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SOME CLASSIFICATION PROBLEM ON WEIL BUNDLES ASSOCIATED TO MONOMIAL WEIL ALGEBRAS

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ABSTRACT. A natural T-function on a natural bundle F is a natural operator transforming vector fields on a manifold M into functions on FM. For a monomial Weil algebra A satisfying $\dim M \geq \mathrm{width}(A) + 1$ we determine all natural T-functions on T^*T^AM, the cotangent bundle to a Weil bundle T^AM.

1. The aim of this paper is the classification of all natural T-functions defined on the cotangent bundle to a Weil bundle T^*T^A corresponding to a monomial Weil algebra A. Roughly speaking, the concept of a monomial Weil algebra denotes an algebra of jets factorized by an ideal generated only by monomial elements. Weil algebras of this kind form a significant class of themselves, since they cover algebras of holonomic and non-holonomic velocities as well as quasivelocities, [11]. The starting point is a general result by Kolář,[4], [5], determining all natural operators $T \to TT^A$ transforming vector fields on manifolds to vector fields on a Weil bundle T^A. Further, partial results of our general problem are solved in [3] and [9]. We follow the basic terminology from [5].

We start from the concept of a natural T-function. For a natural bundle F, a natural T-function f is a natural operator f_M transforming vector fields on a manifold M to functions on FM. The naturality condition reads as follows. For a local diffeomorphism $\varphi : M \to N$ between manifolds M, N and for vector fields X on M and Y on N satisfying $T\varphi \circ X = Y \circ \varphi$ it holds $f_N(Y) \circ F\varphi = f_M(X)$. An absolute natural operator of this kind, i.e. independent of the vector field is called a natural function on F.

There is a related problem of the classification of all natural operators lifting vector fields on m-dimensional manifolds to T^*T^A. The solution of the second problem is given by the solution of the first one as follows ([10]). Natural operators $A_M : TM \to TT^*T^AM$ are in the canonical bijection with natural T-functions $g_M : T^*T^*T^AM \to \mathbb{R}$ linear on fibers of $T^*(T^*T^AM) \to T^*T^AM$. Using natural equivalences $s : TT^* \to T^*T$ by Modugno-Stefani, [7] and $t : TT^* \to T^*T^*$ by Kolář-Radziszewski, [6], we obtain the identification of g_M with natural T-functions $f_M : T^*TT^AM \to \mathbb{R}$ given by $f_M = g_M \circ t_{T^AM} \circ s^{-1}_{T^AM}$. Thus we investigate natural

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T-functions defined on $T^*T^{\mathbb{D}\otimes A}M$ to determine all natural operators $T \to TT^*T^A$, where $\mathbb{D}$ denotes the algebra of dual numbers.

We remind the general result by Kolář, [4], [5]. For a Weil algebra A, the Lie group $\mathcal{A}ut A$ of all algebra automorphisms of A has a Lie algebra $\mathcal{A}ut A$ identified with $\operatorname{Der} A$, the algebra of derivations of A. Thus every $D \in \operatorname{Der} A$ determines a one parameter subgroup $d(t)$ and a vector field D_M on T^AM tangent to $(d(t))_M$. Hence we have an absolute natural operator $\lambda_D : TM \to TT^AM$ defined by $\lambda_D X = D_M$ for any vector field X on M. For a natural bundle F, let $\mathcal{F}$ denote the corresponding flow operator, [5]. Further, let $L_M : A \times TT^AM \to TT^AM$ denote the natural affinor by Koszul, [4], [5]. Then the result by Kolář reads

All natural operators $T \to TT^A$ are of the form $L(c)\mathcal{T}^A + \lambda_D$ for some $c \in A$ and $D \in \operatorname{Der} A$.

Let $\xi : M \to TM$ be a vector field. Kolář in [3] defined an operation $\widetilde{\ }$ transforming a vector field on a manifold M onto a function on T^*M by $\widetilde{\xi}(\omega) = <\xi(p(\omega)), \omega>$, where p is the cotangent bundle projection and $\omega \in T^*M$. One can immediately verify, that for a natural bundle F and a natural operator $A_M : TM \to TFM$ we have a natural T-function $\widetilde{A}_M : T^*FM \to \mathbb{R}$ defined by $\widetilde{A}_M(X) = \widetilde{A_M X}$ for any vector field $X : M \to TM$.

