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ON FINITE T_0 TOPOLOGICAL SPACES

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ABSTRACT. Finite topological spaces became much more essential in topology, with the development of computer science. The task of this paper is to study and investigate some properties of such spaces with the existence of an ordered relation between their minimal neighborhoods. We introduce notations and elementary facts known as Alexandroff space [4, 3, 5]. The family of minimal neighborhoods forms a unique minimal base. We consider T_0 spaces. We give a link between finite T_0 spaces and the related partial order. Finally, we study some properties of multifunctions and their relationships with connected ordered topological spaces.

1. Introduction and Preliminaries

Finite spaces were first studied by P.A. Alexandroff in 1937 in [1]. Actually, finite spaces had been more earlier investigated by many authors under the name of simplicial complexes. There were several other contributions by Flachsmeyer in 1961 [12], Stong in 1966 [22] and L. Lotz in 1970 [19]. Rinow [21] in his book discussed some properties of finite spaces. However, the subject has never been considered as a main field of topology.

With the progress of computer technology, finite spaces have become more important. Herman in 1990 [13], Khalimsky and et. al. in 1990 [14], Kong and Kopperman in 1991 [15] have been applied them to model the computer screen. In this paper we focus on finite spaces with order. The main importance of our study is to offer a new formulations for some topological operators in general topology such as interior, closure, boundary and exterior operators. We present and study comparisons between some topological properties in the case of finite spaces. In what follows, by X we mean always

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mean a finite T_0 space. For each $A \subset X$, the closure (resp. interior, exterior, boundary) of A will be denoted by $\overline{A}$ (resp. int (A), ext (A), ∂A).

For each point x in a space X, there is a smallest neighborhood which is contained in each other neighborhood of x. For each $x \in X$, let

$$U_x = \bigcap \{V : V \text{ is an open set containing } x\}$$

Clearly U_x is the smallest open set containing x since X is finite.

Alexandroff spaces are the topological spaces in which each element is contained in a smallest open set or equivalently the spaces where arbitrary intersections of open sets are open. It is clear that all finite spaces are locally finite and all locally finite spaces are Alexandroff.

Lemma 1.1. *The class $\mathcal{U} = \{U_x : x \in X\}$ is a base for a finite space (X, τ). Each base of τ contains $\mathcal{U}$.*

Notice that if X is Alexandroff, then X is T_1 if and only if $U_x = \{x\}$. It follows that X is discrete and so every point is an isolated point.

Remark 1.2. Observe that if x and y are two points in a space X, then $y \in U_x$ if and only if $U_y \subseteq U_x$.

Definition 1.3. [2] For two points $x, y \in X$, $y \geq x$ if $U_y \subseteq U_x$.

Remark 1.4. From Definition 1.3, the relation $\geq$ is reflexive and transitive since $\subseteq$ is so.

Proposition 1.5. *In a space X, $y \geq x$ if and only if $x \in \overline{\{y\}}$. In such case, x is said to be identified with y.*

Proof. Let $y \geq x$ and $y \neq x$. Then $y \in U_x$ which is the smallest open set containing x. Then for any open set G containing x, we have $(G \setminus \{x\}) \cap \{y\} \neq \phi$. This means x is an accumulation point of y. Therefore $x \in \overline{\{y\}}$. Conversely, let $x \in \overline{\{y\}}$. Then $G \cap \{y\} \neq \phi$ for every open sets G containing x. So $y \in G$ for every open set G. Take $G = U_x$. By Remark 1.2, we get $U_y \subseteq U_x$. This shows that $y \geq x$. □

The following example will be used throughout the paper.

Example 1.6 (The class of topologies on a set with three points)**.** We give a list of all topologies on the set $X = \{x, y, z\}$ up to homeomorphic topologies. There are 9 topologies, with the following U_x, U_y and U_z:

- τ_1 All three points are isolated. This is the discrete topology and $U_x = \{x\}$, $U_y = \{y\}$, $U_z = \{z\}$.

Now, we assume that the topology has only two isolated points , say, $U_x = \{x\}$ and $U_y = \{y\}$. We have two cases concerning U_z:

- τ_2 The neighborhood of U_z has two points. We can assume $U_z = \{x, z\}$.
- τ_3 The neighborhood of U_z has three points, $U_z = X$.

Next, we consider the case that the topology has only one isolated point, say, $U_x = \{x\}$. Again, we have to distinguish between different cases concerning U_y and U_z:

τ_4 Both U_y and U_z have three points. Then $U_y = U_z = X$.

τ_5 One neighborhood, say U_z, has three points and U_y has two points. Then $U_y = \{y, z\}$ is not possible, so $U_y = \{x, y\}$.

τ_6 Both neighborhoods have two points and are equal $U_y = U_z = \{y, z\}$.

τ_7 U_x and U_y have two points and are different. Then they can not be $\{y, z\}$. Thus $U_y = \{x, y\}$ and $U_z = \{x, z\}$.

Finally, we have two cases without isolated points:

τ_8 There is a neighborhood with two points, say, $U_x = \{x, y\}$. Then the neighborhood of z must have three points. Thus $U_x = U_y = \{x, y\}$ and $U_z = X$.

τ_9 All neighborhoods have three points, then $U_x = U_y = U_z = X$.

Figure 1 shows this class of topologies with its minimal neighborhoods. The smallest ellipse refers to a singleton, the middle is a two points neighborhood and the biggest one is the whole space X.

