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CHAINABLE SUBCONTINUA

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Abstract. This paper is concerned with conditions under which a metric continuum (a compact connected metric space) contains a non-degenerate chainable continuum.

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By R.H. Bing's theorem eleven [2] if a metric continuum X contains a non-degenerate subcontinuum H which is hereditarily decomposable, hereditarily unicoherent, and atriodic, then H is chainable.

The following papers give examples of continua with the property that each non-degenerate subcontinuum is not chainable. G.T. Whyburn [16]. R.D. Anderson and G. Choquet [1]. A. Lelek [6] gives an example of a planar weakly chainable continuum each non-degenerate subcontinuum of which separates the plane and thus contains no non-degenerate chainable subcontinuum. W.T. Ingram [5] gives an example of an hereditarily indecomposable tree-like continuum such that each non-degenerate subcontinuum has positive span and hence is not chainable.

C.E. Burgess in [3] shows if a continuum M is almost chainable and K is a proper subcontinuum of M which contains an endpoint p of M, then K is linearly chainable with p as an end point. A continuum M is almost chainable if, for every positive number ε, there exists an ε-covering G of M and a linear chain $C(L_1, L_2, \ldots, L_n)$ of elements of G such that no L_i $(1 \leq i < n)$ intersects an element of $G - C$ and every point of M is within a distance ε of some element of C. He also shows if M is almost chainable, then M is not a triod and M is unicoherent and irreducible between some two points. Examples show M can contain a triod or a non-unicoherent subcontinuum.

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If X and Y are metric continua and if X can be ε-mapped onto Y for all positive ε and Y has a non-degenerate chainable continuum then so does X. This result suggests considering inverse limit spaces. At this stage we refer to a result from the paper of S. Mardešić and J. Segal [9] Theorem 1, p. 148: "Every π-like continuum X is the inverse limit of an inverse sequence $\{P_i; \pi_{ij}\}$ with bonding maps π_{ij} onto and with polyhedra $P_i \in \pi$. A continuum is π-like if it can be ε-mapped onto some polyhedron in π for each positive ε. E. Duda and P. Krupski [4] showed that a k-junctioned metric continuum, k a non-negative integer, has at most k points such that any continuum which contains none of the k points is chainable. A metric continuum is said to be k-junctioned if it is the inverse limit of graphs each of which has at most k branch points, with surjective bonding maps. A continuum is called finitely junctioned if it is k-junctioned for some non negative integer k.

Suppose now X is a tree-like continuum. Then for each $\varepsilon > 0$ X can be mapped onto a tree. By a result quoted above X is the inverse limit of a sequence of trees with surjective bonding maps. $X = \underleftarrow{\lim}\{T_n, f_{nm}\}$. Let $f_n : X \to T_n$ be the standard projection map and let

$$P_n = U\{f_n^{-1}(q) | q \text{ is a branch point}\}.$$

Since T_n has at most a finite number of branch points (points of order ≥ 2) P_n is closed in X. If the union of the P_n is not dense in X then X contains a non-degenerate chainable continuum. Actually it is sufficient that $\{P_n\}$ have a subsequence whose union is not dense in X.

Lets now consider a non-degenerate metric continuum in X with span equal to zero. The notion of span was defined by A. Lelek [7]. In the paper [8] he showed continua with span zero are atriodic and tree-like.

There is a series of papers by L.G. Oversteegen and E.D. Tymchatyn which develop properties of spaces with spans equal to zero or sufficient conditions that a space have a span equal to zero [12, 15, 13, 14]. Also by L.G. Oversteegen [11].

It is interesting to note that a chainable continuum X can be ε-mapped onto any fixed dendrite. Thus for any tree T, by the result of Mardešić and Segal quoted above, X is the inverse of a sequence of T's.

In the paper [10] P. Minc shows an inverse limit of trees with simplicial bonding maps having surjective span zero is chainable.

References

[1] R. D. Anderson and Gustave Choquet, *A plane continuum no two of whose nondegenerate subcontinua are homeomorphic: An application of inverse limits*, Proc. Amer. Math. Soc. **10** (1959), 347–353. MR 21 #3819

[2] R. H. Bing, *Snake-like continua*, Duke Math. J. **18** (1951), 653–663. MR 13,265a

[3] C. E. Burgess, *Homogeneous continua which are almost chainable*, Canad. J. Math. **13** (1961), 519–528. MR 23 #A3551

[4] Edwin Duda and Paweł Krupski, *A characterization of finitely junctioned continua*, Proc. Amer. Math. Soc. **116** (1992), no. 3, 839–841. MR **93a:**54031

[5] W. T. Ingram, *Hereditarily indecomposable tree-like continua. II*, Fund. Math. **111** (1981), no. 2, 95–106. MR **82k:**54056

[6] A. Lelek, *On weakly chainable continua*, Fund. Math. **51** (1962/1963), 271–282. MR 26 #742

[7] ———, *Disjoint mappings and the span of spaces*, Fund. Math. **55** (1964), 199–214. MR 31 #4009

[8] ———, *The span of mappings and spaces*, Topology Proc. **4** (1979), no. 2, 631–633. MR **82c:**54031

[9] Sibe Mardešić and Jack Segal, *ε-mappings onto polyhedra*, Trans. Amer. Math. Soc. **109** (1963), 146–164. MR 28 #1592

[10] Piotr Minc, *On simplicial maps and chainable continua*, Topology Appl. **57** (1994), no. 1, 1–21. MR **95c:**54056

[11] Lex G. Oversteegen, *On span and chainability of continua*, Houston J. Math. **15** (1989), no. 4, 573–593. MR **91g:**54049

[12] Lex G. Oversteegen and E. D. Tymchatyn, *Plane strips and the span of continua. I*, Houston J. Math. **8** (1982), no. 1, 129–142. MR **84h:**54030

[13] ———, *On the span of weakly-chainable continua*, Fund. Math. **119** (1983), no. 2, 151–156. MR **85j:**54051

[14] ———, *On span and weakly chainable continua*, Fund. Math. **122** (1984), no. 2, 159–174. MR **85m:**54034

[15] ———, *Plane strips and the span of continua. II*, Houston J. Math. **10** (1984), no. 2, 255–266. MR **86a:**54042

[16] G. T. Whyburn, *A continuum every subcontinuum of which separate the plane*, Amer. J. Math. **52** (1930), 319–330.

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