected in Y. By elementary theorems a continuous function is always preserving. Quite a few authors noticed—mostly independently from each other—that the converse is also true for real functions: a preserving function $f : \mathbb{R} \to \mathbb{R}$ is continuous. (The first paper we know of is [9] from 1926!) Whyburn proved [13] that a preserving function from a space X into a Hausdorff space is always continuous at a first countability and local connectivity point of X.

Evelyn R. McMillan [6] proved in 1970 that if X is Hausdorff, locally connected and Frèchet, moreover Y is Hausdorff, then any preserving function $f : X \to Y$ is continuous. This is quite a significant and deep result that is surprisingly little known.

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We shall use the notation $Pr(X, T_i)$ ($i = 1, 2, 3$ or $3\frac{1}{2}$) to denote the following statement: Every preserving function from the topological space X into any T_i space is continuous.

The organization of the paper is as follows: In §1 we give some basic definitions and then treat our results that are closely connected to McMillan's theorem. §2 treats several important technical theorems that enable us to conclude that certain preserving functions are continuous. In §3 we prove that certain product spaces X satisfy $Pr(X, T_3)$; in particular, any preserving function from a product of connected linearly ordered spaces into a regular space is continuous. In §4 we discuss some results concerning the continuity of preserving functions defined on compact or sequential spaces. Finally, §5 treats the relation $Pr(X, T_1)$.

Our terminology is standard. Undefined terms can be found in [2] or in [3].

1. Around McMillan's theorem

Our first theorem implies that (at least among Tychonoff spaces) local connectivity of X is a necessary condition for $Pr(X, T_{3\frac{1}{2}})$. For metric spaces this result was proved by Klee and Utz[5].

Theorem 1.1. *If the Tychonoff space X is not locally connected at a point $p \in X$, then there exists a preserving function f from X into the interval $[0, 1]$ which is not continuous at p.*

Proof. Suppose X is not locally connected at the point p, then there is an open neighbourhood U of p such that if K denotes the component of p in U, then K is not a neighbourhood of p. As X is Tychonoff, there exists an open neighbourhood $V \subset U$ of p and a continuous function $\overline{f} : X \to [0, 1]$ such that $\overline{f}$ is identically 0 on V and identically 1 outside of U. Select another continuous function $g : X \to [0, 1]$ such that $g(p) = 1$ and g is identically 0 outside V. Put now

$$f(x) = \begin{cases} \overline{f}(x) + g(x) & \text{if } x \in K; \\ \overline{f}(x) & \text{otherwise.} \end{cases}$$

($f : X \to [0, 1]$ because $\overline{f}[V] = \{0\}$ and $g[X - V] = \{0\}$.) The function f is not continuous at p because $f(p) = 1$ and in every neighbourhood of p there is a point from $V - K$ mapped to 0.

On the other hand, we claim that f is preserving. Indeed, let $C \subset X$ be compact. The restriction of f to the closed set $F = (X - V) \cup K$ is evidently continuous (here $f = \overline{f} + g$), hence $C' = f(C \cap F)$ is compact. But f is 0 on $V - K$ and so $f(C)$ is either C' or $C' \cup \{0\}$, thus $f(C)$ is clearly compact.

Let now C be any connected subset of X. The function f is continuous on $X - K$ (here $f = \overline{f}$), hence $C \cap K \neq \emptyset$ can be assumed. Similarly, f is continuous on $X - V$ ($f(x) = \overline{f}(x)$ for $x \in X - V$), hence we can suppose that C also meets V. If $C \subset U$, then necessarily $C \subset K$ because C is connected and meets the component $K$of U. As f is continuous on K, the image $f(C)$ is then connected.

Thus it remains to check only the case in which C meets both V and $X - U$. In this case $\overline{f}(C)$ is the whole interval $[0, 1]$. But f and $\overline{f}$ are equal at each point where $\overline{f}$ does not vanish, thus $f(C)$ contains the interval $(0, 1]$ and so it is connected. □

It is not a coincidence that the target space in Theorem 1.1 is the interval $[0, 1]$, because of the following result:

Lemma 1.2. *Suppose $f : X \to Y$ is a preserving function into a Tychonoff space Y and f is not continuous at the point $p \in X$. Then there exists a preserving function $h : X \to [0, 1]$ which is also not continuous at p.*

Proof. Since f is not continuous at p, there exists a closed set $F \subset Y$ such that $f(p) \notin F$ but p is an accumulation point of $f^{-1}(F)$. Choose a continuous function $g : Y \to [0, 1]$ such that $g(f(p)) = 0$ and g is identically 1 on F. Then the composite function $h(x) = f(g(x))$ has the stated properties. □

The following Lemmas will be often used in the sequel. They all state simple properties of preserving functions.

Lemma 1.3. *If $f : X \to Y$ is a compactness preserving function, Y is Hausdorff, $M \subset X$ with $\overline{M}$ compact and $f(M)$ infinite then for every accumulation point y of $f(M)$ there is an accumulation point x of M such that $f(x) = y$, i. e. $f(M)' \subset f(M')$.*

Proof. Let $N = M - f^{-1}(y)$ then $f(N) = f(M) - \{y\}$ and so we have $y \in \overline{f(N)} - f(N)$. But $f(\overline{N})$ is also compact, hence closed in Y and so $y \in f(\overline{N}) - f(N)$ as well. Thus there is an $x \in \overline{N} - N$ such that $f(x) = y$ and then x is as required. □

We shall often use the following immediate consequence of this lemma:

Lemma 1.3' (E. R. McMillan [6]). If $f : X \to Y$ is a compactness preserving function, Y is Hausdorff, $\{x_n : n < \omega\} \subset X$ converges to $x \in X$ then either $\{f(x_n) : n < \omega\}$, converges to $f(x)$ or the set $\{f(x_n) : n < \omega\}$ is finite □

Actually, to prove Lemma 1.3' we do not need the full force of the assumption that f is compactness preserving. It suffices to assume that the image of a convergent sequence together with its limit is compact, in other words: the image of a topological copy of $\omega + 1$ is compact. For almost all of our results given below only this very restricted special case of compactness preservation is needed.

Lemma 1.4 ([8]). *If $f : X \to Y$ preserves connectedness, Y is a T_1-space and $C \subset X$ is a connected set, then $f(\overline{C}) \subset \overline{f(C)}$.*

Proof. If $x \in \overline{C}$ then $C \cup \{x\}$ is connected. Thus $f(C \cup \{x\}) = f(C) \cup \{f(x)\}$ is also connected and hence $f(x) \in \overline{f(C)}$. □

The next lemma will also play a crucial role in some theorems of the paper. A weaker form of it appears in [6].

Definition 1.5. We shall say that $f : X \to Y$ is *locally constant* at the point $x \in X$ if there is a neighbourhood U of x such that f is constant on U.

Lemma 1.6. *Let f be a connectivity preserving function from a locally connected space X into a T_1-space Y. If $F \subset Y$ is closed and $p \in \overline{f^{-1}(F)} - f^{-1}(F)$ then p is also in the closure of the set*

$$\{x \in f^{-1}(F) : f \textit{ is not locally constant at } x\}.$$

Proof. Let G be a connected open neighbourhood of p and C be a component of the non-empty subspace $G \cap f^{-1}(F)$. Then C has a boundary point x in the connected subspace G because $\emptyset \neq C \neq G$. Clearly, $f(x) \in F$ by Lemma 1.4. If $V \subset G$ is any connected neighbourhood of x then $V \cup C$ is connected and $V - C \neq \emptyset$ because x is a boundary point of C hence V is not contained in $f^{-1}(F)$, so f is not locally constant at x. □

Lemma 1.7. *Let $f : X \to Y$ be a connectivity preserving function into the T_1-space Y. Suppose that X is locally connected at the point $p \in X$ and f is not locally constant at p. Then $f(U) \cap V$ is infinite for every neighbourhood U of p and for every neighbourhood V of $f(p)$.*

Proof. Choose any connected neighbourhood U of x; then $f(U)$ is connected and has at least two points. Thus if V is any open subset of Y containing $f(p)$ then $f(U) \cap V$ can not be finite because otherwise $f(p)$ would be an isolated point of the non-singleton connected set $f(U)$. □

The following proof of McMillan's theorem is based upon the same ideas as her original proof, although, we think, it is much simpler. We include it here mainly to make the paper self-contained.

Theorem 1.8 (E. R. McMillan [6]). *If X is a locally connected and Frèchet Hausdorff space, then $Pr(X, T_2)$ holds.*

Proof. Assume Y is T_2 and $f : X \to Y$ is preserving but not continuous at the point $p \in X$. Then by Lemma 1.3' there is a sequence $x_n \to p$ such that $f(x_n) = y \neq f(p)$ for all $n < \omega$. Using Lemma 1.6 with $F = \{y\}$ we can also assume that f is not locally constant at the points x_n.

As Y is T_2, there is an open set $V \subset Y$ such that $y \in V$ but $f(p) \notin \overline{V}$. By Lemma 1.7 the image of every neighbourhood of each point x_n contains infinitely many points (different from y) from V.

Now we select recursively sequences $\{x^n_k : k < \omega\}$ converging to x_n for all $n < \omega$. Suppose $n < \omega$ and the points x^m_k are already defined for $m < n$ and $k < \omega$ so that $f(x^m_k) \neq y$. Then x_n is in the closure of the set

$$f^{-1}(V - (\{f(x^m_k) : m, k < n\} \cup \{y\})),$$

hence, as X is Fréchet , the new sequence $\{x_k^n : k < \omega\}$ converging to x_n can be chosen from this set.