2. T_0 Properties and Associated Partial Order

In this section, we investigate some properties of T_0 spaces such that for two distinct points x, y in X, $x \geq y$ and $y \geq x$ only true if $U_x = U_y$. If $\mathcal{U}(x)$ denote the neighborhood system of x. Recall that a space is T_0 if and only if $\mathcal{U}(x) \neq \mathcal{U}(y)$ for $x \neq y$.

Remark 2.1. The topologies τ_i for $i \in \{1, 2, 3, 5, 7\}$, in Example 1.6, are T_0. Any two elements in Figure 1 with the same neighborhood is not T_0 such as τ_j for $j \in \{4, 6, 8, 9\}$.

Definition 2.2. For points x, y in a topological space X, $y \geq x$ if $\mathcal{U}(y) \supseteq \mathcal{U}(x)$. Notice that $\supseteq$ means more numbers of neighborhoods.

Remark 2.3. The relation $\geq$ is reflexive and transitive, since this is true for $\supseteq$. In a T_0 space, it is also antisymmetric. Then $\geq$ is a partial order in T_0 spaces.

Definition 2.4. Let $(X, \leq)$ be a partially ordered set and $U \subseteq X$. We say that U is open if whenever $x \in U$ and $y \geq x$ it is also the case that $y \in U$.

In the following result we denote by the set of open subsets of X by $\mathcal{U}_{\leq}$.

Proposition 2.5. *If $(X, \leq)$ is a partially ordered set, then $(X, \mathcal{U}_{\leq})$ is a finite T_0 space.*

Proof. Clearly, X and ϕ are elements in $\mathcal{U}_{\leq}$. Let $U_i \in \mathcal{U}_{\leq}$ for every $i \in I$. For any $x \in \bigcup_{i \in I} U_i$ and $y \geq x$, there is $i_0 \in I$ such that $x \in U_{i_0}$. By openness of U_{i_0}, we have $y \in U_{i_0}$ and $y \in \bigcup_{i \in I} U_i$. Therefore $\bigcup_{i \in I} U_i \in \mathcal{U}_{\leq}$. Also, if A and

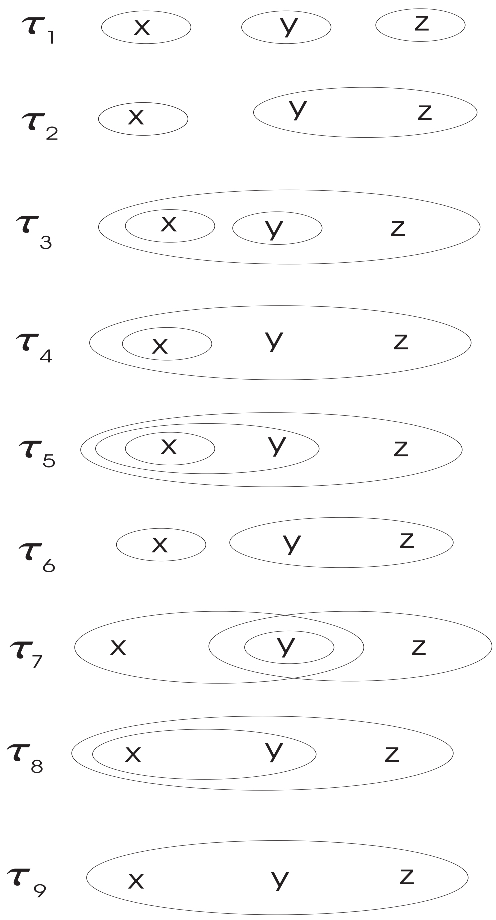

FIGURE 1. Topological spaces with minimal neighborhoods.

B are elements in $\mathcal{U}_{\leq}$, then $A \cap B \in \mathcal{U}_{\leq}$. To show T_0, consider two distinct elements $x, y \in X$. Clearly, $x \in U_x$ and $y \in U_y$. If $y \notin U_x$, the proof is complete. If $y \in U_x$, $y \geq x$, by the antisymmetry of $\leq$, $x \ngeq y$ and so $x \notin U_y$ which also completes the proof. □

Remark 2.6. The closure of any singleton p of a finite T_0 space $(X, \mathcal{U}_{\leq})$ has the form $\overline{\{p\}} = \{x : x \leq p\}$.

Remark 2.7. The order relation $\leq$ is not useful in T_1 spaces, because there are no comparable elements. In other words, since for every two distinct points x and y in a T_1 space, $\overline{\{x\}} \neq \overline{\{y\}}$, then the relation $x \leq y$ is never satisfied. Thus $\leq$ is a good tool only for T_0 spaces which are not T_1.

Proposition 2.8. *A finite T_0 space contains an isolated point.*

Proof. Obvious. □

Remark 2.9. Finite spaces with isolated points need not be T_0. Each of τ_4 and τ_6 in figure 1 of example 1.6 has an isolated points, but not T_0.

Proposition 2.10. *Every open set in a finite T_0 space contains an isolated point.*

Proof. Let G be an arbitrary open set in X. Then G is a finite open T_0 subspace. By Proposition 2.8, there is an isolated point x in G. Since G is open in X, then x must be isolated point in X. □

3. Some Topological Properties

In this paper, some of topological operators which are well known for topologists have new forms with respect to the Alexandroff's notion of order.

