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CONCERNING THE DUAL GROUP OF A DENSE SUBGROUP

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ABSTRACT. Throughout this Abstract, G is a topological Abelian group and $\widehat{G}$ is the space of continuous homomorphisms from G into $\mathbb{T}$ in the compact-open topology. A dense subgroup D of G *determines* G if the (necessarily continuous) surjective isomorphism $\widehat{G} \twoheadrightarrow \widehat{D}$ given by $h \mapsto h|D$ is a homeomorphism, and G is *determined* if each dense subgroup of G determines G. The principal result in this area, obtained independently by L. AUßENHOFER and M. J. CHASCO, is the following: Every metrizable group is determined. The authors offer several related results, including these.

(1) There are (many) nonmetrizable, noncompact, determined groups.
(2) If the dense subgroup D_i determines G_i with G_i compact, then $\oplus_i D_i$ determines $\Pi_i G_i$. In particular, if each G_i is compact then $\oplus_i G_i$ determines $\Pi_i G_i$.
(3) Let G be a locally bounded group and let G^+ denote G with its Bohr topology. Then G is determined if and only if G^+ is determined.
(4) Let $\mathrm{non}(\mathcal{N})$ be the least cardinal κ such that some $X \subseteq \mathbb{T}$ of cardinality κ has positive outer measure. No compact G with $w(G) \geq \mathrm{non}(\mathcal{N})$ is determined; thus if $\mathrm{non}(\mathcal{N}) = \aleph_1$ (in particular if CH holds), an infinite compact group G is determined if and only if $w(G) = \omega$.

Question. Is there in ZFC a cardinal κ such that a compact group G is determined if and only if $w(G) < \kappa$? Is $\kappa = \mathrm{non}(\mathcal{N})$? $\kappa = \aleph_1$?

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0. Terminology, Notation and Preliminaries

For X a set and κ a cardinal, we write $[X]^\kappa = \{A \subseteq X : |A| = \kappa\}$.

For each space $X = (X, \mathcal{T})$ we write

$$\mathcal{K}(X) := \{K \subseteq X : K \text{ is } \mathcal{T}\text{-compact}\}.$$

All groups considered here, whether or not equipped with a topology, are Abelian groups written additively. The identity of a group G is denoted 0 or 0_G, and the torsion subgroup of G is denoted $\mathrm{tor}(G)$. The reals, rationals, and integers are denoted $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$, respectively, and the "unit circle" group $\mathbb{T}$ is the group $(-\frac{1}{2}, \frac{1}{2}]$ with addition mod 1. Except when we specify otherwise, these groups carry their usual metrizable topology.

The symbol $\mathbb{P}$ denotes the set of positive prime integers.

The set of homomorphisms $h : G \to \mathbb{T}$, a group under pointwise operation, is denoted $\mathrm{Hom}(G, \mathbb{T})$. For a subgroup A of $\mathrm{Hom}(G, \mathbb{T})$ we denote by $(G, \mathcal{T}_A)$ the group G with the topology induced by A. Evidently $(G, \mathcal{T}_A)$ is a Hausdorff topological group if and only if A separates points of G. The topology $\mathcal{T}_A$ is the coarsest topology on G for which the homomorphism $e_A : G \to \mathbb{T}^A$ given by $(e_A(x))_h = h(x)$ $(x \in G,\ h \in A)$ is continuous. When $G = (G, \mathcal{T})$ is a topological group, the set of $\mathcal{T}$-continuous functions in $\mathrm{Hom}(G, \mathbb{T})$ is a subgroup of $\mathrm{Hom}(G, \mathbb{T})$ denoted $\widehat{G}$ or $\widehat{(G, \mathcal{T})}$; in this case the topology $\mathcal{T}_{\widehat{G}}$ is the *Bohr topology* associated with $\mathcal{T}$, and $(G, \mathcal{T}_{\widehat{G}})$ is denoted G^+ or $(G, \mathcal{T})^+$. When $\widehat{(G, \mathcal{T})}$ separates points we say that G is a *maximally almost periodic* group and we write $G = (G, \mathcal{T}) \in \mathbf{MAP}$. Whether or not $(G, \mathcal{T}) \in \mathbf{MAP}$, the closure of $e[G]$ in $\mathbb{T}^{\widehat{G}}$, denoted $b(G)$ or $b(G, \mathcal{T})$, is the *Bohr compactification* of $(G, \mathcal{T})$.

The Bohr compactification $b(G)$ of a topological group G is characterized by the condition that each continuous homomorphism from G into a compact Hausdorff group extends continuously to a homomorphism from $b(G)$. From this and the uniform continuity of continuous homomorphisms it follows that if D is a dense subgroup of G then $b(D) = b(G)$. It is conventional to suppress mention of the function $e_{\widehat{G}}$ and to write simply $\widehat{G} = \widehat{G^+}$. When $G \in \mathbf{MAP}$ we write $G^+ \subseteq b(G) \subseteq \mathbb{T}^{\widehat{G}}$, the inclusions being both algebraic and topological.

A group G with its discrete topology is denoted G_d. For notational convenience, and following van Douwen [34], for every (Abelian) group G we write $G^\# = (G_d)^+ \subseteq b(G_d)$.