Since the sequence $\{x_k^n : k < \omega\}$ converges to x_n and $\{x_n : n < \omega\}$ converges to the (Frèchet) point p, there is also a sequence $\{x_{k_l}^{n_l} : l < \omega\}$ converging to p. But then the sequence $\{n_l : l < \omega\}$ must tend to infinity so, by passing to a subsequence if necessary, we can assume that $n_{l+1} > \max(n_l, k_l)$ for all $l < \omega$. However, the sequence $\{f(x_{k_l}^{n_l}) : l < \omega\}$ does not converge to $f(p)$ because $f(p) \notin \overline{\{f(x_{k_l}^{n_l}) : l < \omega\}} \subset \overline{V}$, while the points $f(x_{k_l}^{n_l})$ are all distinct, contradicting Lemma 1.3'. $\square$

We could prove the following local version of McMillan's theorem:

Theorem 1.9. *If X is a locally connected Hausdorff space, p is a Frèchet point of X and f is a preserving function from X into a Tychonoff space Y, then f is continuous at p.*

Proof. By Lemma 1.2 it suffices to prove this in the case when Y is the interval $[0, 1]$. Assume, indirectly, that f is not continuous at p then, since p is a Fréchet point and by Lemma 1.6, we can choose a sequence $x_n \to p$ and a $y \in [0, 1]$ with $y \neq f(p)$ such that $f(x_n) = y$ and f is not locally constant at x_n for all $n < \omega$.

For each n choose a neighbourhood U_n of x_n with $p \notin \overline{U}_n$ and put $A_n = \{x \in U_n : 0 < |f(x) - y| < 1/n\}$. For any connected neighbourhood W of x_n its image $f(W)$ is a non-singleton interval containing y, hence the local connectivity of X implies that $x_n \in \overline{A}_n$ for all $n < \omega$ and so p belongs to the closure of $\bigcup\{A_n : n < \omega\}$. As p is a Frèchet point, there is a sequence $z_k \in A_{n_k}$ converging to p. But n_k necessarily tends to infinity because $p \notin \overline{A}_n \subset \overline{U}_n$ for each $n < \omega$, hence $f(z_k) \to y \neq f(p)$, contradicting Lemma 1.3'. Indeed, the set $\{f(z_k) : k < \omega\}$ is infinite because $f(z_k) \neq y$. $\square$

Theorem 1.9 is not a full local version of Theorem 1.8 because local connectivity is assumed in it globally for X. This leads to the following natural question:

Problem 1.10. *Let X be a Hausdorff space, f be a preserving function from X into a Tychonoff space Y and let X be locally connected and Frèchet at the point $p \in X$. Is it true then that f is continuous at p?*

We do not know the answer to this problem, however we can prove some partial affirmative results.

Definition 1.11 ([1])**.** A point x of a space X is called an (α_4) point if for any sequence $\{A_n : n < \omega\}$ of countably infinite sets with $A_n \to x$ for each $n < \omega$ there is a countably infinite set $B \to x$ such that $\{n < \omega : A_n \cap B \neq \emptyset\}$ is infinite.

An (α_4) and Frèchet point will be called an (α_4)-F point in X.

Theorem 1.12. *Let f be a preserving function from a topological space X into a Hausdorff space Y and let p be a point of local connectivity and an (α_4)-F point in X. Then f is continuous at p.*

Proof. Assume not. Then by the Lemma 1.3' there is a point $y \in Y$ such that $y \neq f(p)$ but p is in the closure of $f^{-1}(y)$. Choose an open neighbourhood V of y in Y with $f(p) \notin \overline{V}$. By Lemma 1.7 and Lemma 1.3' we can recursively choose pairwise distinct points $y_n \in V$ such that p is in the closure of $f^{-1}(y_n)$ for all $n \in \omega$. As the point p is an (α_4)-F point in X, there is a "diagonal" sequence $\{x_m : m \in M\}$ converging to p, where $f(x_m) = y_m$ and $M \subset \omega$ is infinite, contradicting Lemma 1.3'. □

The next result yields a different kind of partial answer to Problem 1.10:

Theorem 1.13. *Let f be a preserving function from a topological space X into a Tychonoff space Y and let p be a Frèchet point of local connectivity of X with character $\leq 2^\omega$. Then f is continuous at p.*

Proof. Assume not, f is discontinuous at the point $p \in X$. By Lemmas 1.2 and 1.3' we can suppose that $Y = [0,1]$, $f(p) = 0$ and every neighbourhood of p is mapped onto the whole interval $[0,1]$. Let $\mathcal{U}$ be a neighbourhood-base of p of size $\leq 2^\omega$ and choose for each $U \in \mathcal{U}$ a point $x_U \in U$ such that $f(x_U) \in [1/2, 1]$ and the points $f(x_U)$ are all distinct. Put $A = \{x_U : U \in \mathcal{U}\}$, then $p \in \overline{A}$ and so there exists a sequence $\{x_n : n < \omega\} \subset A$ converging to p, contradicting Lemma 1.3'. □

There is a variant of this result in which the assumption that Y be Tychonoff is relaxed to T_3, however the assumption on the character of the point p is more stringent. Its proof will make use of the following (probably well-known) lemma:

Lemma 1.14. *Let Z be an infinite connected regular space, then any non-empty open subset G of Z is uncountable.*

Proof. Choose a point $z \in G$ and an open proper subset V of G with $z \in V \subset \overline{V} \subset G$. If G would be countable then, as a countable regular space, G would be Tychonoff, and so there would be a continuous function $f : G \to [0,1]$ such that $f(z) = 1$ and f is identically zero on $G - V$. Extend f to a function $\overline{f} : Z \to [0,1]$ by putting $\overline{f}(y) = 0$ if $y \in Z - G$. Then $\overline{f}$ is continuous and hence $\overline{f}(Z)$ is also connected. Consequently we have $f(G) = \overline{f}(Z) = [0,1]$ implying that $|G| \geq |[0,1]| > \omega$, and so contradicting that G is countable. □

Theorem 1.15. *Let f be a preserving function from a topological space X into a regular space Y and let $p \in X$ be a Frèchet point of local connectivity with character $\leq \omega_1$. Then f is continuous at p.*

Proof. Assume f is discontinuous at the point $p \in X$. As p is a a Frèchet point, there is a sequence $x_n \to p$ such that $f(x_n)$ does not converge to

$f(p)$. Taking a subsequence if necessary, we can suppose by Lemma 1.3' that $f(x_n) = y \neq f(p)$ for all $n < \omega$.

Choose now an open neighbourhood V of the point $y \in Y$ with $f(p) \notin \overline{V}$. Let $\mathcal{U}$ be a neighbourhood base of the point p in X such that $|\mathcal{U}| \leq \omega_1$ and the elements of $\mathcal{U}$ are connected.

Choose now points x_U from the sets $U \in \mathcal{U}$ such that $f(x_U) \in V$ and the points $f(x_U)$ are all distinct. This can be accomplished by an easy transfinite recursion because for each $U \in \mathcal{U}$ the set $f(U)$ is connected and infinite, hence $f(U) \cap V$ is uncountable by the previous lemma. Put $A = \{x_U : U \in \mathcal{U}\}$. Then $p \in \overline{A}$ and so there exists a sequence $\{y_n : n < \omega\} \subset A$ converging to p, contradicting Lemma 1.3'. □

We shall now consider some further topological properties and prove several results about them saying that preserving functions are sequentially continuous. Since in a Fréchet point sequential continuity implies continuity, these results are clearly relevant to McMillan's theorem. Their real significance, however, will only become clear in the following two sections.

Definition 1.16. A point x in a topological space X is called a *sequentially connectible (in short: SC)* point, if $x_n \in X$, $x_n \to x$ implies that there are an infinite subsequence $\langle x_{n_k} : k < \omega \rangle$ and a sequence $\langle C_k : k < \omega \rangle$ consisting of connected subsets of X such that $\{x_{n_k}, x\} \subset C_k$ for all $k < \omega$, (i.e. C_k "connects" x_{n_k} with x, this explains the terminology,) moreover $C_k \to x$, i.e. every neighbourhood of the point x contains all but finitely many C_k's. A space X is called an SC space if all its points are SC points.

Remark 1.17. It is clear that the SC property is closely related to local connectivity. Let us say that a point x in space X is a strong local connectivity point if it has a neighbourhood base $\mathcal{B}$ such that the intersection of an arbitrary (non-empty) subfamily of $\mathcal{B}$ is connected. For example, local connectivity points of countable character or any point of a connected linearly ordered space has this property.

We claim that *if x is a strong local connectivity point of X then x is an SC point in X.* Indeed, assume that $x_n \to x$ and for every $n \in \omega$ let C_n denote the intersection of all those members of $\mathcal{B}$ which contain both points x_n and x. (As the sequence $\{x_n : n < \omega\}$ converges to x, we can suppose that some element $B_0 \in \mathcal{B}$ contains all the x_n's.) Then $\{x, x_n\} \subset C_n$, moreover $C_n \to x$. Indeed, the latter holds because if $x \in B \in \mathcal{B}$ then, by definition, $x_n \in B$ implies $C_n \subset B$. □

The SC property does not imply local connectivity. (If every convergent sequence is eventually constant then the space is trivially SC.) However, the following simple lemma shows that if there are "many" convergent sequences then such an implication is valid.