Proposition 3.1. *In any finite space, the open (closed) points are the maximal (minimal) elements. Therefore U is an open set if it contains its upper bounds. By the complement, F is closed if it contains its lower bounds.*

Proof. By Definition 1.3 and Proposition 1.5, the partial order $x \leq y$ for points x, y in a finite T_0 space X was defined by $y \in U_x$ or $x \in \overline{\{y\}}$. This means that for open points $\{x\} = U_x$, there is no strictly larger y. Thus the open points are the maximal elements in X. Similarly, for a closed point y, if $U_x = \{x\}$, then $x \leq y$ implies $y = x$. Hence the closed points are the minimal elements. □

Proposition 3.2. *In a finite topological space X, the interior of a subset $A \subset X$, int A, is the set of all points $a \in A$ such that for every $b \geq a$ it is also true that $b \in A$. In other words, the elements of int A are all upper bounds of X belonging to A.*

Proof. We prove that $\text{int } A = \{a \in A : b \geq a \text{ implies } b \in A\}$. For "$\subseteq$", let $a \in \text{int}(A)$ that is $U_a \subseteq A$ for $\text{int } A$ is the greatest open set contained in A. But $U_a = \{b : b \geq a\}$. For "$\supseteq$", if $b \in A$ for all $b \geq a$. That means $U_a \subseteq A$, then $a \in \text{int } A$. □

Proposition 3.3. *For a subset A in a finite topological space X, the closure of A, $\overline{A}$, is the set of points $b \in X$ such that $b \leq a$ for some $a \in A$. In other words, $\overline{A}$ is the set of all lower bounds of points of A.*

Proof. In a finite topological space X, $\overline{A} = \bigcup_{a \in A} \overline{\{a\}}$. By the definition of $\leq$, we have $\overline{\{a\}} = \{b : b \leq a\}$ and the claim. □

Proposition 3.4. *For a finite topological space X, the closure of an interior of a subset A of X, $\overline{\text{int } A}$, is the set of all x in X such that there exists a maximal element $a \in A$ with $a \geq x$.*

Proof. By Proposition 3.3, $\overline{\text{int } A} = \bigcup\{b : b \leq a \text{ for some } a \in \text{ int } A\}$. Since $a \in \text{int } A$, then all $c \geq a$ are also in A. Therefore $\overline{\text{int } A} = \bigcup\{b : b \leq a \text{ for } a \in A\}$. Since we have a finite ordered set, there is a maximal $c \geq a$. Hence $\overline{\text{int } A}$ is the lower bounds of maximal elements in A. □

Proposition 3.5. *In a finite topological space X, the interior of a closure of $A \subset X$, int $\overline{A}$, is the points $x \in X$ such that all maximal elements $y \geq x$ belong to A.*

Proof. From 3.2 and 3.3, we have

$$\begin{aligned} \text{int } \overline{A} &= \{a \in \overline{A} : b \geq a \text{ implies } b \in \overline{A}\} \\ &= \{a \in \overline{A} : b \geq a \text{ implies that there is } c \in A, c \geq b\} \\ &= \{a \in \overline{A} : \text{ all maximal elements } b \geq a \text{ must belong to } A\} \end{aligned}$$

To prove the equality, if b is maximal, then $b \in A$ since there is no other c in A. Hence $b \in \text{int}\overline{A}$. On the other hand, if all elements in A are maximal and $b \geq a$, then there is a maximal $c \geq b$ with $c \in A$. $\square$

In finite topological spaces, From Propositions 3.4 and 3.5, we have $\text{int}\overline{A} \subseteq \overline{\text{int } A}$. The following example shows that the inverse inclusion is not always true.

Example 3.6. Consider the topology τ_3 in Example 1.6. If $A = \{y, z\}$, we get $\text{int } \overline{A} = \{y\}$ and $\overline{\text{int } A} = \{y, z\}$. Therefore $\text{int}\overline{A} \not\supseteq \overline{\text{int } A}$.

In 1961, Levine [16] introduced the notion of semi-open set in any topological space. A subset A in a topological space X is called semi-open if and only if $A \subset \overline{\text{int } A}$. In 1982, Mashhour et. al. [20] defined the concept of preopen set. A subset A is preopen in X if and only if $A \subset \text{int } \overline{A}$. There are no implications between these two concepts. This means that semiopen sets need not be preopen and conversely. In 1997, Abd El-Monsef et. al. [9] defined the concept of γ-open set as a union of semi-open and preopen sets which is equivalent to $A \subset \text{int } \overline{A} \cup \overline{\text{int } A}$, for any subset A of X. From the relation $\text{int}\overline{A} \subseteq \overline{\text{int } A}$, we obtain a new implication between types of near openness in finite topological spaces.

Remark 3.7. In a finite topological space X, the following are true:

(i) Every preopen set in X is semi-open. The converse may not be true as it is shown in Example 3.6
(ii) Semiopen sets coincide with γ-open sets. The following diagram shows the relation between these notions in a finite case.

$$\begin{array}{ccc} \textbf{openness} & \Longrightarrow & \textbf{preopenness} \\ & & \Downarrow \\ \gamma\textbf{-openness} & \Longleftrightarrow & \textbf{semi-openness} \end{array}$$

The axiom T_0 is necessary for satisfying the previous implications. The following examples show this fact.

Example 3.8. Any subset of an indiscrete space is preopen but not semi-open, since the indiscrete spaces are not T_0.