A subset S of a topological group G is said to be *bounded* in G if for every nonempty open $V \subseteq G$ there is finite $F \subseteq G$ such that $S \subseteq F + V$; G is *locally bounded* [resp., *totally bounded*] if some nonempty open subset of G is bounded [resp., G itself is bounded]. It is a theorem of Weil [36] that each locally bounded group G embeds as a dense topological subgroup of a locally compact group $W(G)$, unique in the obvious sense; the group $W(G)$ is compact if and only if G is totally bounded. We denote by **LCA** [resp., **LBA**] the class of locally compact [resp., locally bounded] Hausdorff Abelian groups. The relation $\mathbf{LCA} \subseteq \mathbf{MAP}$ is a well known consequence of the Gel′fand-Raĭkov Theorem (cf. [18, 22.17]); since each subgroup $S \subseteq G \in \mathbf{MAP}$ clearly satisfies $S \in \mathbf{MAP}$, we have in fact the relations $\mathbf{LCA} \subseteq \mathbf{LBA} \subseteq \mathbf{MAP}$.

Lemma 0.1. *Let S be a subgroup of $G \in \mathbf{LBA}$. Then*

(a) *S is dual-embedded in G in the sense that each $h \in \widehat{S}$ extends to an element of $\widehat{G}$;*
(b) *if $h \in \widehat{S}$ and $x \in G \backslash \overline{S}^G$, the extension $k \in \widehat{G}$ of h may be chosen so that $k(x) \neq 0$.*

It follows that for each subgroup S of a group $G \in \mathbf{LBA}$ the topology of S^+ coincides with the topology inherited by S from G^+. This validates the following notational convention. For $S \subseteq G \in \mathbf{LBA}$, S not necessarily a subgroup of G, we denote by S^+ the set S with the topology inherited from G^+. When G is discrete, so that $G^+ = G^\#$, we write $S^\#$ in place of S^+ when $S \subseteq G$.

Theorem 0.2. *(Glicksberg [16]). Let $K \subseteq G \in \mathbf{LBA}$. Then $K \in \mathcal{K}(G)$ if and only if $K^+ \in \mathcal{K}(G^+)$. Hence if $K \in \mathcal{K}(G)$, then K and K^+ are homeomorphic.*

Theorem 0.3. *(Flor [14]. See also Reid [28]). Let $G \in \mathbf{LBA}$ and let $x_n \to p \in b(G) = b(W(G))$ with each $x_n \in G^+ \subseteq (W(G))^+ \subseteq b(G)$. Then*

(a) *$p \in (W(G))^+$, and*
(b) *not only $x_n \to p$ in $(W(G))^+ \subseteq b(G)$ but also $x_n \to p$ in $W(G)$.*

Remark 0.4. Strictly speaking, the papers cited above in connection with Theorems 0.2 and 0.3 deal with groups $G \in \mathbf{LCA}$. Our modest generalization to the case $G \in \mathbf{LBA}$ is justified by 0.2 and 0.3 as originally given and by these facts about $G \in \mathbf{LBA}$:

(i) G is a (dense) topological subgroup of $W(G) \in \mathbf{LCA}$;
(ii) G^+ is a (dense) topological subgroup of $(W(G))^+$; and
(iii) $b(G) = b(W(G))$.

In what follows, groups of the form $\widehat{G}$ will be given the *compact-open* topology. This is defined as usual: the family

$$\{U(K, \epsilon) : K \in \mathcal{K}(G), \epsilon > 0\}$$

is a base at $0 \in \widehat{G}$, where for $A \subseteq G$ one writes

$$U(A, \epsilon) = \{h \in \widehat{G} : x \in A \Rightarrow |h(x)| < \epsilon\}.$$

We have noted already that for $G \in \mathbf{MAP}$ the groups $\widehat{G}$ and $\widehat{G^+}$ are identical; that is, $\widehat{G} = \widehat{G^+}$ as groups. Our principal interest in Theorem 0.2 is that for $G \in \mathbf{LBA}$ it gives a topological consequence, as follows.

Corollary 0.5. *Let $G \in \mathbf{LBA}$. Then $\widehat{G} = \widehat{G^+}$ as topological groups. That is, the compact-open topology on $\widehat{G}$ determined by $\mathcal{K}(G)$ coincides with the compact-open topology on $\widehat{G}$ determined by $\mathcal{K}(G^+)$.*

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1. The Groups $\widehat{G}$ for G Metrizable

If D is a dense subgroup of an Abelian topological group $G = (G, \mathcal{T})$ then every $h \in \widehat{D}$ extends (uniquely) to an element of $\widehat{G}$; and of course, each $h \in \widehat{G}$ satisfies $h|D \in \widehat{D}$. Accordingly, abusing notation slightly, we have $\widehat{D} = \widehat{G}$ as groups. Since groups of the form $\widehat{G}$ carry the compact-open topology, it is natural to inquire whether the identity $\widehat{G} = \widehat{D}$ is topological as well as algebraic. Informally: do $\mathcal{K}(G)$ and $\mathcal{K}(D)$ induce the same topology on the set $\widehat{G} = \widehat{D}$? The question provokes this definition.

Definition 1.1. Let G be an Abelian topological group.

(a) Let D be a dense subgroup of G. Then D *determines* G (alternatively: G is *determined* by D) if $\widehat{G} = \widehat{D}$ as topological groups.
(b) G is *determined* if every dense subgroup of G determines G.