Lemma 1.18. *Let x be a both Frèchet and SC point in a space X. Then x is also a point of local connectivity in X.*

Proof. Let G be any open set containing x and set

$$H = \cup\{\, C : x \in C \subset G \text{ and } C \text{ is connected}\,\}.$$

We claim that (the obviously connected) set H is a neighbourhood of x. Indeed, otherwise, as x is a Frèchet point, we could choose a sequence $x_n \to x$ from the set $G - H$ while for every point $y \in G - H$ no connected set containing both x and y is a subset of G, contradicting the SC property of x. □

If SC holds globally, i.e. in an SC space, then in the above result the Frèchet property can be replaced with a weaker property that will turn out to play a very important role in the sequel.

Definition 1.19. A point p in a topological space X is called an s point if for every family $\mathcal{A}$ of subsets of X such that $p \in \overline{\bigcup \mathcal{A}}$ but $p \notin \overline{A}$ for all $A \in \mathcal{A}$ there is a sequence $\langle\langle x_n, A_n\rangle : n < \omega\rangle$ such that $x_n \in A_n \in \mathcal{A}$, the sets A_n are pairwise distinct and $\{x_n\}$ converges to some point $x \in X$ (that may be different from p).

A Frèchet point is evidently an s point, moreover any point that has a sequentially compact neighbourhood is also an s point. Other examples of s points will be seen later.

Theorem 1.20. *Any s point in a T_3 and SC space is a point of local connectivity.*

Proof. Let p be an s point in the regular SC space X and let G be an open neighbourhood of p. We have to prove that the component K_0 of the point p in G is a neighbourhood of p. Assume this is false and choose an open set U such that $p \in U \subset \overline{U} \subset G$.

Put

$$\mathcal{A} = \{K \cap \overline{U} : K \text{ is a component of } G,\ K \neq K_0\}.$$

Then $p \in \overline{\bigcup \mathcal{A}}$ and $p \notin \overline{A}$ for $A \in \mathcal{A}$ (because a component of G is relatively closed in G), hence, by the definition of an s point, there exists a sequence $\{\langle x_n, A_n\rangle : n < \omega\}$ such that $x_n \in A_n \in \mathcal{A}$, $x_n \to x$ for some $x \in X$ and if $A_n = K_n \cap \overline{U}$ then the components K_n are distinct. Note that $x \in \overline{U} \subset G$. As distinct components are disjoint, we can assume that $x \notin K_n$ for all $n < \omega$. As x is an SC point, there are a connected set C and some $n < \omega$ such that $\{x, x_n\} \subset C \subset G$. However, this is impossible, because then $K_n \cup C$ would be a connected set in G larger then the component K_n. □

The significance of the SC property in our study of continuity properties of preserving functions is revealed by the following result.

Theorem 1.21. *A preserving function $f : X \to Y$ into a Hausdorff space Y is sequentially continuous at each SC point of X.*

Proof. Let $x \in X$ be an SC point and assume that $x_n \to x$ but $f(x_n)$ does not converge to $f(x)$ for a sequence $\{x_n : n < \omega\}$ in X. We can assume by Lemma 1.3' that $f(x_n) = y \neq f(x)$ for all $n < \omega$. Choose an open neighbourhood V of y in Y such that $f(x) \notin \overline{V}$.

As x is an SC point in X, we can also assume that there is a sequence of connected sets C_n such that $C_n \to x$ and $x, x_n \in C_n$ for $n < \omega$. We can now define a sequence $z_n \in C_n$ such that $f(z_n) \in V$ and the points $f(z_n)$ are all distinct. Indeed, assume $n < \omega$ and the points z_i are already defined for $i < n$ in this way. As $f(C_n)$ is connected and $f(C_n) \cap V$ is a non-empty open proper subset, this intersection $f(C_n) \cap V$ is not closed and hence is infinite. Consequently there exists a point $z_n \in C_n$ with $f(z_n) \in f(C_n) \cap V - \{f(z_i) : i < n\}$. But then the sequence $\{z_n\}$ contradicts Lemma 1.3'. □

Corollary 1.22. *Let f be a preserving function from a topological space X into a Hausdorff space Y and let p be both an SC point and a Frèchet-point in X. Then f is continuous at p.* □

The following example (due to E. R. McMillan [6]) yields a locally connected SC space with a discontinuous preserving function. (Compare this with Theorem 1.21.)

Example 1.23. Take ω_1 copies of the interval $[0,1]$ and identify the 0 points. We get in this way a "hedgehog" $X = \{0\} \cup \{R_\xi : \xi \in \omega_1\}$, where the spikes $R_\xi = (0,1] \times \{\xi\}$ are disjoint copies of the half closed interval $(0,1]$. A basic neighbourhood of a point $x \in R_\xi$ is an open interval around x in R_ξ. A basic neighbourhood of 0 is a set of the form $\{0\} \cup \bigcup\{J_\xi : \xi < \omega_1\}$, where each J_ξ is a non-empty initial interval of R_ξ and $J_\xi = R_\xi$ holds for all but countably many ordinals ξ.

It is easy to see that in this way we get a locally connected Tychonoff SC space X. The function $f : X \to [0,1]$ defined by $f((x,\xi)) = x, f(0) = 0$ is preserving but not continuous at the point 0 because every neighbourhood of 0 is mapped onto $[0,1]$. □

The next example is locally connected, hereditary Lindelof, $T_{3\frac{1}{2}}$, of countable tightness and with a preserving function given on it that is not even sequentially continuous. It follows that this space is not an SC space.

Example 1.24. The underlying set of our space X consists of a point p, of a sequence of points p_n for $n < \omega$ and countably many arcs $\{I(n,m) : m < \omega\}$ with disjoint interiors connecting the points p_n and p_{n+1} for every $n < \omega$.

If x is an inner point of some arc, then its basic neighbourhoods are the open intervals around it on the arc. The basic neighbourhoods of a point p_n are the unions of initial (or final) segments of the arcs containing p_n. Finally, basic neighbourhoods of p are the sets which contain all but finitely many p_n's together with their basic neighbourhoods and for any two consecutive p_n's in the set all but finitely many of the arcs $I(n,m)$. Note that the subspace $X - \{p\}$ can be realised as a subspace of the plane, hence it is easy to check that X has the above stated properties.

Now let $f : X \to [0,1]$ be defined as follows: $f(p) = 0$, $f(p_n) = 0$ if n is even, $f(p_n) = 1$ if n is odd, and f is continuous on each arc $I(n,m)$. Then f is not sequentially continuous at p as shown by the sequence$\{p_n : n \text{ odd }\}$ converging to p, but it is preserving. Indeed, an infinite sequence whose members are from the interiors of different arcs is closed discrete and so a compact subset of X can meet only finitely many open arcs. It follows then that its f-image is the union of finitely many compact subsets of [0,1]. Moreover, if a connected set contains both p and some other point, then it also contains an arc $I(n,m)$, and thus its image is the whole $[0,1]$. □

In the rest of this section we shall consider a slight weakening of the sequential continuity property that comes up naturally in the proof of McMillan's theorem or Theorems 2.3 and 2.4 below.

Definition 1.25. A function $f : X \to Y$ is said to be *weakly sequentially continuous* at the point x if $f(x_n) \to f(x)$ whenever $x_n \to x$ in X and f is not locally constant at x_n for all $n < \omega$.

We shall give below two types of points in which preserving functions turn out to be weakly sequentially continuous. In the next section then these will be used to yield "real" continuity of preserving functions on some interesting classes of spaces.

Definition 1.26. A point x in a space X is called an *inflatable point* if $x_n \to x$ with $x_n \neq x$ for all $n < \omega$ implies that there are a subsequence $\{x_{n_k} : k < \omega\}$ and neighbourhoods U_k of x_{n_k} for $k < \omega$ such that $U_k \to x$ (i.e. every neighbourhood of x contains all but finitely many U_k's). The space X will be called inflatable if all its points are.

It is obvious that any generalized ordered space is inflatable.

Theorem 1.27. *Any preserving function $f : X \to Y$ from a locally connected space X into a T_2-space Y is weakly sequentially continuous at an inflatable point x.*

Proof. Assume, indirectly, that $x_n \to x$ but $f(x_n)$ does not converge to $f(x)$, while f is not locally constant at x_n for all $n < \omega$. By Lemma 1.3' we can assume that $f(x_n) = y \neq f(x)$ for all $n < \omega$. Choose an open neighbourhood $V \subset Y$ of y such that $f(x) \notin \overline{V}$. As x is inflatable, we may also assume to have open sets U_n with $x_n \in U_n$ such that $U_n \to x$. Using Lemma 1.7 we can recursively choose points $z_n \in U_n$ with distinct f-images such that $f(z_n) \in V$ for all $n < \omega$, contradicting Lemma 1.3' again. □

The other property is both a weakening of the Frèchet property and a variation on the s property.

Definition 1.28. We call a point x in a space X a *set-Fréchet* point if whenever $A = \bigcup\{A_n : n < \omega\}$ with $x \in \overline{A}$ but $x \notin \overline{A_n}$ for all $n < \omega$ then there is a sequence $\{x_n\} \subset A$ such that $x_n \to x$. Of course, a space is set-Fréchet if all its points are.