Proposition 3.9. *In a finite space X, the boundary of a subset A of X is the set $b \in X$ such that $b \leq a$ for some $a \in A$ and $b \leq c$ for some $c \notin A$.*

Proof. It follows from the definition, $\partial A = \overline{A} \cap \overline{X \setminus A}$, and Proposition 3.3. □

Proposition 3.10. *In a finite space X, the exterior of $A \subset X$ is the set of points $x \in X$ such that for $b \geq x$ implies $b \notin A$.*

Proof. Since for any topological space ext (A) $= X \setminus \overline{A} =$ int $(X \setminus A)$. By Proposition 3.2, we have

$$\begin{aligned} \text{ext (A)} &= \{x \in X : b \geq x \text{ implies } b \in X \setminus A\} \\ &= \{x \in X : b \geq x \text{ implies } b \notin A\} \end{aligned}$$

□

4. The Dimension for Finite Spaces

The notion of a topological dimension has a sense for finite spaces. Although we have only a finite number of points, all finite dimensions are possible. We use the notion of inductive dimension as stated by Engelking ([11], Chapter 7). A discrete space X has dimension zero denoted by $\dim X = 1$, since the neighborhoods have no boundaries. On the real line R, the open intervals form a basis. The boundary of an interval consist of two points which is a discrete space. Analogously, the plane has dimension ≤ 2 since a circumference has dimension ≤ 1. This is the typical situation in three-dimensional spaces. This leads to the following definition.

Definition 4.1. ([7], p.195) Let X be a topological space. $\dim X = -1$ if and only if $X = \phi$. Let n be a positive integer and $\dim X \leq k$ be defined for each $k \leq n-1$. Then $\dim X \leq n$ if X has a base β such that $\dim \partial B \leq n-1$ for all $B \in \beta$.

A base β of any finite space will be replaced by the set of all minimal neighborhoods for each of its points.

Proposition 4.2. *For a finite space X, $\dim X \leq n$ if and only if the minimal base $\mathcal{U}$ fulfils $\dim \partial U_x \leq n-1$ for each $U_x \in \mathcal{U}$.*

Proof. If $\dim X \leq n$, then by definition 4.1 it has a base β such that $\dim \partial B \leq n-1$ for all $B \in \beta$. By Lemma 1.1, each β contains $\mathcal{U}$. Then $\dim \partial U_x \leq n-1$. The converse is obvious. □

Proposition 4.3. *A topological space X has dimension zero if and only if X has a base of clopen sets.*

Proof. It is clear that $\partial B = \phi$ if and only if B is clopen. Dimension zero means existence of a base with $\partial B = \phi$. This condition is equivalent to B being clopen. Since ∂B is containing those x which are accumulation points of B and its complement. For clopen sets B, we have $\partial B = \phi$. On the other hand, if $\partial B = \partial(X \setminus B) = \phi$, then B and $X \setminus B$ are closed and so B is clopen. □

Proposition 4.4. *Every finite T_0 space with base β of clopen sets is discrete.*

Proof. Since every zero dimensional T_0 space is T_1. then it is closed as a consequence of the finiteness of the space. □

Finiteness and T_0 axiom are necessary. It can be satisfied from the following examples.

Example 4.5. The Cantor set $\{0,1\}^\infty$ with product topology has a base of clopen sets, but this space is not discrete.

Example 4.6. In Example 1.6, the topology τ_6 has a base $\{\{x\},\{y,z\}\}$ with clopen sets. X is not discrete, because X is not T_0.

Lemma 4.7. *Let $X = C \cup V$ such that every $c \in C$ is closed and $v \in V$ is open. Then each of C and V is a discrete subspace of X.*

Proof. Since C is a finite closed subspace of X, then each $c \in C$ is a closed point in C. Then $C \setminus \{c\}$ is a finite closed subset in C. So $\{c\}$ is an open point in C. This means that C is a discrete set. It is clear that V is discrete. □

The following Proposition describes the finite one-dimensional T_0 space.

Theorem 4.8. *Let X be a finite T_0 space. Then $\dim X \leq 1$ if and only if every singleton in X is either open or closed.*

Proof. Let X be a finite T_0 space. By Proposition 2.8, X has an open point say x_0 and $U_{x_0} = \{x_0\}$. Since $\dim X \leq 1$, then $\dim \partial(\{x_0\}) = 0$ and so $\partial(\{x_0\})$ is discrete. This means that each $y_0 \in \partial(\{x_0\})$ is closed in $\partial(\{x_0\})$. Since $\partial(\{x_0\})$ is closed in X, then $\{y_0\}$ is also closed in X. Take $X^{'} = X \setminus \mathrm{cl}(\{x_0\})$ which is an open finite T_0 subspace of X. By Proposition 2.10, $X^{'}$ has an open point set x_1 and $U_{x_1} = \{x_1\}$ which is also open in X. Also $\dim \partial(\{x_1\}) = 0$, then $\partial(\{x_1\})$ is discrete. So, each $y_1 \in \partial(\{x_1\})$ is closed in $\partial(\{x_1\})$. Then $\{y_1\}$ is closed in X. Take $X^{''} = X \setminus \mathrm{cl}(\{x_0, x_1\})$. By continuing, the proof is completed in one direction. Conversely, Suppose that singletons are open or closed. By Lemma 4.7, we have in either cases discrete subspaces of X. Therefore the dimension of each subspace is $X \leq 1$. Hence $\dim X \leq 1$. □

Definition 4.9. The height of a partially ordered set $(X, \leq)$ is the degree of a longest element of the increasing sequence $x_1 < x_2 < .... < x_n$ of elements of X.

Example 4.10. The discrete space has a height 1, since there are no comparable elements i.e. no $x < y$. The space in which each point is either open or closed of height 2, for $x < y$ is satisfied only for a closed point x and an open point y.

Theorem 4.8 can be reformulate as follows: The dimension of a finite space X is the height of a partially ordered set $(X, \leq)$ minus 1.