Remarks 1.2.

(a) It is a theorem of Kaplan [21, 2.9] (cf. Banasczyzk [4, 1.3] and Raczkowski [26, §3.1] for alternative treatments, and Außenhofer [2] and [3, 3.4] for a generalization) that for each G the family $\{U(K, \frac{1}{4}) : K \in \mathcal{K}(G)\}$ is basic at $0 \in \widehat{G}$. (For notational simplicity, henceforth we write $U(K) := U(K, \frac{1}{4}) \subseteq \widehat{G}$ for $K \in \mathcal{K}(G)$.) Thus the condition that a group G is determined by its dense subgroup D reduces to (i.e., is equivalent to) the condition that $\mathcal{K}(D)$ is cofinal in $\mathcal{K}(G)$ in the sense that for each $K \in \mathcal{K}(G)$ there is $E \in \mathcal{K}(D)$ such that $U(E) \subseteq U(K)$.
(b) Let D and S be dense subgroups of a topological group G such that $D \subseteq S \subseteq G$. Then since $\mathcal{K}(D) \subseteq \mathcal{K}(S) \subseteq \mathcal{K}(G)$, D determines G if and only if D determines S and S determines G. In particular, a dense subgroup of a determined group is determined.
(c) The principal theorem in this corner of mathematics is the following result, obtained independently by Außenhofer [3, 4.3] and Chasco [7]. This is the point of departure of the present inquiry.

Theorem 1.3. *Every metrizable, Abelian group is determined.*

Discussion 1.4. Is every topological group determined? Is every **MAP** group determined? Are there nonmetrizable, determined groups? Is every closed (or, open) subgroup of a determined group itself determined? Is the class of determined groups closed under passage to continuous homomorphisms? Continuous isomorphisms? The formation of products? These are some of the questions we address.

2. Determined Groups: G vs. G^+

Lemma 2.1. *Let D be a subgroup of $G \in \mathbf{LBA}$. Then D is dense in G if and only if D^+ is dense in G^+.*

Corollary 2.2. *Let D be a subgroup of $G \in \mathbf{LBA}$. Then D determines G if and only if D^+ determines G^+.*

Theorem 2.3. *Let $G \in \mathbf{LBA}$. Then G is determined if and only if G^+ is determined.*

Theorem 2.4. *Let G be an $\mathbf{LBA}$ group such that G^+ determines $b(G)$. Then*

(a) *G is totally bounded (and hence $G = G^+$); and*
(b) *if also $G \in \mathbf{LCA}$ then G is compact (and hence $G = G^+ = b(G)$).*

Corollary 2.5. *Let $G \in \mathbf{LBA}$. Then $b(G)$ is determined if and only if $W(G)$ is compact and determined; in this case $W(G) = b(G)$.*

Corollary 2.6. *Let $G \in \mathbf{LCA}$. If G is noncompact then G^+ does not determine $b(G)$ (and hence $b(G)$ is not determined).*

Theorem 2.7. *Let G be a closed subgroup of a product of $\mathbf{LBA}$ groups. Then a dense subgroup D of G determines G if and only D^+ determines G^+. Thus G is determined if and only if G^+ is determined.*

3. Determined Groups: Some Examples

Theorem 3.1. *There are totally bounded, nonmetrizable, determined groups.*

Proof. Let G be an arbitrary determined $\mathbf{LBA}$ group such that G is not totally bounded. (Appealing to Theorem 1.3, one might choose $G \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$.) That G^+ is as required follows from three facts:

(a) G^+ is determined (Theorem 2.3);
(b) a group with a dense metrizable subgroup is itself metrizable [6, Prop. IX §2.1.1];
(c) $b(G)$ is not metrizable.

□

Theorem 3.2. *A nondetermined group may have a dense, determined subgroup.*

Theorem 3.3. *The image of a nondetermined group under a continuous homomorphism may be determined.*

Proof. We see in Theorem 4.8 below that compact groups of weight $\geq \mathfrak{c}$ are nondetermined. Each such group maps by a continuous homomorphism onto either the group $\mathbb{T}$ or a group of the form $(\mathbb{Z}(p))^\omega$ ($p \in \mathbb{P}$) [10], and such groups are determined by Theorem 1.3. ☐

Discussion 3.4. Obviously an **LBA** group with no proper dense subgroup is vacuously determined. We mention three classes of such groups.

- (i) Discrete groups.
- (ii) Groups of the form $G^\# = (G_d)^+$. (It is well known [12, 2.1] that every subgroup of such a group is closed.)
- (iii) **LCA** groups of the type given by Rajagopalan and Subrahmanian [27]. Specifically, let $\kappa \geq \omega$, fix $p \in \mathbb{P}$, and topologize the group $G := (\mathbb{Z}(p^\infty))^\kappa$ so that its subgroup $H := (\mathbb{Z}(p))^\kappa$ in its usual compact topology is open-and-closed in G.

Theorem 3.5.

- (a) *A determined group may contain a nondetermined open-and-closed subgroup.*
- (b) *There are non-totally bounded, nonmetrizable, determined* **LBA** *groups.*

Although compact groups of the form K^κ with $\kappa \geq \mathfrak{c}$ are not determined, we see in Corollary 3.11 below that such groups do contain nontrivial determining subgroups.