2. In this section, we find all natural T-functions $f_M : T^*T^AM \to \mathbb{R}$ for any manifold M for $m = \dim M \geq \operatorname{width}(A) + 1$. For some cases of A, [11], all natural T-functions in question are of the form

$$h(\widetilde{L(c)\mathcal{T}^A}, \widetilde{\lambda_D})) \qquad c \in C,\ D \in \mathcal{D}$$

where C is a basis of A, $\mathcal{D}$ is a basis of $\operatorname{Der} A$ and h is any smooth function $\mathbb{R}^{\dim A + \dim \operatorname{Der} A} \to \mathbb{R}$. Let $\mathbb{D}^r_k$ denote the algebra of jets $J^r_0(\mathbb{R}^k, \mathbb{R})$. It can be also considered as the algebra of polynomials of variables $\tau_1, \dots, \tau_k$. By [6], any Weil algebra A is obtained as the factor of $\mathbb{D}^r_k$ by an ideal I of itself, i.e. $A = \mathbb{D}^r_k / I$.

The contravariant approach to the definition of a Weil bundle by Morimoto sets $M_A = \operatorname{Hom}(C^\infty(M, \mathbb{R}), A)$ and was studied by many authors as Muriel, Munoz, Rodriguez, Alonso,([1] [8]). The covariant approach (Kolář, [3], [5]) defines T^AM as the space of A-velocities. Let $\varphi, \psi : \mathbb{R}^k \to M$, $\varphi(0) = \psi(0)$. Then φ and ψ are said to be I-equivalent iff for any $germ_x f$, $f : M \to \mathbb{R}$ it holds $germ(f \circ \varphi - f \circ \psi) \in I$. Classes of such an equivalence $j^A\varphi$ are said to be A-velocities. For a smooth map $g : M \to N$ define $T^Ag(j^A\varphi) = j^A(g \circ \varphi)$. Since T^A preserves products, we have $T^A\mathbb{R} = A$, $T^A\mathbb{R}^m = A^m$. The identification $F : M_A \to T^AM$ between those two approaches to the definition of Weil bundle is given by

$$F(j^A\varphi)(f) = j^A(f \circ \varphi) \qquad \text{for any } f \in C^\infty(M, \mathbb{R}) \tag{1}$$

We are going to construct natural T-functions defined on T^*T^A from natural operators $T \to TT^r_k$, since there are some additional ones on T^*T^A, which cannot be constructed from natural operators $T \to TT^A$.

Let $p : \mathbb{D}^r_k \to A$ be the projection homomorphism of Weil algebra inducing the natural transformation $\tilde{p}_M : T^r_k M \to T^A M$. There is a linear map $\iota : A \to \mathbb{D}^r_k$ such that $p \circ \iota = \mathrm{id}_A$. By ι we construct an embedding $T^A M \to T^r_k M$. Consider any $j^A\varphi \in T^A M$ as an element of $\mathrm{Hom}(C^\infty(M,\mathbb{R}), A)$. Then domains of $j^A\varphi \in T^A_{x_0} M$ can be replaced by $J^r_{x_0}(M,\mathbb{R})$. Indeed, for any $f \in C^\infty(M,\mathbb{R})$ it holds $j^A\varphi(f) = j^A(f \circ \varphi) = [germ_{x_0} f \circ germ_0 \varphi]_I$, where $x_0 = \varphi(1)$, $0 \in \mathbb{R}^k$. Since any ideal I in the algebra $E(k)$ of finite codimension contains the r-th power of the maximal ideal of $E(k)$, the last expression can be replaced by $[j^r_0(f \circ \varphi)]_J = j^A\varphi(j^r_{x_0} f)$, where J is an ideal of $\mathbb{D}^r_k$ corresponding to I.

Further, any element $j^r_{x_0} f \in J^r_{x_0}(M,\mathbb{R})$ can be decomposed onto $f(x_0) + j^r_{x_0}(t^{-1}_{f(x_0)} \circ f) = f(x_0) + j^r_{x_0}\tilde{f}$, where $t_y : \mathbb{R} \to \mathbb{R}$ denotes in general a translation mapping 0 onto y. The second expression is an element of the bundle of covelocities of type $(1,r)$, namely an element of $(T^{r*})_{x_0} M = (T^{r*}_1)_{x_0} M$, the bundle of covelocities of type (k,r) being defined as $T^{r*}_k M = J^r(M,\mathbb{R}^k)_0$, [5] .

Select any minimal set of generators $\mathcal{B}_{x_0}$ of the algebra $T^{r*}_{x_0} M$. For any $j^r_{x_0}\tilde{f} \in \mathcal{B}_{x_0}$ define $\tilde{\iota}_{x_0} : T^A_{x_0} M \to (T^r_k)_{x_0} M$ by $(\tilde{\iota}_{x_0}(j^A\varphi))(j^r_{x_0}\tilde{f}) = \tilde{\iota}((j^A\varphi)(j^r_{x_0}\tilde{f}))$. In the second step, $\tilde{\iota}$ can be extended onto the homomorphism $J^r_{x_0}(M,\mathbb{R}) \to \mathbb{D}^r_k$.