Remark 4.11. Recall that a space (X, τ) is called $T_{\frac{1}{2}}$ if every generalized closed subset of X is closed [18] or equivalently if every singleton is either open or closed [8].
Observe that by Theorem 4.8, every finite $T_{\frac{1}{2}}$ space is a one dimensional T_0. It is also clear that a space X is $T_{\frac{1}{2}}$ if its non-closed singletons are isolated.

5. Density in Finite Spaces

In this section, we introduce some characterizations of density in finite spaces.

Proposition 5.1. *In finite T_0 spaces, the set of all open points is dense.*

Proof. Let X be a finite T_0. By Proposition 2.8, X contains an isolated point. If $O = \{x : \{x\} \text{ is open }\}$. Then by Proposition 2.10, every open set G containing x must intersect O. This means that $x \in \overline{O}$. Hence O is dense. □

Remark 5.2. The condition T_0 is necessary in Proposition 5.1. The singleton $\{x\}$ in τ_6 of example 1.6 is open but not dense.

Proposition 5.3. *A subset A is dense in a finite T_0 space X if and only if it contains all open points of X.*

Proof. Let $O = \{x : \{x\} = U_x\}$ be the set of open points in X. If A is a dense subset of X, by density, then A must intersect U_x. So, it must contain x. The converse case follows readily from Proposition 5.1. □

Corollary 5.4. *In finite T_0 spaces, each dense set contains an open dense set.*

Proof. Obvious from Proposition 5.1 and Proposition 5.3. □

In Corollary 5.4, The condition T_0 is necessary.

Example 5.5. Consider the topology τ_6 in Example 1.6. A subset $\{a, c\}$ is dense. But it does not contain an open dense subset.

Some topological concepts induced by density. Recall that A is said to be codense (resp. nowhere dense, dense-in-itself) if $\text{int}\,(A) = \phi$ (resp. $\text{int}\,\overline{A} = \phi$, $A = \text{d}(A)$), where $\text{d}(A)$ denotes the set of accumulation points of A. Dense-in-itself of any $A \subseteq X$, equivalently that A does not have any isolated points. We state these notions in the finite spaces.

Proposition 5.6. *For arbitrary finite topological space X, the following are true for any subset A of X,*

(i) *It is co-dense if there is no upper bounds belongs to A.*
(ii) *It is nowhere dense if there is no element has a maximal element in A.*
(iii) *It is dense in-itself if A contains all of its lower bounds.*

Proof. (i) and (ii) are obvious from Proposition 3.2 and Proposition 3.5 respectively.
(iii) Clear by Proposition 3.3 and using the equality $A = \overline{A}$. □

Recall that a topological space (X, τ) is said to be a submaximal if each of its dense subsets is open.

Proposition 5.7. *Any finite T_0 space X is submaximal if and only if it contains at most a non-isolated point.*

Proof. The set of open points is dense by Proposition 5.1. Also by the submaximality of X, every dense subset is open. Then for every points $x, y, z \in X$ such that $x \leq y \leq z$, $X \setminus \{y\}$ is not open. Therefore y is not isolated. Conversely, if all the points of X are isolated, then the only dense set is X itself. This means that X is submaximal. Otherwise, if X contains only non-isolated point y. Let A be any dense subset of X. By Proposition 5.3, A contains all isolated points of X. □

Corollary 5.8. *Every finite T_0 submaximal spaces is $T_{\frac{1}{2}}$. It is also one-dimensional space.*

Proof. It is a consequence Theorem 4.8. □

6. Some Weaker Forms of Continuity in Finite Spaces

Continuous functions play an important role in topology. The following proposition shows a corresponding definition of continuity between finite spaces using the minimal neighborhood of each point and its image.

Proposition 6.1. *Let X and Y be finite topological spaces, then the function $f : X \longrightarrow Y$ is continuous at x if and only if $f(U_x) \subseteq U_{f(x)}$.*

Proof. Let $f : X \longrightarrow Y$ be a continuous function. Fix a point $x \in X$. Since $U_{f(x)}$ is an open neighborhood of $f(x)$, then by the continuity of f, $f^{-1}(U_{f(x)})$ is an open neighborhood of x and so $f(U_x) \subseteq U_{f(x)}$. Conversely, let W be an open set containing $f(x)$. By assumption, $f(U_x) \subseteq U_{f(x)} \subseteq W$. Take $U_x = U$, then $f(U) \subseteq W$ which shows that f is continuous at x. □

Recall that a function $f : X \longrightarrow Y$ from a topological space X into a topological space Y is precontinuous [20] (resp. semicontinuous [17], γ-continuous [9]) if the inverse image of each open set in Y is preopen (resp. semiopen, γ-open) in X.

In an arbitrary topological space, there is no connection between precontinuity and semicontinuity. Each of them implies γ-continuity. In the finite case, by Remark 3.7, every precontinuous function is semicontinuous which coincides with γ-continuous. The following implications show a new connection between continuity and some kinds of near continuity.

$$\begin{array}{ccc} \textbf{Continuity} & \Longrightarrow & \textbf{Precontinuity} \\ & & \Downarrow \\ \gamma\textbf{-continuity} & \Longleftrightarrow & \textbf{Semicontinuity} \end{array}$$

The following example shows that the converse is not always true.

Example 6.2. Let $X = Y = \{x, y, z\}$ as in Example 1.6. The mapping $f : (X, \tau_3) \longrightarrow (Y, \tau_5)$ which defined by $f(x) = x$, $f(y) = z$ and $f(z) = y$ is γ-continuous, but not precontinuous.

A function $f : X \longrightarrow Y$ from a topological space X into a topological space Y is called preopen [20] (resp. semiopen[6], γ-open[9]) if the image of each open set in X is preopen (resp. semiopen, γ-open).