Theorem 1.29. *Let f be a preserving function from a locally connected T_2-space X into the interval $[0,1]$. Then f is weakly sequentially continuous at every set-Fréchet point x of X.*

Proof. Assume $x_n \to x$ but $f(x_n)$ does not converge to $f(x)$, moreover f is not locally constant at each x_n for $n < \omega$. By Lemma 1.3', we can assume that $f(x) = 1$ and $f(x_n) = 0$ for all $n < \omega$. Note that then for any connected neighbourhood G of any point x_n the image $f(G)$ is a proper interval containing 0. For every $n < \omega$ choose an open sets U_n such that $x_n \in U_n$ and $x \notin \overline{U}_n$ and put $A_n = \{z \in U_n : 0 < f(z) < 1/n+1\}$. By Lemma 1.7 the conditions in the definition of a set-Fréchet point are satisfied for the sets A_n so there is a sequence of points $z_n \in A = \bigcup\{A_n : n < \omega\}$ converging to x. It is immediate that $f(z_n)$ converges to $0 \neq 1 = f(x)$ while the set $\{f(z_n) : n < \omega\}$ is infinite because $f(z_n) \neq 0$ for all n, contradicting Lemma 1.3'. □

2. From sequential continuity to continuity

The aim of this section is to prove that if a locally connected space X fulfills one of the "convergence-type" conditions of the first section (i.e. X is an SC-space or it is inflatable or set-Fréchet) and $f : X \to Y$ is a preserving function then, assuming in addition appropriate separation axioms for X and Y, f is continuous at every s-point of X. The proofs of these theorems, just like their formulations, are very similar.

Theorem 2.1. *A preserving function $f : X \to Y$ from a locally connected SC-space X into a regular space Y is continuous at every s-point of X.*

Proof. Assume indirectly that f is not continuous at the s-point $p \in X$. Then there exists a closed set $F \subset Y$ such that $p \in \overline{f^{-1}(F)}$ but $f(p) \notin F$. Choose an open set $V \subset Y$ such that $F \subset V$ and $f(p) \notin \overline{V}$.

Let $\mathcal{K}$ be the family of the components of $f^{-1}(\overline{V})$ and put $\mathcal{A} = \{K \cap f^{-1}(F) : K \in \mathcal{K}\}$. As $p \notin \overline{K}$ for $K \in \mathcal{K}$ by Lemma 1.4 and $p \in \overline{f^{-1}(F)}$, the conditions given in the definition of an s point are fulfilled for the family $\mathcal{A}$. Thus there is a sequence $\{x_n : n < \omega\} \subset f^{-1}(F)$ such that $x_n \to x$ for some $x \in X$ and if K_n is the component of x_n in $f^{-1}(\overline{V})$, then $K_m \neq K_n$ for $m \neq n$.

As the components K_n are pairwise disjoint, we can suppose that $x \notin K_n$ for all $n < \omega$. It follows that if C is a connected set which contains both x and some x_n then $C \not\subset f^{-1}(\overline{V})$, because otherwise $K_n \cup C$ would be a connected subset of $f^{-1}(\overline{V})$ strictly larger than the component K_n, a contradiction. Hence, using that x is an SC-point, we may assume to have a sequence C_n of connected sets such that $C_n \to x$ and $C_n \not\subset f^{-1}(\overline{V})$. We can choose points $z_n \in C_n - f^{-1}(\overline{V})$ for all $n < \omega$, then $z_n \to x$ and $f(z_n) \notin \overline{V}$. But by Theorem 1.21 f is sequentially continuous, consequently $f(x) = \lim f(z_n) \in Y - V$. On the other hand, $f(x_n) \in F$ for $n < \omega$ and so, using again the sequential continuity of f at the point x, we get that $f(x) = \lim f(x_n) \in F$, a contradiction. $\square$

The proofs of the other two analogous theorems for inflatable and set-Fréchet spaces, respectively, make essential use of the following lemma:

Lemma 2.2. *Assume that $f : X \to Y$ is a preserving and weakly sequentially continuous function from a locally connected T_3-space X into a T_3-space Y and f is not continuous at some s-point of X. Then there are two sets $F \subset V \subset Y$, F closed and V open in Y and a convergent sequence $x_n \to x$ in X such that for all $n < \omega$ we have $x_n \neq x$, $f(x_n) \in F$ but $f(U) \not\subset \overline{V}$ whenever U is a neighbourhood of x_n. It follows that f is not locally constant at the points x_n and x.*

Proof. Assume f is not continuous at the s-point $p \in X$, then there exists a closed set $F \subset Y$ such that $p \in \overline{f^{-1}(F)}$ but $f(p) \notin F$. Choose an open set $V \subset Y$ such that $F \subset V$ and $f(p) \notin \overline{V}$.

Put

$$B = \{x \in f^{-1}(F) : f \text{ is not locally constant at } x\},$$

then $p \in \overline{B}$ by Lemma 1.6. Now let

$$H = \{x \in f^{-1}(F) : f(U) \not\subset \overline{V} \text{ for every neighbourhood } U \text{ of } x\},$$

clearly $H \subset B$. We assert that $p \in \overline{H}$ as well. To show this, fix a closed neighbourhood W of p. Let $\mathcal{K}$ be the family of the components of $f^{-1}(\overline{V})$ and put $\mathcal{A} = \{K \cap B \cap W : K \in \mathcal{K}\}$. Since $p \notin \overline{K}$ for $K \in \mathcal{K}$ by Lemma 1.4, the conditions in the definition of an s-point are satisfied for p and $\mathcal{A}$. Thus there is a sequence $\{y_n : n < \omega\} \subset B \cap W$ such that $y_n \to y$ for some $y \in X$ and if K_n is the component of y_n in $f^{-1}(\overline{V})$, then $K_m \neq K_n$ if $m \neq n$.

We claim that $y \in W \cap H$. As W is closed, trivially $y \in W$. The sets K_n are disjoint so we can assume that $y \notin K_n$ for all $n < \omega$. Using that f is weakly sequentially continuous and $y_n \in B$, we get that $f(y) = \lim f(y_n) \in F$, hence $y \in B$. We have yet to show that $f(U) \not\subset \overline{V}$ if U is any neighbourhood of y. By local connectivity, we can suppose that U is connected. But the connected set U meets (infinitely many) distinct components of $f^{-1}(\overline{V})$, so indeed $U \not\subset f^{-1}(\overline{V})$.

Now applying the s-point property of p to the family $\mathcal{A} = \{\{x\} : x \in H\}$ there is an infinite sequence of points $x_n \in H$ converging to some point x, completing the proof. $\square$

Theorem 2.3. *A preserving function from a locally connected and inflatable T_3 space X into a T_3 space Y is continuous at every s-point of X.*

Proof. Assume, indirectly, that the preserving function $f : X \to Y$ is not continuous at an s-point of X. By Theorem 1.27 f is weakly sequentially continuous, hence we can apply the preceding lemma and choose appropriate sets F, V in Y and points x_n and x in X. As X is inflatable and locally connected, we can also assume that there is a sequence of connected open sets U_n such that $U_n \to x$ and $x_n \in U_n$ for $n < \omega$. Then for all n we have $f(U_n) - \overline{V} \neq \emptyset$, hence by Lemma 1.14 these sets are uncountable. Consequently we can recursively select another sequence of points $y_n \in U_n$ (and so converging to x) such that $f(y_n) \in Y - \overline{V}$ and the $f(y_n)$'s are pairwise distinct. But then the sequence $f(y_n)$ does not converge to $f(x)$ because $f(x) = \lim f(x_n) \in F \subset V$ by the weak sequential continuity of f, contradicting Lemma 1.3' again. □

Corollary 2.4. *If X is locally compact, locally connected, and monotonically normal (in particular if X is a locally connected LOTS) then $Pr(X, T_3)$ holds.*

Proof. See [4, theorem 3.12] for a proof that a (locally) compact, monotonically normal space is both inflatable and radial. Consequently, it is also locally sequentially compact, and thus an s-space. □

Theorem 2.5. *A preserving function from a locally connected and set-Frèchet T_3 space X into a $T_{3\frac{1}{2}}$ space is continuous at every s-point of X.*

Proof. By Lemma 1.2, it suffices to prove this for preserving functions $f : X \to [0,1]$. Assume, indirectly, that the function f is not continuous at some s-point $p \in X$. Then, by Theorem 1.29, f is weakly sequentially continuous, hence we may again apply Lemma 2.2 to choose appropriate sets F and V in $Y = [0,1]$ and points x_n and x in X. There is a continuous function $g : [0,1] \to [0,1]$ which is identically 1 on F and 0 on $[0,1] - V$. Then the composite function $h = gf : X \to [0,1]$ is also preserving, $h(x_n) = 1$ for all n, and the h-image of any neighbourhood of a point x_n is the whole interval $[0,1]$.