Notation 3.6. Let $\{G_i : i \in I\}$ be a set of groups, let $S_i \subseteq G_i$, and let $p \in G := \Pi_{i \in I} G_i$. Then

- (i) $s(p) = \{i \in I : p_i \neq 0_i\}$;
- (ii) $\oplus_{i \in I} G_i = \{x \in G : |s(x)| < \omega\}$; and
- (iii) $\oplus_{i \in I} S_i = (\Pi_{i \in I} S_i) \cap (\oplus_{i \in I} G_i)$.

In this context we often identify S_i with the subset $S_i \times \{0_{I \setminus \{i\}}\}$ of G. In particular we write $G_i \subseteq G$ and we identify $\widehat{G_i}$ with $\{h|G_i : h \in \widehat{G}\}$.

We use the following property to find some determining subgroups of certain (nondetermined) products.

Definition 3.7. A topological group G has the *cofinally zero* property if for all $K \in \mathcal{K}(G)$ there is $F \in \mathcal{K}(G)$ such that every $h \in U(F)$ satisfies $h|K \equiv 0$.

Remark 3.8. We record two classes of groups with the cofinally zero property.

- (i) G is a determining subgroup of a compact Abelian group. (There is $F \in \mathcal{K}(G)$ such that $U(F) = \{0\}$, so each $h \in U(F)$ satisfies $h|K \equiv 0$ for all $K \in \mathcal{K}(G)$.)
- (ii) G is a torsion group of bounded order. (Given $K \in \mathcal{K}(G)$, let $n > 4$ satisfy $nx = 0$ for all $x \in G$ and use Remark 1.2(a) to choose $F \in \mathcal{K}(G)$ such that $U(F) \subseteq U(K, \frac{1}{n})$.)

Lemma 3.9. *Let $\{G_i : i \in I\}$ be a set of* **LBA** *groups with the cofinally zero property and let $G = \Pi_{i\in I} G_i$. If D_i is a dense, determining subgroup of G_i, then $D := \oplus_{i\in I} D_i$ determines G.*

Corollary 3.10. *Let $\{G_i : i \in I\}$ be a set of determined* **LBA** *groups with the cofinally zero property and let $G = \Pi_{i\in I} G_i$. If D_i is a dense subgroup of G_i, then $\oplus_{i\in I} D_i$ determines G.*

Corollary 3.11. *Let $\{G_i : i \in I\}$ be a set of compact Hausdorff groups and let $G = \Pi_{i\in I} G_i$. Then $\oplus_{i\in I} G_i$ determines G.*

Theorem 3.12. *The image under a continuous homomorphism of a compact determined group is determined.*

Remark 3.13. It is easily checked that if a locally compact space X is σ-compact then it is *hemicompact*, i.e., some countable subfamily $\{K_n : n < \omega\}$ of $\mathcal{K}(X)$ is cofinal in $\mathcal{K}(X)$ in the sense that for each $K \in \mathcal{K}(X)$ there is $n < \omega$ such that $K \subseteq K_n$. It follows that if an **LCA** group G is σ-compact (equivalently: Lindelöf) then $w(\widehat{G}) \leq \omega$, so $\widehat{G}$ in this case is determined by Theorem 1.3.

4. Nondetermined groups: Some Examples

The principal result of this section is that compact Abelian groups of weight $\geq \mathfrak{c}$ are nondetermined.

Lemma 4.1. *Let G be an* **LBA** *group with a proper dense subgroup D such that each $K \in \mathcal{K}(D)$ satisfies either*

(i) *K is finite or*
(ii) *$\langle K \rangle$ is closed in D.*

Then D does not determine G.

We noted in Theorem 2.4(b) that if G^+ determines $b(G)$ with $G \in$ **LCA**, then G is compact (in fact $G = G^+ = b(G)$). Lemma 4.1 allows a more direct proof in the case that G is discrete.

Corollary 4.2. *Let G be an infinite Abelian group. Then $G^\#$ does not determine $b(G_d)$.*

Lemma 4.3. *(*[11]*). Let G be an Abelian group.*

(a) *If A is a point-separating subgroup of $\mathrm{Hom}(G, \mathbb{T})$, then $(G, \mathcal{T}_A)$ is a totally bounded, Hausdorff topological group with $\widehat{(G, \mathcal{T}_A)} = A$;*
(b) *for every totally bounded Hausdorff topological group topology $\mathcal{T}$ on G the subgroup $A := \widehat{(G, \mathcal{T})}$ of $\mathrm{Hom}(G, \mathcal{T})$ is point-separating and satisfies $\mathcal{T} = \mathcal{T}_A$.*

Discussion 4.4. It is easily checked that for each Abelian group G the set $\mathrm{Hom}(G, \mathbb{T})$ is closed in the compact space $\mathbb{T}^G$. Thus $\mathrm{Hom}(G, \mathbb{T})$, like every Hausdorff (locally) compact group, carries a Haar measure. Our convention here is that Haar measure is complete, so in particular every subset of a measurable set of measure 0 is itself measurable (and of measure 0).