In arbitrary topological space, there is no connection between preopen and semiopen functions. In a finite case, by Remark 3.7, a preopen function is semiopen which coincides with γ-open. Each of them belongs to γ-open function. The implications between these types of functions and other corresponding ones are given by the following diagram:

$$\begin{array}{ccc} \textbf{Open function} & \Longrightarrow & \textbf{Preopen function} \\ & & \Downarrow \\ \gamma\textbf{-open function} & \Longleftrightarrow & \textbf{Semiopen function} \end{array}$$

The converse of these implications are not true, in general, as the following example illustrates.

Example 6.3. Let $X = \{x, y, z\}$ with a topology τ_5 as in Example 1.6 and $Y = \{a, b, c, d\}$ with the topology of minimal neighborhoods $U_a = \{a\}$, $U_b = \{b\}$, $U_c = U_d = Y$. The function $f : X \longrightarrow Y$ is γ-open but not preopen.

7. Continuity of Multifunctions in Finite Topological Spaces

Let X and Y be two nonempty sets and $P(Y)$ be the collection of all subsets of Y. Recall that the function $F : X \longrightarrow P(Y)$ is called a multifunction ([7], p.186).

Definition 7.1. A function $F : X \longrightarrow P(Y)$ from a topological space X into a topological space Y is called:

(1) *upper semicontinuous mulitifunction* (abb.U.S.C) at a point $x_0 \in X$ if for every open set V in Y such that $F(x_0) \subset V$, there exists an open set U containing x_0 such that $F(U) \subset V$.

(2) *lower semicontinuous multifunction* (abb.L.S.C)at a point $x_0 \in X$ if for every open set V in Y such that $F(x_0) \cap V \neq \phi$, there exists an open set containing x_0 such that $F(x) \cap V \neq \phi$ for each $x \in U$.

Remark 7.2. U.S.C is not necessarily L.S.C, and L.S.C is not necessarily U.S.C.

Now, the following theorems and examples give the relation beteen the usual definition of continuity for single valued functions and upper and lower semicontinuity for multifunctions.

Definition 7.3. [14] A space X is said to be a connected ordered topological space (abb. COTS) if for every three points subset Y in X, there exists $y \in Y$ such that Y meets two connected components of $X \setminus \{y\}$. In other words, for any three points one of them separates the other two.

Remark 7.4. Let $X = [0,1]$ with usual topology and f be a continuous function from X into itself. $\pi : X \longrightarrow Y$ is a quotient function from X into a finite COTS Y. The function g from Y into itself is defined by $g(y) = \pi f \pi^{-1}\{y\}$ is continuous. The following examples discuss the continuity of g in the case of single valued functions.

Example 7.5. Let $X = [0,1]$ and $f : X \longrightarrow X$ be a continuous function defined by

$$f(x) = \begin{cases} x + 1/2 & : \quad 0 \leq x \leq 1/2 \\ 1 & : \quad 1/2 \leq x \leq 1 \end{cases}$$

and $Y = \{a, b, c, d, e\}$ is a COTS. π is a quotient function defined by

$$\pi(x) = \begin{cases} a & : \quad x = 0 \\ b & : \quad 0 < x < 1/2 \\ c & : \quad x = 1/2 \\ d & : \quad 1/2 < x < 1 \\ e & : \quad x = 1 \end{cases}$$

Since $g(y) = \pi f \pi^{-1}(y)$ for each $y \in Y$, so $g(a) = c$, $g(b) = d$, $g(c) = e$, $g(d) = e$, $g(e) = e$. Then g is continuous at each point $y \in Y$.

The following theorem is considered as an equivalent definition for U.S.C and L.S.C in finite topological spaces with respect to the minimal neighborhood of each point.

Theorem 7.6. *For finite topological spaces X and Y and multifunction $F : X \longrightarrow P(Y)$, we have:*

(i) *F is U.S.C at x if and only if $F(\mathcal{U}_x) \subset \bigcup_{y \in F(x)} \mathcal{U}_y$.*

(ii) *F is L.S.C at x if and only if $F(x') \cap \mathcal{U}_y \neq \phi$ for all $x' \in \mathcal{U}_x$ and $y \in F(x)$.*

Proof. (i). Let F be U.S.C at $x \in X$. Then for every open set Vsuch that $F(x) \subset V$. There exits an open set U with $x \in U$ such that $F(U) \subset V$. Since $x \in U$, then $\mathcal{U}_x \subset \mathcal{U}$. By assumption, we obtain $F(\mathcal{U}_x) \subset V$. Take $V = \bigcup_{y \in F(x)} \mathcal{U}_y$, then $F(\mathcal{U}_x) \subset \bigcup_{y \in F(x)} \mathcal{U}_y$. Conversely, let $F(\mathcal{U}_x) \subset \bigcup_{y \in F(x)} \mathcal{U}_y$. For $x \in \mathcal{U}_x$, take any $V \supset F(x)$. As $\mathcal{U}$, we choose $\mathcal{U}_x$, then $F(\mathcal{U}_x) \subset \bigcup_{y \in F(x)} \mathcal{U}_y \subset V$. Therefore $F(\mathcal{U}_x) \subset V$ shows that F is U.S.C.