Let us choose open sets G_n for $n < \omega$ such that $x_n \in G_n$ and $x \notin \overline{G}_n$. If $A_n = G_n \cap f^{-1}((0, \frac{1}{n}))$ and $A = \bigcup A_n$, then $x \notin \overline{A}_n$ but $x \in \overline{A}$. So, as X is a set-Frèchet space, there is a sequence of points $y_n \in A$ converging to x. But then the set $\{h(y_n) : n < \omega\}$ is infinite and $h(y_n) \to 0 \neq 1 = f(x)$, contradicting Lemma 1.3'. □

Corollary 2.6. *If X is a locally connected, locally countably compact, and set-Frèchet T_3-space then $Pr(X, T_{3\frac{1}{2}})$ holds.*

Proof. It is enough to note that a countably compact set-Frèchet space is also sequentially compact and so an s-space. □

3. Some product theorems

The aim of this section is to prove that an arbitrary product X of certain "good" spaces has the property $Pr(X, T_3)$. To achieve this, we shall make use of Theorem 2.1. It is well-known that the product of connected and locally connected spaces is locally connected, moreover a very similar argument (based on the productivity of connectedness) implies that the product of connected SC spaces is SC. Hence two of the assumptions of Theorem 2.1 are productive if the factors are connected. Nothing like this can be expected, however, about the third assumption of Theorem 2.1, namely the s-property. To make up for this, we are going to consider a stronger property that is at least countably productive, and use this stronger property to establish what we want, first for Σ-products and then for arbitrary products. Now, this stronger property will require the existence of a winning strategy for player **I** in the following game.

Definition 3.1. Fix a space X and a point $p \in X$. The game $G(X,p)$ is played by two players **I** and **II** in ω rounds. In the n-th round first **I** chooses a neighbourhood U_n of p and then **II** chooses a point $x_n \in U_n$. **I** wins if the produced sequence $\{x_n : n < \omega\}$ has a convergent subsequence, otherwise **II** wins.

We shall say that X is *winnable* at the point p if **I** has a winning strategy in the game $G(X,p)$. X is *winnable* if $G(X,p)$ is winnable for all $p \in X$. Note that, formally, a winning strategy for **I** is a map $\sigma : X^{<\omega} \to \mathcal{V}(p)$, where $\mathcal{V}(p)$ is the family of neighbourhoods of p, such that if $\langle x_n : n < \omega \rangle$ is a sequence played following σ, i.e. $x_n \in \sigma\langle x_0, x_1, ..., x_{n-1}\rangle$ for all $n < \omega$, then $\langle x_n : n < \omega\rangle$ has a convergent subsequence.

Lemma 3.2. *A winnable point p of a space X is always an s point.*

Proof. Let σ be a winning strategy of **I** in the game $G(X,p)$ and fix a family $\mathcal{A}$ of subsets of X with $p \in \overline{\bigcup \mathcal{A}}$ but $p \notin \overline{A}$ for all $A \in \mathcal{A}$. Let us play the game $G(X,p)$ in such a way that **I** follows σ and assume that the first n rounds of the game have been played with the points $x_i \in U_i$ and the distinct sets $A_i \in \mathcal{A}$ with $x_i \in A_i$ chosen by player **II** for $i < n$. Let $U_n = \sigma\langle x_0, x_1, ..., x_{n-1}\rangle$ be the next winning move of **I**, then **II** can choose a set $A_n \in \mathcal{A}$ with $A_n \cap (U_n - \cup_{i<n}\overline{A_i}) \neq \emptyset$ and then pick $x_n \in A_n \cap (U_n - \cup_{i<n}\overline{A_i})$ as his next move. But player **I** wins hence, a suitable subsequence of $\{x_n : n < \omega\}$, whose members were picked from distinct elements of $\mathcal{A}$, will converge. □

In order to prove the desired product theorem for winnable spaces we shall consider *monotone strategies* for player **I**. A strategy σ of player **I** is said to be *monotone* if for every subsequence $\langle x_{i_0}, ..., x_{i_{r-1}}\rangle$ of a sequence $\langle x_0, x_1, ..., x_{n-1}\rangle$ of points in X we have

$$\sigma\langle x_0, x_1, ..., x_{n-1}\rangle \subset \sigma\langle x_{i_0}, ..., x_{i_{r-1}}\rangle.$$

Lemma 3.3. *If player* **I** *has a winning strategy in the game $G\langle X, p\rangle$ then he also has a monotone winning strategy.*

Proof. Let σ be a winning strategy for player **I**; we define a new strategy σ_0 as follows : for any sequence $s = \langle x_0, x_1, ..., x_{n-1}\rangle$ put

$$\sigma_0(s) = \bigcap\{\sigma\langle x_{i_0}, ..., x_{i_{r-1}}\rangle : 0 \le i_0 < i_1 < ... < i_{r-1} < n\}.$$

The function σ_0 is clearly a monotone winning strategy for **I**. □

It is easy to see that if **I** plays using the monotone strategy σ_0 then any infinite subsequence of the sequence chosen by player **II** is also a win for

I, i. e. has a convergent subsequence. Another important property of a monotone winning strategy σ_0 is the following: If $\langle x_n : n < \omega \rangle$ is a sequence of points in X such that we have $x_n \in \sigma_0 \langle x_0, x_1, ..., x_{n-1} \rangle$ only for $n \geq m$ for some fixed $m < \omega$ then this is still a winning sequence for **I**. Indeed, this holds because for every $n \geq m$ we have

$$x_n \in \sigma_0 \langle x_0, x_1, ..., x_{n-1} \rangle \subset \sigma_0 \langle x_m, x_{m+1}, ..., x_{n-1} \rangle$$

by monotonicity, hence the "tail" sequence $\langle x_n : m \leq n < \omega \rangle$ is produced by a play of the game where **I** follows the strategy σ_0.

For a family of spaces $\{X_s : s \in S\}$ and a fixed point p (called the *base point*) of the product $X = \prod \{X_s : s \in S\}$ let $T(x)$ denote the support of the point x in X: this is the set $\{s \in S : x(s) \neq p(s)\}$. Then $\Sigma(p)$ (or simply Σ if this does not lead to misunderstanding) denotes the Σ-product with base point p: it is the subspace of X of the points with countable support, i.e.

$$\Sigma = \{x \in X : |T(x)| \leq \omega\}.$$

In the proof of the next result we shall use two lemmas. The first one is an easy combinatorial fact:

Lemma 3.4. *Let $\langle H_k : k < \omega \rangle$ be a sequence of countable sets, then for every $n < \omega$ there is a finite set F_n depending only on the first n many sets $\langle H_k : k < n \rangle$ such that $F_n \subset \bigcup_{i<n} H_i$, $F_n \subset F_{n+1}$ and $\bigcup F_n = \bigcup H_n$.*

Proof. Fix an enumeration $H_i = \{x(i,j) : j < \omega\}$ of the set H_i for all $i < \omega$ and then let $F_n = \{x(i,j) : i, j < n\}$. □

The second lemma is about the sequences which play a crucial role in the games $G\langle X, p \rangle$. Let us call a sequence $\{x_n\}$ in the space X *good* if every infinite subsequence of it has a convergent subsequence.

Lemma 3.5. *If $x_n \in X = \prod \{X_i : i < \omega\}$ for $n < \omega$ and $\{x_n(i) : n < \omega\}$ is a good sequence in X_i for all $i \in \omega$ then $\{x_n\}$ is a good sequence in X.*

Proof. We shall prove that if N is any infinite subset of ω then there is in X a convergent subsequence of $\{x_n : n \in N\}$.

We can choose by recursion on $k < \omega$ infinite sets N_k such that $N_{k+1} \subset N_k \subset N$ and $\{x_n(k) : n \in N_k\}$ converges to a point $x(k)$ in X_k. Then there is a diagonal sequence $\{n_k : k < \omega\}$ such that $n_k \in N_k$ and $n_k < n_{k+1}$ for all $k < \omega$. The sequence $\{n_k : k < \omega\}$ is eventually contained in N_i , hence $x_{n_k}(i) \to x(i)$ in X_i, for all $i < \omega$. It follows that $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n : n \in N\}$ in X. □

The following result says a little more than that winnability is a countably productive property.

Lemma 3.6. *Let $p \in X = \prod \{X_s : s \in S\}$ and suppose that $G\langle X_s, p(s) \rangle$ is winnable for every $s \in S$. Then $G\langle \Sigma(p), p \rangle$ is also winnable.*

Proof. We have to construct a winning strategy σ for player **I** in the game $G\langle\Sigma(p),p\rangle$. By Lemma 3.3, we can fix a monotone winning strategy σ_s of **I** in the game $G\langle X_s,p(s)\rangle$ for each $s\in S$. Given a sequence $\langle x_0,...,x_{n-1}\rangle\in[\Sigma(p)]^{<\omega}$, let H_i denote the support of x_i and F_n be the finite set assigned to the sequence $\{H_i\ :i<n\}$ as in Lemma 3.4. Now, if π_s is the projection from the product X onto the factor X_s for $s\in S$ then set

$$\sigma\langle x_i:i<n\rangle=\bigcap\{\pi_s^{-1}(\sigma_s\langle x_i(s):i<n\rangle):s\in F_n\rangle\}.$$

Now let $\langle x_n:n<\omega\rangle$ be a sequence of points in $\Sigma(p)$ produced by a play of the game $G\langle\Sigma(p),p\rangle$ in which **I** followed the strategy σ. Then for every $s\in H=\bigcup H_n$ the sequence $\langle x_n(s):n<\omega\rangle$ is a win for player **I** in the game $G\langle X_s,p(s)\rangle$ because there is an $m<\omega$ with $s\in F_m$ and then $x_n(s)\in\sigma_s\langle x_i(s):i<n\rangle$ is valid for all $n\geq m$. Consequently, by Lemma 3.5, the sequence $\langle x_n|H:n<\omega\rangle$ has a convergent subsequence in $\prod\{X_s:s\in H\}$, while for $s\in S-H$ we have $x_n(s)=p(s)$ for all $n<\omega$, and so $\langle x_n:n<\omega\rangle$ indeed has a convergent subsequence in $\Sigma(p)$. □

The following two statements both easily follow from the fact that any product of connected spaces is connected.