Concerning Haar measure λ on a **LCA** group G we appeal frequently to the *Steinhaus-Weil Theorem*: *If $S \subseteq G$ is λ-measurable and $\lambda(S) > 0$, then the difference set $S - S := \{x - y : x, y \in S\}$ contains a nonempty open subset of G; thus S, if a subgroup of G, is open in G.*

Lemma 4.5. *(*[13, 3.10]*). Let G be an Abelian group, let $\{x_n : n < \omega\}$ be a faithfully index sequence in G, and let*

$$S := \{h \in \mathrm{Hom}(G, \mathbb{T}) : h(x_n) \to 0 \in \mathbb{T}\}.$$

Let λ be the Haar measure of $\mathrm{Hom}(G, \mathbb{T})$. Then S is a λ-measurable subgroup of $\mathrm{Hom}(G, \mathbb{T})$, with $\lambda(S) = 0$.

Theorem 4.6. *Let X be a compact Hausdorff space such that $|X| < 2^{\aleph_1}$. Then*

(a) *(*[17]*) X contains a closed, countably infinite subspace; and*
(b) *X contains a nontrivial convergent sequence.*

Theorem 4.7. *Let G be an Abelian group such that $|G| < 2^{\aleph_1}$ and let A be a dense subgroup of $\mathrm{Hom}(G, \mathbb{T})$ such that either*

(i) *A is non-Haar measurable, or*
(ii) *A is Haar measurable, with $\lambda(A) > 0$.*

Then $(G, \mathcal{T}_A)$ does not determine $W(G, \mathcal{T}_A)$.

Theorem 4.8. *Let G be a compact, Abelian group such that $w(G) \geq \mathfrak{c}$. Then G is not determined.*

Proof. (Outline) Step 1. $\mathbb{T}^{\mathfrak{c}}$ is not determined. [Proof. There is a non-measurable subgroup A of $\mathbb{T}$ algebraically of the form $\oplus_{\mathfrak{c}}\mathbb{Z}$. Apply Theorem 4.7(i) to the (dense) subgroup $e_A(\mathbb{Z})$ of $\mathbb{T}^A = \mathbb{T}^{\mathfrak{c}}$.]

Step 2. $F^{\mathfrak{c}}$ is not determined (F a finite Abelian group, $|F| > 1$). [Proof. We have $b((\oplus_\omega F)_d) = F^{\mathfrak{c}}$, so Corollary 4.2 applies.]

Step 3. There is a continuous isomorphism $\phi : G \twoheadrightarrow K^{\mathfrak{c}}$ with either $K = \mathbb{T}$ or $K = F$ as in Step 2, so Theorem 3.12 applies. □

Corollary 4.9 (CH)**.** *Let G be a compact Abelian group. Then G is determined if and only if G is metrizable.*

Corollary 4.10 (CH)**.** *Let $\{G_i : i \in I\}$ be a set of compact Abelian groups with each $|G_i| > 1$, and let $G = \Pi_{i \in I} G_i$. Then G is determined if and only if $|I| \leq \omega$ and each G_i is determined.*

We close this section with an example indicating that the intersection of dense, determining subgroups may be dense and nondetermining.

Theorem 4.11. *There are dense, determining subgroups D_i $(i = 0, 1)$ of $\mathbb{T}^{\mathfrak{c}}$ such that $D_0 \cap D_1$ is dense in $\mathbb{T}^{\mathfrak{c}}$ and does not determine $\mathbb{T}^{\mathfrak{c}}$.*

Proof. Let Z be a dense, cyclic, nondetermining subgroup of $\mathbb{T}^{\mathfrak{c}}$, let A_i $(i = 0, 1)$ be dense, torsion subgroups of $\mathbb{T}$ such that $A_0 \cap A_1 = \{0_{\mathbb{T}}\}$, and set $D_i := Z + \oplus_{\mathfrak{c}} A_i \subseteq \mathbb{T}^{\mathfrak{c}}$ $(i = 0, 1)$. Then A_i determines $\mathbb{T}$ by Theorem 1.3 so $\oplus_{\mathfrak{c}} A_i$ determines $\mathbb{T}^{\mathfrak{c}}$ by Lemma 3.9, so D_i determines $\mathbb{T}^{\mathfrak{c}}$ $(i = 0, 1)$; but the dense subgroup $Z = D_0 \cap D_1$ of $\mathbb{T}^{\mathfrak{c}}$ does not determine $\mathbb{T}^{\mathfrak{c}}$. □

5. Concerning Topological Linear Spaces

Remark 5.1. Let κ be a cardinal number and denote by l^1_κ the space of real κ-sequences $x = \{x_\xi : \xi < \kappa\}$ such that $||x||_1 := \sum_{\xi<\kappa} |x_\xi| < \infty$. The additive topological group l^1_κ respects compactness (cf. Remus and Trigos-Arrieta [29]).

The group $\widehat{(l^1_\kappa)^+}$ is not discrete, so the Weil completion $W((l^1_\kappa)^+)$ is another example of a compact nondetermined group.

Definition. A topological group G is *(group) reflective* if the evaluation mapping $\Omega_G : G \to \widehat{\widehat{G}}$ defined by $\Omega_G(x)(h) := h(x)$ for $x \in G$, $h \in \widehat{G}$ is a topological isomorphism of G onto $\widehat{\widehat{G}}$.

Theorem 5.2. *Let G be a noncomplete, reflective group and let $R(G)$ be its Raĭkov completion. Then $R(G) \in$ **MAP** and G does not determine $R(G)$.*

Example 5.4 *infra* illustrates Theorem 5.2.