(ii). Let F be L.S.C at $x \in X$. That means for every open set V such that $F(x) \cap V \neq \phi$, there is an open set U with $x \in U$ such that $F(x') \cap V \neq \phi$ for all $x' \in U$. Applying for a special V, take $V = \mathcal{U}_y$ and $y \in F(x)$. Since $\mathcal{U}_x \subset \mathcal{U}$, then $F(x') \cap \mathcal{U}_y \neq \phi$ for all $x' \in \mathcal{U}_x$ and $y \in F(x)$. Conversely, let V be an arbitrary open set with $F(x) \cap V \neq \phi$. Take $y \in F(x) \cap V$, then $\mathcal{U}y \subset V$. As $\mathcal{U}$, we choose $\mathcal{U}_x$ and by assumption $\phi \neq F(x') \cap \mathcal{U}_y \subset F(x') \cap V$. Then $F(x') \cap V \neq \phi$ for all $x' \in \mathcal{U}_x$ and F is L.S.C. □

U.S.C may not be L.S.C. It can be satisfied from the following example.

Example 7.7. Let $X = Y = \{a, b\}$. Their minimal neighborhoods are $\mathcal{U}_a = \{a\}$ and $\mathcal{U}_b = \{a, b\}$. The multifunction $F : X \longrightarrow P(Y)$ is defined by $F(a) = \{a, b\}, F(b) = \{a\}$. Then F is L.S.C. It is not U.S.C since we have $F(\mathcal{U}_b) = \{a, b\} \not\subset \bigcup\limits_{y \in F(b)} \mathcal{U}_y = \{a\}$ by Theorem 7.6(i), $\mathcal{U}_b = \{a, b\}, F(b) = \{a\}$. Therefore F is not U.S.C. If the multifunction $G : X \longrightarrow P(Y)$ is defined by $G(a) = \{a\}, G(b) = \{a, b\}$, then G is U.S.C. It is not L.S.C at the point b. Since by Theorem 7.6(ii), we have $\mathcal{U}_b = \{a, b\}, F(b) = \{a, b\}$ and $F(a) \cap \mathcal{U}_a = \{b\} \cap \{a\} = \phi$.

The following examples discuss the upper and lower semi-continuity of g.

Example 7.8. Let $X = [0, 1]$ with usual topology. The function f from X into itself defined by

$$f(x) = \begin{cases} 3/4 - 2x & : \quad 0 \leq x < 1/4 \\ 1/3x + 1/6 & : \quad 1/4 \leq x \leq 1 \end{cases}$$

and the quotient function $\pi : X \longrightarrow Y$, where Y is a finite COTS with five points defined by

$$\pi(x) = \begin{cases} a & : \quad x = 0 \\ b & : \quad x = 0 < x < 1/2 \\ c & : \quad x = 1/2 \\ d & : \quad 1/2 < x < 1 \\ e & : \quad x = 1 \end{cases}$$

The multifunction $g : Y \longrightarrow P(Y)$ is defined by $g(y) = \pi f \pi^{-1} y$. Then $g(a) = d, g(b) = \{b, c, d\}, g(c) = b, g(d) = b, g(e) = c$. g is L.S.C, but not U.S.C since the point $y = a, \mathcal{U}_a = \{a, b\}$, $g(a) = \{d\}$ and $g(\mathcal{U}_a) = \{b, c, d\} \not\subset \bigcup\limits_{y \in g(a)} \mathcal{U}_y) = \{d\}$.

Now, we consider some properties of U.S.C and L.S.C for multifunctions. These make a connection between multifunctions and dimension theory.

Theorem 7.9. *Let $\pi : X \longrightarrow Y$ be a continuous function from a topological space X into a one-dimensional T_0 space Y. For any closed point in Y, $\pi^{-1}(y)$ is only one point and for any open point $z \in \mathcal{U}_y$, $\pi^{-1}(y) \in cl(\pi^{-1}(z))$, then π is an open function.*

Proof. Let W be an open set in X. Our aim is to prove that $\pi(W) = \bigcup_{y\in\pi(W)} \mathcal{U}_y$. It is clear that $\pi(W) \subset \bigcup_{y\in\pi(W)} \mathcal{U}_y$. So, it is enough to prove that $\mathcal{U}_y \subset \pi(W)$ for each $y \in \pi(W)$. There are two cases: If y is an open point, then $\mathcal{U}_y = \{y\} \subset \pi(W)$ and we are done. If y is a closed point, by the assumption, $\pi^{-1}(y) = x$. Let z be an open point such that $z \in \mathcal{U}(y)$, by hypothesis, $x \in cl(\pi^{-1}(z))$. Since W is an open neighborhood of x, then $\pi^{-1}(z)\cap W \neq \phi$. For a point $v \in \pi^{-1}(z)\cap W \neq \phi$, we have $z = \pi(v) \in \pi(W)$. Therefore, $\mathcal{U}_y \subset \pi(W)$. This completes the proof. □

Under the previous assumptions of Theorem 7.9, π is not a closed function.

Example 7.10. Let $X = [0, 1]$ with usual topology and Y is a COTS with three points. The function $\pi : X \longrightarrow Y$ is defined by

$$\pi(x) = \begin{cases} a & : \quad x = 0 \\ b & : \quad 0 < x < 1 \\ c & : \quad x = 1 \end{cases}$$

The assumption of Theorem 7.9 hold. π is not closed because if we take a closed set $F \subset (0, 1)$, then $\pi(F) = \{b\}$ is not closed.