Lemma 3.7. *A Σ-product of connected SC spaces is also an SC space.* □

Lemma 3.8. *A Σ-product of connected and locally connected spaces is also locally connected.* □

We now have all the necessary ingredients needed to prove our main product theorem.

Theorem 3.9. *Let $f:X=\prod\{X_s:s\in S\}\to Y$ be a preserving function from a product of connected and locally connected SC spaces into a regular space Y. If $p\in X$ and $G\langle X_s,p(s)\rangle$ is winnable for all $s\in S$ then f is continuous at the point p .*

Proof. Let Σ denote the sigma-product with base point p. Then, by Lemma 3.6, $G\langle\Sigma,p\rangle$ is winnable and so p is an s point in Σ. Moreover, by Lemmas 3.7 and 3.8, Σ is also a locally connected SC space. Hence Theorem 2.1 implies that the restriction of the function f to the subspace Σ of X is continuous at p.

To prove that f is also continuous at the point p in X, fix a neighbourhood V of $f(p)$ in Y. As the restriction $f|\Sigma$ is continuous at p, there is an elementary neighbourhood U of p in the product space X such that $f(U\cap\Sigma)\subset V$. Since the factors X_s are connected and locally connected, we can assume that U and $U\cap\Sigma$ are also connected and hence, by Lemma 1.4, we have

$$f(U)\subset f(\overline{U\cap\Sigma})\subset\overline{f(U\cap\Sigma)}\subset\overline{V}.$$

The regularity of Y then implies that f is continuous at p. □

Corollary 3.10. *$Pr(X, T_3)$ holds whenever X is any product of connected and locally connected, winnable SC spaces. In particular, if $X = \prod\{X_s : s \in S\}$ where each factor X_s is either a connected linearly ordered space (with the order topology) or a connected and locally connected first countable space then $Pr(X, T_3)$ is valid.* □

For the proof of the next Corollary we need a general fact about the relations $Pr(X, T_i)$.

Lemma 3.11. *If $q: X \to Y$ is a quotient mapping of X onto Y then, for any i, $Pr(X, T_i)$ implies $Pr(Y, T_i)$.*

Proof. Let $f: Y \to Z$ be a preserving function into the T_i space Z. The function $fq: X \to Z$, as the composition of a continuous (and so preserving) and of a preserving function is also preserving, hence, by $Pr(X, T_i)$, it is continuous. But then f is continuous because q is quotient. □

Corollary 3.12. *Let $X = \prod\{X_s : s \in S\}$ where all the spaces X_s are compact, connected, locally connected, and monotonically normal. Then $Pr(X, T_2)$ holds.*

Proof. It follows from the recent solution by Mary Ellen Rudin of Nikiel's conjecture [10], combined with results of L.B.Treybig [11] or J.Nikiel [7], that *every compact, connected, locally connected, monotonically normal space is the continuous image of a compact, connected, linearly ordered space.* Hence our space X is the continuous image of a product of compact, connected, linearly ordered spaces. But any T_2 continuous image of a compact T_2 space is a quotient image (and T_3), hence Corollary 3.10 and Lemma 3.11 imply our claim. □

Comparing this result with Corollary 2.4, the following question is raised naturally.

Problem 3.13. *Let X be a product of locally compact, connected and locally connected monotonically normal spaces. Is then $Pr(X, T_3)$ true?*

The following is result is mentioned here mainly as a curiosity.

Corollary 3.14. *Let $X = \prod\{X_s : s \in S\}$ be a product of linearly ordered and/or first countable spaces. Then the following are equivalent:*

a) $Pr(X, T_3)$;
b) X is locally connected;
c) the spaces X_s are all locally connected and all but finitely many of them are also connected.

Proof. a)⇒b) : Lemma 1.1.
b)⇒c) : [2, 6.3.4]
c)⇒a) : Corollary 3.10. □

Remark 3.15. E. R. McMillan raised the following question in [6]: does $Pr(X, T_2)$ imply that X is a k-space? We do not know the (probably negative) answer to this question, however we do know that the answer is negative if T_2 is replaced by T_3 in it. Indeed, for instance $\mathbb{R}^{\omega_1}$ is not a k-space (see e.g. [2, exercise 3.3.E],), but, by Corollary 3.10, $Pr(\mathbb{R}^{\omega_1}, T_3)$ is valid.

4. The sequential and the compact cases

The two examples given in section 1 of (locally connected) spaces on which there are non-continuous preserving functions both lack the properties of (i) sequentiality and (ii) compactness. Here (i) arises naturally as a weakening of the Frèchet property from McMillan's theorem while the significance of (ii) needs no explanation. This leads us naturally to the following problem.

Problem 4.1. *Assume that the locally connected space X is (i) sequential and/or (ii) compact. Is then $Pr(X, T_2)$ (or $Pr(X, T_{3\frac{1}{2}})$) true?*

The answer in case (i) turns out to be positive if we assume the SC property instead of local connectivity. The following result reveals why local connectivity need not be assumed in it.(Compare this also with Lemma 1.18.)

Theorem 4.2. *Any sequential SC space X is locally connected.*

Proof. We have to prove that if K is a component of an open set $G \subset X$ then K is open. Assume not, then $X - K$ is not closed, hence as X is sequential there is a sequence $\{x_n\} \subset X - K$ such that $x_n \to x \in K$. Since X is an SC space and G is a neighbourhood of x there is a connected set $C \subset G$ such that $\{x, x_n\} \subset C$ for some $n < \omega$. But this is impossible because then the connected set $K \cup C \subset G$ would be larger than the component K of G. □

Now we give the above promised partial solution to Problem 4.1 in case (i), i. e. for sequential spaces.

Theorem 4.3. *If X is a sequential SC space then $Pr(X, T_2)$ holds.*

Proof. Let $f : X \to Y$ be a preserving map into a T_2 space Y. Since X is sequential it suffices to show that the function f is sequentially continuous but this immediate from Theorem 1.21. □

Our next result implies that a counterexample to Problem 4.1 (in either case) can not be very simple in the sense that discontinuity of a preserving function cannot occur only at a single point (as it does in both examples 1.23 and 1.24). In order to prepare this result we first introduce a topological property that generalizes both sequentiality and (even countable) compactness.

Definition 4.4. X is called a *countably k* space if for any set $A \subset X$ that is not closed in X there is a countably compact subspace C of X such that $A \cap C$ is not closed in C.

This condition means that the topology of X is determined by its countably compact subspaces. All countably compact and all k (hence also all sequential) spaces are countably k. It is easy to see that the countably k property is always inherited by closed subspaces and for regular spaces by open subspaces as well.

Theorem 4.5. *Let X be countably k and locally connected and Y be T_3, moreover let $f\colon X \to Y$ be a preserving function. Then the set of points of discontinuity of f is not a singleton. So if X is also T_3 then the discontinuity set of f is dense in itself.*

Proof. Assume, indirectly, that f is not continuous at $p \in X$ but it is continuous at all other points of X. Then we can choose a closed set $F \subset Y$ with $A = f^{-1}(F)$ not closed. Evidently, then $\overline{A} - A = \{p\}$. As A is not closed in X and X is countably k, there is a countably compact set C in X such that $A \cap C$ is not closed in C; clearly then $p \in C$ and p is in the closure of $A \cap C$.

Let $V \subset Y$ be an open set with $F \subset V$ and $f(p) \notin \overline{V}$ and put $H = f^{-1}(V)$. Then H is open in X and contains the set A. For every component L of H we have $p \notin \overline{L}$ by $f(p) \notin \overline{L} \subset \overline{V}$ and Lemma 1.4, hence $p \in \overline{A \cap C}$ implies that there are infinitely many components of H that meet $A \cap C$. Thus we may choose a sequence $\{L_n : n < \omega\}$ of distinct such components with points $x_n \in L_n \cap A \cap C$.

We claim that $x_n \to p$. As the x_n's are chosen from the countably compact set C, it is enough to prove for this that if $x \neq p$ then x is not an accumulation point of the sequence $\{x_n\}$. If $x \notin \overline{A} = A \cup \{p\}$ this is obvious so we may assume that $x \in A \subset H$. But then, as H is open and X is locally connected, the connected component of x in H is a neighbourhood of x that contains at most one of the points x_n.

The point $f(p)$ is not in the closure of the set $\{f(x_n)\colon n < \omega\} \subset F$, hence, by Lemma 1.3', we can suppose that $f(x_n) = y \neq f(p)$ for each $n < \omega$.

Now we choose a sequence of neighbourhoods V_n of y in Y with $V_0 = V$ and $\overline{V_{n+1}} \subset V_n$ for all $n < \omega$ and then put $U_n = f^{-1}(V_n)$. Clearly U_n is open in X and $U_0 = H$, hence, as was noted above, the closure of any connected set contained in U_0 contains at most one of the points x_n.