Definition 5.3. A reflexive locally convex vector space (LCS) in which every closed bounded subset is compact is called a *Montel space.*

Reflexivity and boundedness (Schaefer[30] §I.5, §IV.5) are meant here in the sense of topological vector spaces. By a *Montel group* we mean the underlying (additive) topological group of a Montel space. Since by definition these are reflexive LCS, Montel groups are reflective as proven by Smith [31].

Example 5.4. Kōmura [23] and Amemiya and Kōmura [1] construct by induction three different noncomplete Montel spaces, the completion of each being a "big product" of copies of $\mathbb{R}$, and one of them being exactly $\mathbb{R}^{\mathfrak{c}}$. These groups indicate that Theorem 5.2 is not vacuous. One of the groups constructed in [1] is separable. Thus in particular, again by Theorem 5.2, we see that $\mathbb{R}^{\mathfrak{c}}$ has a countable dense subgroup which does not determine $\mathbb{R}^{\mathfrak{c}}$.

The remarks above show again that the property of being determined is not $\mathfrak{c}$-productive.

6. Cardinals κ Such That $\omega < \kappa \leq \mathfrak{c}$

It is well known (cf. for example [24, 2.18] or [8, 8.2.4]) that under Martin's Axiom [MA] every cardinal κ with $\omega \leq \kappa < \mathfrak{c}$ satisfies $2^\kappa = \mathfrak{c}$. In particular under MA $+\neg$CH it follows from Theorem 4.6(b) that every compact Hausdorff space X such that $|X| < 2^{\aleph_1} = \mathfrak{c}$ contains a nontrivial convergent sequence. Malykhin and Šapirovskiĭ [25] have achieved a nontrivial extension of this result: Under MA, every compact Hausdorff space X with $|X| \leq \mathfrak{c}$ contains a nontrivial convergent sequence.

Theorem 6.1 (MA)**.** *Let G be a group with $|G| \leq 2^\omega$, and let A be a dense nonmeasurable subgroup of $\widehat{G_d}$. Then every compact subset of $(G, \mathcal{T}_A)$ is finite, so its completion $W(G, \mathcal{T}_A)$ is not determined.*

If we denote by λ_G the (completed) Haar measure on a LCA group G, let λ_G^* stand for the associated outer measure. The existence of a nonmeasurable subset X of $\mathbb{T}$ (with $|X| = \mathfrak{c}$) is well known, so the case $\kappa = \mathfrak{c}$ of the following theorem generalizes the statement in Step 1 of the proof of Theorem 4.8.

Theorem 6.2. *Let $\omega < \kappa \leq \mathfrak{c}$. If there is $X \in [\mathbb{T}]^\kappa$ such that $\lambda_{\mathbb{T}}^*(X) > 0$, then there is a nonmeasurable, free Abelian subgroup A of $\mathbb{T}$ algebraically of the form $A = \oplus_\kappa \mathbb{Z}$.*

Responding to a question on a closely related matter, Stevo Todorčević [33] proposed and proved the above result for $\kappa = \aleph_1$. His proof additionally yields that $X \backslash \mathrm{tor}(\mathbb{T})$ can be broken into ω-many pairwise disjoint independent sets, each of cardinality $\aleph_1$.

For torsion groups of prime order, we obtain the following.

Theorem 6.3. *Let F be a finite group of prime order p, and let κ_1, κ_2 be infinite cardinals such that $\kappa_1 \leq 2^{\kappa_2}$. If there is $X \in [F^{\kappa_2}]^{\kappa_1}$ such that $\lambda_{F^{\kappa_2}}^*(X) > 0$, then there is a nonmeasurable subgroup A of F^{κ_2} algebraically of the form $A = \oplus_{\kappa_1} F$.*

Discussion 6.4. For an ideal $\mathcal{I}$ of subsets of a set S we write as usual

$$\mathrm{non}(\mathcal{I}) = \min\{|Y| : Y \subseteq S,\ Y \notin \mathcal{I}\}.$$

Let F be a finite group ($|F| > 1$), let $\mathcal{N}(\mathbb{T})$ and $\mathcal{N}(F^\omega)$ denote the σ-algebra of $\lambda_{\mathbb{T}}$- and λ_{F^ω}-measurable sets of measure zero. As with any two compact metric spaces of equal cardinality equipped with atomless ("continuous") probability measures, the spaces $\mathbb{T}$ and F^ω are Borel-isomorphic in the sense that there is a bijection $\phi : \mathbb{T} \twoheadrightarrow F^\omega$ such that the associated bijection $\overline{\phi} : \mathcal{P}(\mathbb{T}) \twoheadrightarrow \mathcal{P}(F^\omega)$ carries the Borel algebra $\mathcal{B}(\mathbb{T})$ onto the Borel algebra $\mathcal{B}(F^\omega)$ in such a way that $\lambda_{F^\omega}(\overline{\phi}(B)) = \lambda_{\mathbb{T}}(B)$ for each $B \in \mathcal{B}(\mathbb{T})$. (See [22, 17.41] or [32, 3.4.23] for a proof of this "Borel isomorphism Theorem for measures".)