Proposition 7.11. *For a topological space X and one-dimensional T_0-space Y, consider a continuous function f and a quotient a function π from X into Y. If the multifunction $g : Y \longrightarrow P(Y)$ is defined by $g = \pi f\pi^{-1}$, then $g(y)$ is a singleton for each closed point $y \in Y$.*

Proof. Obvious, since f and π are single -valued functions. □

Theorem 7.12. *For a space X and one-dimensional T_0-space Y, assume the following assumptions hold.*

(1) *The function $\pi : X \longrightarrow Y$ is continuous.*
(2) *For every closed point y, $\pi^{-1}(y)$ is one point.*
(3) *For every open point $z \in \mathcal{U}_y$, $\pi^{-1}(z) = x$.*
(4) *A multifunction $g : Y \longrightarrow P(Y)$ defined by $g = \pi f\pi^{-1}$.*

Then g is L.S.C.

Proof. Let $y_0 \in Y$ and V be an open set in Y such that $g(y_0) \cap V \neq \phi$. If y_0 is an open point, then we can take $\mathcal{U}_{y_0} = \{y_0\}$ and so g is L.S.C. If y_0 is a closed point, by assumption, for any open point $y \in \mathcal{U}_{y_0}$, $\pi^{-1}(y_0) = x \in cl(\pi^{-1}(y))$. By Proposition 30, $g(y_0)$ is a singleton and so $f^{-1}\pi^{-1}(V)$ is an open neighborhood of x. Therefore $\pi{-}1(V) \cap W \neq \phi$ for all $y \in \mathcal{U}_{y_0}$ and $W = f^{-1}\pi^{-1}(V)$. This shows that g is L.S.C. □

Under the conditions of Theorem 7.12, the multifunction is not necessarily to be U.S.C.

Example 7.13. Let X and Y and the quotient function as in Example 7.8. The function f is defined by

$$f(x) = \begin{cases} 2x - 1 & : \quad 0 < x < 1/2 \\ 1 - 2x & : \quad 1/2 < x < 1 \\ 1 & : \quad x \in \{0, 1\} \end{cases}$$

We have $g(a) = e, g(b) = \{b, c, d\}, g(c) = a, g(d) = \{b, c, d\}, g(e) = e$. g is not U.S.C since at the point $y_0 = a$, $\mathcal{U}_a = \{a, b\}$, $g(a) = \{e\}$, we have $g(\mathcal{U}_a) = \{b, c, d, e\} \not\subset \bigcup\limits_{y \in g(a)} \mathcal{U}_y) = \{d, e\}$.

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References

[1] P. S. Alexandroff, *Diskrete Räume*, Rec. Math. [Mat. Sbornik] N.S. **2** (1937), 501–518.
[2] ———, *Combinatorial topology. Vol. 1*, Graylock Press, Rochester, N. Y., 1956. MR 17,882a
[3] Francisco G. Arenas, *Some results on Alexandroff spaces*, Preprint, Topology Atlas, 1997, `http://at.yorku.ca/p/a/a/k/12.htm`.
[4] Francisco. G. Arenas, *Alexandroff spaces*, Acta Math. Univ. Comenian. (N.S.) **68** (1999), no. 1, 17–25. MR **2000i:**54028
[5] Francisco G. Arenas, Julian Dontchev, and Maximilian Ganster, *On some weaker forms of Alexandroff spaces*, Arab. J. Sci. Eng. Sect. A Sci. **23** (1998), no. 1, 79–89. MR **99b:**54056
[6] Nanigopal Biswas, *On some mappings in topological spaces*, Bull. Calcutta Math. Soc. **61** (1969), 127–135. MR 43 #6886
[7] George L. Cain, *Introduction to general topology*, Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1994. MR 1 248 821
[8] William Dunham, *$T_{1/2}$-spaces*, Kyungpook Math. J. **17** (1977), no. 2, 161–169. MR 57 #10677
[9] Abd. El-Fattah El-Atik, *A study on some types of mappings on topological spaces*, Master's thesis, Tanta University, 1997.
[10] ———, *Approximation of fractals by finite topological spaces*, Preprintreihe Mathematik 22, Ernst-Moritz-Arndt-Universität Greifswald, 2001.
[11] Ryszard Engelking, *General topology*, PWN—Polish Scientific Publishers, Warsaw, 1977. MR 58 #18316b
[12] Jürgen Flachsmeyer, *Zur Spektralentwicklung topologischer Räume*, Math. Ann. **144** (1961), 253–274. MR 26 #735
[13] G. T. Herman, *On topology as applied to image analysis*, Comp.Vision, Graphics Image Process. **52** (1990), 409–415.
[14] Efim Khalimsky, Ralph Kopperman, and Paul R. Meyer, *Computer graphics and connected topologies on finite ordered sets*, Topology Appl. **36** (1990), no. 1, 1–17. MR **92c:**54037
[15] T. Yung Kong, Ralph Kopperman, and Paul R. Meyer, *A topological approach to digital topology*, Amer. Math. Monthly **98** (1991), no. 10, 901–917. MR **93b:**68092

[1] A short version of the thesis, supervised by C. Bandt, was distributed as a preprint at Ernst-Moritz-Arndt-Universität Greifswald [10].

[16] Norman Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly **68** (1961), 44–46. MR 23 #A3548

[17] ______, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41. MR 29 #4025

[18] ______, *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo (2) **19** (1970), 89–96. MR 46 #4471

[19] L. Lotz, *Endliche Topologische Räume*, Tech. report, Greifswald, Germany, 1970, Diplomarbeit.

[20] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deep, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt (1982), no. 53, 47–53 (1983). MR **87c:**54002

[21] W. Rinow, *Lehrbuch der Topologie*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1975, Hochschulbücher für Mathematik, Band 79. MR 58 #24157

[22] R. E. Stong, *Finite topological spaces*, Trans. Amer. Math. Soc. **123** (1966), 325–340. MR 33 #3247

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