Next, let K_n be the component of x_n in U_n for $n < \omega$. We claim that every boundary point of K_n is mapped by f to a boundary point of V_n, i. e.

$$f(FrK_n) \subset FrV_n.$$

Indeed, $f(\overline{K_n}) \subset \overline{V_n}$ by Lemma 1.4 (or by continuity at all points distinct from p). Moreover, we have

$$FrK_n = \overline{K_n} - K_n \subset FrU_n = \overline{U_n} - U_n$$

because K_n, U_n are open and K_n is a component of U_n, and $f(FrU_n) \cap V_n = \emptyset$ because $U_n = f^{-1}(V_n)$. Thus indeed $f(FrK_n) \subset \overline{V_n} - V_n = FrV_n$.

Every connected neighbourhood of p meets all but finitely many K_n's hence also FrK_n, consequently $K = \bigcup\{FrK_n : n < \omega\}$ is not closed. So there exists a countably compact set D with $D \cap K$ not closed in D. But the sets FrK_n are closed, thus D must meet infinitely many of them, i. e. the set

$$N = \{n < \omega : D \cap FrK_n \neq \emptyset\}$$

is infinite. Let us choose a point z_n from each nonempty $D \cap K_n$.

Again we claim that the only accumulation point of the sequence $\{z_n : n \in N\}$ is p. Indeed, if $x \neq p$ would be such an accumulation point, then $f(z_n) \in \overline{V_n} \subset \overline{V_1}$ for all $n \in N - \{0\}$ would also also imply $f(x) \in \overline{V_1} \subset V_0$. By continuity and local connectivity at x then there is a connected neighbourhood W of x with $f(W) \subset V_0$. But then the set

$$W \cup \bigcup\{K_n : W \cap K_n \neq \emptyset\}$$

would be a connected subset of U_0 that has p in its closure, a contradiction.

Consequently, the sequence $\{z_n : n \in N\}$ must converge to p, while $\{f(z_n) : n \in N\} \subset \overline{V}$ does not converge to $f(p)$, contradicting Lemma 1.3' because $f(z_n) \in FrV_n$ for all $n \in N$ and the boundaries FrV_n are pairwise disjoint, hence the $f(z_n)$'s are pairwise distinct.

The last statement of the theorem now follows easily because an isolated discontinuity of f yields an open subspace of X on which the restriction of f has a single point of discontinuity, although if X is T_3 then any open subspace of X is both countably k and locally connected. $\square$

Noting that Problem 4.1 really comprises three different questions, and having shown above that, in a certain sense, it seems to be hard to find counterexamples to any of these, we now turn our attention to the case in which both (i) and (ii) are assumed. In this case we can provide a positive answer, at least consistently and with the extra assumption that the cellularity of the space in question is "not too large". In fact, what we can prove is that if $2^\omega < 2^p$ then any locally connected, compact T_2 space X that is sequential and does not contain a cellular family of size p satisfies $Pr(X, T_2)$. Of course, here p stands for the well-known cardinal invariant of the continuum whose definition is recalled below.

A set $H \subset \omega$ is called a *pseudo intersection* of the family $\mathcal{A} \subset [\omega]^\omega$ if H is almost contained in every member of $\mathcal{A}$, i.e. $H - A$ is finite for each $A \in \mathcal{A}$. Then p is the minimal cardinal κ such that there exists a family $\mathcal{A} \subset [\omega]^\omega$ of size κ which has the finite intersection property but does not have an infinite pseudo intersection. (Here the finite intersection property means that any finite subfamily of $\mathcal{A}$ has *infinite* intersection.)

It is well-known (see e.g.[12]) that the cardinal p is regular, $\omega_1 \leq p \leq 2^\omega$ and $2^\kappa = 2^\omega$ for $\kappa < p$. The condition "$2^p > 2^\omega$" of our result is satisfied if $p = 2^\omega$ (hence Martin's axiom implies it), but it is also true if $2^{\omega_1} > 2^\omega$.

Now, our promised consistency result on compact sequential spaces will be a corollary of a ZFC result of somewhat technical nature. Before formulating this, however, we shall prove two lemmas that may have some independent interest in themselves.

Lemma 4.6. *Let X be a compact T_2 space of countable tightness and $f : X \to [0,1]$ be a compactness preserving map of X into the unit interval. If $x \in X$ is a point in X and $[a,b]$ is a subinterval of $[0,1]$ such that for every neighbourhood U of x we have $[a,b] \subset f(U)$ then for any $G_{<p}$ set H containing x we also have $[a,b] \subset f(H)$.*

Proof. Without loss of generality we may assume that H is closed. Now the proof will proceed by induction on κ where $\omega \leq \kappa < p$ and H is a (closed) G_κ set, or equivalently, the character $\chi(H,X) = \kappa$. If $\kappa = \omega$ then we can write $H = \bigcap\{G_n : n < \omega\}$ with G_n open and $\overline{G_{n+1}} \subset G_n$ for all $n < \omega$. Fix a countable dense subset $\{c_n : n < \omega\}$ of $[a,b]$ and then pick $x_n \in G_n$ with $f(x_n) = c_n$, this is possible by assumption. Note that then every accumulation point of the set $M = \{x_n : n < \omega\}$ is in H, hence by Lemma 1.3 we have

$$[a,b] = f(M)' \subset f(M') \subset f(H).$$

Next, if $\omega < \kappa < p$ then we have $x \in H = \bigcap\{S_\xi : \xi < \kappa\}$, where $S_\xi \supset S_\eta$ if $\xi < \eta$ and the S_ξ are closed sets of character $< \kappa$. By induction, we have $f(S_\xi) \supset [a,b]$ for all $\xi < \kappa$, and we have to prove that $f(H) \supset [a,b]$ as well. In fact, it suffices to show that $f(H) \cap [a,b] \neq \emptyset$ because applying this to all (non-singleton) subintervals of $[a,b]$ we actually get that $f(H) \cap [a,b]$ is dense in $[a,b]$ while $f(H)$ is also compact, hence closed.

We do this indirectly; assume $f(H) \cap [a,b] = \emptyset$ then we can choose points $x_\xi \in S_\xi - H$ for all $\xi < \kappa$ such that the images $f(x_\xi) \in [a,b]$ are all distinct. Let $\bar{x}$ be a complete accumulation point of the set $\{x_\xi : \xi < \kappa\}$. Then $\bar{x} \in H$ and $t(X) = \omega$ implies that there is a countable subset $A \subset \{x_\xi : \xi < \kappa\}$ such that $\bar{x} \in \overline{A} - A$. Choose now a neighbourhood base $\mathcal{B}$ of H in X of size $\kappa < p$. The family $\{A \cap B : B \in \mathcal{B}\} \subset [A]^\omega$ has the finite intersection property hence it has an infinite pseudo intersection $P \subset A$, i. e. the set $P - B$ is finite for each $B \in \mathcal{B}$. This implies that every accumulation point of P is contained in H. But $\overline{P}$ is compact, hence by Lemma 1.3 we have

$$\emptyset \neq f(P)' \subset f(P') \cap [a,b] \subset f(H) \cap [a,b],$$

which is a contradiction. □

Before we give the other lemma, let us recall that for any space X we use $\widehat{c}(X)$ to denote the smallest cardinal κ such that X does *not* contain κ disjoint open sets.

Lemma 4.7. *Let $f : X \to Y$ be a connectivity preserving map from a locally connected space X into a T_2 space Y. Then for every $x \in X$ with $\chi(f(x), Y) < \widehat{c}(X)$ there is a $G_{<\widehat{c}(X)}$ set H in X such that $x \in H$ and if $z \in H$ is any point of continuity of f then $f(z) = f(x)$.*

Proof. Let $\kappa = \widehat{c}(X)$ and fix a neighbourhood base $\mathcal{V}$ of the point $f(x)$ in Y with $|\mathcal{V}| < \kappa$. For every $V \in \mathcal{V}$ let us then set

$$G_V = \bigcup\{G : G \text{ is open in } X \text{ and } f(G) \cap V = \emptyset\}.$$

For every component K of the open set G_V we have $f(x) \notin \overline{f(K)}$ and therefore $x \notin \overline{K}$ by Lemma 1.4, moreover the components of G_V form a cellular family because X is locally connected, hence their number is less than κ. Consequently,

$$H_V = \bigcap\{X - \overline{K} : K \text{ is a component of } G_V\}$$

is a $G_{<\kappa}$ set with $x \in H_V$ and $H_V \cap G_V = \emptyset$.

The cardinal κ is regular (see e.g. [3, 4.1]), hence $H = \cap\{H_V : V \in \mathcal{V}\}$ is also a $G_{<\kappa}$ set that contains the point x. Now, suppose that z is a point of continuity of f with $f(z) \neq f(x)$. Then there is a basic neighbourhood $V \in \mathcal{V}$ of $f(x)$ and a neighbourhood W of $f(z)$ with $V \cap W = \emptyset$, and there is an open neighbourhood U of x in X with $f(U) \subset W$. But then, by definition, we have $z \in U \subset G_V$, hence $z \notin H_V \supset H$. □

Theorem 4.8. *Let X be a locally connected compact T_2 space of countable tightness. If, in addition, we also have $|X| < 2^{\mathfrak{p}}$ and $\widehat{c}(X) \leq \mathfrak{p}$ then $Pr(X, T_2)$ holds.*

Proof. Using Lemma 1.2 it suffices to show that any preserving function $f : X \to [0,1]$ is continuous. To this end, first note that if f is not continuous at a point $x \in X$ then the oscillation of f at x is positive, hence , by local connectivity at x and because f is preserving there are $0 \leq a < b \leq 1$ such that $f(U) \supset [a,b]$ holds for every neighbourhood U of x. Consequently, by Lemma 4.6 we also have $f(H) \supset [a,b]$ whenever H is any $G_{<\mathfrak{p}}$ set containing the point x. In particular, this implies that if the singleton $\{x\}$ is a $G_{<\mathfrak{p}}$ set (equivalently, if the character of x in X is less than $\mathfrak{p}$) then f is continuous at x.