Lemma 6.5. *The cardinals $\mathrm{non}(\mathcal{N}(\mathbb{T}))$ and $\mathrm{non}(\mathcal{N}(F^\omega))$ are equal.*

We write

$$\mathrm{non}(\mathcal{N}) := \mathrm{non}(\mathcal{N}(\mathbb{T})) = \mathrm{non}(\mathcal{N}(F^\omega)),$$

a definition justified by Lemma 6.5.

Theorem 6.6. *Let G be a compact Abelian group such that $w(G) \geq \mathrm{non}(\mathcal{N})$. Then G is nondetermined.*

7. Questions

Question 7.1. Is there a compact group G with a countable dense subgroup D such that $w(G) > \omega$ and D determines G?

Question 7.2. If $\{G_i : i \in I\}$ is a set of topological Abelian groups and D_i is a dense determining subgroup of G_i, must $\oplus_{i\in I} D_i$ determine $\Pi_{i\in I} G_i$? In particular, does $\oplus_{i\in I} G_i$ determine $\Pi_{i\in I} G_i$? In particular, does $\oplus_{\mathfrak{c}} \mathbb{R}$ determine $\mathbb{R}^{\mathfrak{c}}$?

Discussion 7.3. Consider the following cardinals:

(a) $\mathfrak{m}_{\mathbb{T}} :=$ the least cardinal κ such that $\mathbb{T}^\kappa$ is nondetermined;
(b) $\mathfrak{m}_{f\exists}$ [resp., $\mathfrak{m}_{f\forall}$] $:=$ the least cardinal κ such that some [resp., each] finite group F has F^κ nondetermined;
(c) $\mathfrak{m}_{c\exists}$ [resp., $\mathfrak{m}_{c\forall}$] $:=$ the least cardinal κ such that some [resp., each] compact abelian group of weight κ is nondetermined;
(d) $\mathfrak{m}_{p\exists}$ [resp., $\mathfrak{m}_{p\forall}$] $:=$ the least cardinal κ such that some [resp., each] product of κ-many compact determined groups is nondetermined.

It follows from Theorems 1.3 and 6.6 that each $\mathfrak{m}_x$, with the possible exception of $\mathfrak{m}_{p\exists}$, satisfies $\aleph_1 \leq \mathfrak{m}_x \leq \mathrm{non}(\mathcal{N})$. Further if $\mathrm{non}(\mathcal{N}) = \aleph_1$, then all seven cardinals $\mathfrak{m}_x$ are equal to $\aleph_1$. The condition $\mathrm{non}(\mathcal{N}) = \aleph_1$ is clearly consistent with CH, and it has been shown to be consistent as well with $\neg$CH (see for example [5], [15] and [20, Example 1, page 568]), so in particular there are models of ZFC + $\neg$CH in which every compact (Abelian) group G satisfies: G is determined if and only if G is metrizable. (Without appealing to the cardinal $\mathrm{non}(\mathcal{N})$, Michael Hrušák [19] in informal conversation suggested the existence of models of ZFC + $\neg$CH in which $\{0,1\}^{\aleph_1}$ is nondetermined.)

Question 7.4. Are the various cardinal numbers $\mathfrak{m}_x$ equal in ZFC? Are they equal to one of the familiar "small cardinals" conventionally noted in the Cichoń diagram (cf. [5], [35])? Is each $\mathfrak{m}_x = \mathrm{non}(\mathcal{N})$? Is each $\mathfrak{m}_x = \aleph_1$? Is each $\mathrm{cf}(\mathfrak{m}_x) > \omega$?

We know of no models of ZFC in which $\mathbb{T}^{\aleph_1}$, or some group of the form $F^{\aleph_1}$ (F finite, $|F| > 1$), is determined, so we are forced to consider the possibility that the following questions have an affirmative answer.

Question 7.5. Are the following (equivalent) statements theorems of ZFC?

(a) The group $\mathbb{T}^{\aleph_1}$ and groups of the form $F^{\aleph_1}$ (F finite, $|F| > 1$) are nondetermined.
(b) A compact abelian group G is determined if and only if G is metrizable.

The following question is suggested by those above.

Question 7.6. Is there in ZFC a cardinal κ such that a compact group G is determined if and only if $w(G) < \kappa$?

Question 7.7. Is it consistent with ZFC that $\mathfrak{m}_{p\exists} = 2$? Is it consistent with ZFC that $\mathfrak{m}_{p\exists} = \omega$?

Question 7.7 has analogues in the context of groups which are not assumed to be compact, as follows.

Question 7.8. In ZFC alone or in augmented axiom systems: Is the product of finitely many determined groups necessarily determined? If G is determined, is $G \times G$ necessarily determined?

References

[1] Ichiro Amemiya and Yukio Kōmura, *Über nicht-vollständige Montelräume*, Math. Ann. **177** (1968), 273–277. MR 38 #508

[2] L. Außenhofer, *Contributions to the duality theory of abelian topological groups and to the theory of nuclear groups*, Ph.D. thesis, Universität Tübingen, 1998.

[3] ———, *Contributions to the duality theory of abelian topological groups and to the theory of nuclear groups*, Dissertationes Math. (Rozprawy Mat.) **384** (1999), 113. MR **2001c:**43003

[4] Wojciech Banaszczyk, *Additive subgroups of topological vector spaces*, Lecture Notes in Mathematics, no. 1466, Springer-Verlag, Berlin, 1991. MR **93b:**46005

[5] Tomek Bartoszyński and Haim Judah, *Set theory*, A K Peters Ltd., Wellesley, MA, 1995. MR **96k:**03002

[6] Nicolas Bourbaki, *Elements of mathematics. General topology. Part 2*, Hermann, Paris, 1966. MR 34 #5044b

[7] M. J. Chasco, *Pontryagin duality for metrizable groups*, Arch. Math. (Basel) **70** (1998), no. 1, 22–28. MR **98k:**22003

[8] Krzysztof Ciesielski, *Set theory for the working mathematician*, Cambridge University Press, Cambridge, 1997. MR **99c:**04001

[9] W. W. Comfort, S. U. Raczkowski, and F. Javier Trigos-Arrieta, *The dual group of a dense subgroup*, Manuscript submitted for publication, 2001.

[10] W. W. Comfort and Dieter Remus, *Pseudocompact refinements of compact group topologies*, Math. Z. **215** (1994), no. 3, 337–346. MR **95f:**54035

[11] W. W. Comfort and K. A. Ross, *Topologies induced by groups of characters*, Fund. Math. **55** (1964), 283–291. MR 30 #183

[12] W. W. Comfort and Victor Saks, *Countably compact groups and finest totally bounded topologies*, Pacific J. Math. **49** (1973), 33–44. MR 51 #8321

[13] W. W. Comfort, F. Javier Trigos-Arrieta, and Ta Sun Wu, *The Bohr compactification, modulo a metrizable subgroup*, Fund. Math. **143** (1993), no. 2, 119–136. MR **94i:**22013

[14] Peter Flor, *Zur Bohr-Konvergenz von Folgen*, Math. Scand. **23** (1968), 169–170 (1969). MR 40 #4685

[15] D. H. Fremlin, *Consequences of Martin's axiom*, Cambridge University Press, Cambridge, 1984. MR **86i:**03001

[16] Irving Glicksberg, *Uniform boundedness for groups*, Canad. J. Math. **14** (1962), 269–276. MR 27 #5856

[17] A. Hajnal and I. Juhász, *Remarks on the cardinality of compact spaces and their Lindelöf subspaces*, Proc. Amer. Math. Soc. **59** (1976), no. 1, 146–148. MR 54 #11263

[18] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations*, Academic Press Inc., Publishers, New York, 1963. MR 28 #158

[19] Michael Hrušák, November 20 2000, Personal Communication.

[20] Thomas Jech, *Set theory*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978. MR **80a:**03062

[21] Samuel Kaplan, *Extensions of the Pontrjagin duality. I. Infinite products*, Duke Math. J. **15** (1948), 649–658. MR 10,233c

[22] Alexander S. Kechris, *Classical descriptive set theory*, Springer-Verlag, New York, 1995. MR **96e:**03057

[23] Yukio Kōmura, *Some examples on linear topological spaces*, Math. Ann. **153** (1964), 150–162. MR 32 #2884

[24] Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, no. 102, North-Holland Publishing Co., Amsterdam, 1980. MR **82f:**03001

[25] V. I. Malyhin and B. È. Šapirovskiĭ, *Martin's axiom, and properties of topological spaces*, Dokl. Akad. Nauk SSSR **213** (1973), 532–535. MR 49 #7985

[26] Sophia U. Raczkowski-Trigos, *Totally Bounded Groups*, Ph.D. thesis, Wesleyan University, Middletown, Connecticut, USA, 1998.

[27] M. Rajagopalan and H. Subrahmanian, *Dense subgroups of locally compact groups*, Colloq. Math. **35** (1976), no. 2, 289–292. MR 54 #5381

[28] G. A. Reid, *On sequential convergence in groups*, Math. Z. **102** (1967), 227–235. MR 36 #3069

[29] Dieter Remus and F. Javier Trigos-Arrieta, *Locally convex spaces as subgroups of products of locally compact abelian groups*, Math. Japon. **46** (1997), no. 2, 217–222. MR **98f:**46003

[30] Helmut H. Schaefer, *Topological vector spaces*, Graduate Texts in Mathematics, no. 3, Springer-Verlag, New York, 1986. MR 49 #7722

[31] Marianne Freundlich Smith, *The Pontrjagin duality theorem in linear spaces*, Ann. of Math. (2) **56** (1952), 248–253. MR 14,183a

[32] S. M. Srivastava, *A course on Borel sets*, Graduate Texts in Mathematics, no. 180, Springer-Verlag, New York, 1998. MR **99d:**04002

[33] Stevo Todorčević, August 2001, Personal Communication.

[34] Eric K. van Douwen, *The maximal totally bounded group topology on G and the biggest minimal G-space, for abelian groups G*, Topology Appl. **34** (1990), no. 1, 69–91. MR **91d:**54044

[35] Jerry E. Vaughan, *Small uncountable cardinals and topology*, Open problems in topology, North-Holland, Amsterdam, 1990, pp. 195–218. MR 1 078 647

[36] André Weil, *Sur les Espaces à Structure Uniforme et sur la Topologie Générale*, Hermann et Cie., Paris, 1938, Actualités Sci. Ind., no. 551 = Publ. Inst. Math. Univ. Strasbourg, 1.

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