On the other hand, by Lemma 4.7, for every point $x \in X$ there is a closed $G_{<\mathfrak{p}}$ set H_x with $x \in H_x$ such that for any point of continuity $z \in H_x$ of f we have $f(z) = f(x)$. We claim that f is constant on every such set H_x and then, by the above, f is continuous at every point $x \in X$.

For this it suffices to show that f has a point of continuity in every (non-empty) closed $G_{<\mathfrak{p}}$ set H. Indeed, for any point $y \in H_x$ then the intersection $H_x \cap H_y$ contains a point of continuity z for which $f(x) = f(z) = f(y)$ must hold. By the Čech-Pospišil theorem (see e.g. [3, 3.16]) and by $|H| < 2^{\mathfrak{p}}$ there is a point $z \in H$ with $\chi(z, H) < \mathfrak{p}$ and so $\chi(z, X) < \mathfrak{p}$ as well, for H is a $G_{<\mathfrak{p}}$ set in X. But we have seen above that then z is indeed a point of continuity of f. □

Theorem 4.9. *Assume that $2^{\omega} < 2^{\mathfrak{p}}$ and X is a locally connected and sequential compact T_2 space with $\widehat{c}(X) \leq \mathfrak{p}$. Then $Pr(X, T_2)$ holds.*

Proof. By a slight strengthening of some well-known results of Shapirovski (see e.g. [3, 2.37 and 3.14]), for any compact T_2 space X we have both $\pi\chi(X) \leq t(X)$ and

$$d(X) \leq \pi\chi(X)^{<\widehat{c}(X)}.$$

Consequently, for our space X we have

$$d(X) \leq \omega^{<p} = 2^{\omega}$$

and so by sequentiality $|X| \leq 2^{\omega}$ as well. But this shows that all the conditions of Theorem 4.8 are satisfied by our space X. □

To conclude, let us emphasize again that Lemma 1.3, i.e. the full force of compactness preservation, as opposed to just the preservation of the compactness of convergent sequences, was only used in this section (cf. the remark made after 1.3').

5. The relation $Pr(X, T_1)$

The main aim of this section is to prove that if $Pr(X, T_1)$ holds and X is T_3 then X is discrete. Note the striking contrast between $Pr(X, T_1)$ and $Pr(X, T_2)$: the latter holds for a large class of (non-discrete) spaces (see Theorem 1.8 or Corollary 3.11).

Let us recall that the *cofinite topology* on an underlying set X is the coarsest T_1 topology on X: the open sets are the empty set and the complements of the finite subsets of X. It is not hard to see that such a space is hereditarily compact and any infinite subset in it is connected. Let us start with a result that gives several different characterizations of T_1 spaces X that satisfy $Pr(X, T_1)$.

Theorem 5.1. *For a T_1 space X the following conditions are equivalent:*

a) If Y is T_1 and $f : X \to Y$ is a connectedness preserving function then f is continuous.

b) If Y is T_1 and $f : X \to Y$ is a preserving function then f is continuous (i.e. $Pr(X, T_1)$ holds).

c) If Y has the cofinite topology and $f : X \to Y$ is a preserving function then f is continuous.

d) If $A \subset X$ is not closed then there exists a connected set $H \subset X$ such that $H \cap A \neq \emptyset \neq H - A$ and $H - A$ is finite.

Proof. a) $\Rightarrow$ b) and b)$\Rightarrow$ c) are obvious.

c) $\Rightarrow$ d) Assume that $A \subset X$ is not closed. Let Y denote the space with the cofinite topology on the underlying set of X. Choose a point $a_0 \in A$. (A is not closed so it is not empty, either.) Define now the function $f : X \to Y$ by

$$f(x) = \begin{cases} a_0 & \text{if } x \in A, \\ x, & \text{otherwise.} \end{cases}$$

Then f is not continuous because the inverse image of the closed set $\{a_0\}$ is the non-closed set A hence ,by c), f is not preserving. As an arbitrary subset of Y is compact, f preserves compactness, so there is a connected set $H \subset X$ such that $f(H)$ is not connected. It follows that H is infinite and $f(H)$ is finite but not a singleton. As f is the identity map on $X - A$, the set $H - A$ is finite and so $H \cap A \neq \emptyset$.Finally, $H \subset A$ is impossible because $f(H)$ is not a singleton.

d) $\Rightarrow$ a) Assume $f : X \to Y$ is not continuous for a T_1-space Y, hence there is a closed set $F \subset X$ such that $A = f^{-1}(F)$ is not closed in X. By d), there is a connected set H such that $H \cap A \neq \emptyset$ and $\emptyset \neq H - A$ is finite. But then $f(H)$ is not connected because it is the the disjoint union of two non-empty relatively closed sets, namely of $f(H) \cap F$ and of the finite set $f(H) - F$. Consequently, f does not preserve connectedness. □

Corollary 5.2. *If $Pr(X, T_1)$ holds for a T_1-space X then every closed subspace of X is the topological sum of its components.*

Proof. Let K be a component of the closed subset $F \subset X$. It is enough to prove that K is relatively open in F. Assume this is false; then $A = F - K$ is not closed in X, and thus, by condition d) of Theorem 5.1, there is a connected set H in X such that $H \cap A \neq \emptyset$ and $\emptyset \neq H - A$ is finite. Then $H - F$ is also finite, consequently $H \subset F$ because H is connected and F is closed. Thus H is a connected subset of F which meets the component K of F, contradicting that $H \cap A \neq \emptyset$. □

Corollary 5.3. *If $Pr(X, T_1)$ holds for a T_3-space X then every closed subspace of X is locally connected.*

Proof. By 5.2 it is enough to prove that if every closed subset of a regular space X is the topological sum of its components then X is locally connected.

Let U be a closed neighbourhood of a point $x \in X$. By our assumption if K denotes the component of x in U then K is open in U, hence $K \subset U$ is a connected neighbourhood of x in X. As the closed neighbourhoods of a point form a neighbourhood-base of the point in a regular space, X is locally connected. □

Theorem 5.4. *If $Pr(X, T_1)$ holds for a T_3-space X then X is discrete.*

By Corollary 5.3 it is enough to prove the following result that, we think, is interesting in itself:

Theorem 5.5. *If X is T_3 and every regular closed subspace of X is locally connected then X is discrete.*

Proof. We can assume without any loss of generality that X is connected. Suppose, indirectly, that x is a non-isolated point in X. By regularity, there is a sequence of non-empty open sets $\{G_n : n < \omega\}$ such that $x \notin G_n$ and $\overline{G}_n \subset G_{n+1}$ for all $n < \omega$. Then the open set $G = \bigcup\{G_n : n < \omega\}$ can not be also closed in the connected space X, so there is a point $p \in \overline{G} - G$.

Put $U_0 = G_0$ and $U_n = G_n - \overline{G}_{n-1}$ for $0 < n < \omega$. If $H_0 = \bigcup\{U_n :$ n is even$\}$ and $H_1 = \bigcup\{U_n : n$ is odd$\}$, we claim that then $\overline{G} = \overline{H_0 \cup H_1} = \overline{H_0} \cup \overline{H_1}$. Indeed, if $z \in \overline{G}$ and W is any open neighbourhood W of z then there is a $k < \omega$ such that $W \cap G_k \neq \emptyset$. Clearly, if n is the least such k then we have $\emptyset \neq W \cap U_n$ as well, and so $x \in \overline{H_0 \cup H_1}$.

Consequently, $p \in \overline{H}_0$ or $p \in \overline{H}_1$; assume e. g. that $p \in \overline{H}_0$. We shall show that then $\overline{H}_0$ is not locally connected at p, although it is a regular closed set, arriving at a contradiction.

Indeed, let U be any neighbourhood of p in $\overline{H_0}$ and fix an even number $n < \omega$ with $U_n \cap U = \emptyset$. Then $U \cap U_{n+1} \subset \overline{H_0} \cap H_1 = \emptyset$ implies $U \subset \overline{G_n} \cup (X \setminus G_{n+1})$, where $\overline{G_n}$ and $X \setminus G_{n+1}$ are disjoint closed sets both meeting U, hence U is disconnected. □

With a little more effort it can also be shown that for any non-isolated point p in a T_3 space X there is a regular closed set H in X with $p \in H$ such that p is not a local connectivity point in H.

We do not know if every T_2 space X with $Pr(X, T_1)$ has to be discrete. Also, the following T_2 version of Theorem 5.5 seems to be open: *If X is T_2 and all closed subspaces of X are locally connected then X has to be discrete.* Note that if X has the cofinite topology then it is hereditarily locally connected and satisfies $Pr(X, T_1)$.

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