We extend the map $\tilde{\iota}_{x_0}$ to $\tilde{\iota} : T^A M \to T^r_k M$. For a general Weil algebra B we show that any element $j^B\varphi \in T^B_{\bar{x}} M$ corresponds bijectively to some element $j^B\varphi_0 \in T^B_{x_0} M$. Indeed, $j^B\varphi(j^r_{\bar{x}} f) = j^B(f \circ \varphi) = j^B(f \circ t^{-1}_{\bar{x}} \circ t_{\bar{x}} \circ \varphi_0) = j^B\varphi_0(j^r_{x_0} f_0)$. This general property extends $\tilde{\iota}_{x_0}$ onto $\tilde{\iota} : T^A M \to T^r_k M$. We proved the following assertion

Proposition 1. *Let $A = \mathbb{D}^r_k / I$ be a Weil algebra, $p : \mathbb{D}^r_k \to A$ the projection homomorphism with its associated natural transformation $\tilde{p} : T^r_k \to T^A$ and $\iota : A \to \mathbb{D}^r_k$ a linear map satisfying $p \circ \iota = \mathrm{id}_A$. For a manifold M and $x_0 \in M$ let $\mathcal{B}_{x_0}$ be a minimal set of generators of the algebra $J^r_{x_0}(M,\mathbb{R})_0 = T^{r*}_{x_0} M$. Then there is an embedding $\tilde{\iota} : T^A M \to T^r_k M$ satisfying $\tilde{p}_M \circ \tilde{\iota} = \mathrm{id}_{T^A M}$ such that $(\tilde{\iota}(j^A\varphi))(j^r_{x_0}\tilde{f}) = \iota((j^A\varphi)(j^r_{x_0}\tilde{f}))$ for any $j^A\varphi \in T^A_{x_0} M$ and $j^r_{x_0}\tilde{f} \in \mathcal{B}_{x_0}$.*

In the following investigations, we limit ourselves to monomial Weil algebras. A Weil algebra $A = \mathbb{D}^r_k / I$ is said to be monomial if I is generated only by monomials. We shall need the coordinate expression of some operators used later for the construction of natural T-functions in question. Thus we introduce coordinates on $T^A M$ and $T^*T^A M$. Consider the polynomial approach to the definition of $\mathbb{D}^r_k$. Then its elements are of the form $\frac{1}{\alpha!} x_\alpha \tau^\alpha$, where $\tau_1, \dots, \tau_k$ are variables and α are multiindices satisfying $0 \le |\alpha| \le r$. Define a linear map $\iota : A \to \mathbb{D}^r_k$ as follows. For τ^α, put $\iota(p(\tau^\alpha)) = 0$ if $\tau^\alpha \in I$ and $\iota(p(\tau^\alpha)) = \tau^\alpha$ otherwise. As a matter of fact, $\iota : A \to \mathbb{D}^r_k$ is a zero section. Similarly as $p : \mathbb{D}^r_k \to A$, the map ι can be extended to $\tilde{\iota} : A^m \to (\mathbb{D}^r_k)^m$ by components. Then it coincides with the map $\tilde{\iota}$

from Proposition 1, if we put $M=\mathbb{R}^m$, choose $x_0\in\mathbb{R}^m$ and substitute $j^r_{x_0}x^i$ for the elements of $\mathcal{B}_{x_0}$, where x^i are canonical coordinates on $\mathbb{R}^m$. Further, define the additional coordinates on T^*T^AM by $p^\alpha_i dx^i_\alpha$.

Let us define operators $T\to TT^A$ by means of $\tilde\iota$ and natural operators $T\to TT^r_k$ as follows. Every natural operator $\lambda: T\to TT^r_k$ defines an operator

$$\Lambda: T\to TT^A \quad \text{by} \quad \Lambda = T\tilde p\circ\lambda\circ\tilde\iota \tag{2}$$

which does not to have be natural and neither does the functions $\widetilde\Lambda: T^*T^A\to\mathbb{R}$ Consider a basis of natural operators $T\to TT^r_k$. The non-absolute natural operators λ together with some of the absolute ones in this basis induce natural operators $\Lambda: T\to TT^A$, while the others will be used for the construction of the additional natural functions defined on T^*T^A.

By general theory, [5], searching for natural T-functions defined on T^*T^A, we are going to investigate G^{r+2}_m-invariant functions defined on $(J^{r+1}T)_0\mathbb{R}^m\times(T^*T^A)_0\mathbb{R}^m$. Therefore we state some assertions, concerning the action of G^{r+2}_m and some of its subgroups on this space. It will be necessary to consider the coordinate expression of this action as well as that of base operators $\Lambda: T\to TT^A$ and their associated functions $\widetilde\Lambda: T^*T^A\to\mathbb{R}$.

Denote by λ^β_j a natural operator $\lambda_{D^\beta_j}$ associated to a derivation of $\mathbb{D}^r_k$ defined by $\tau_i\to\delta^j_i\tau^\beta$ for $j\in\{1,\dots,k\}$ and $1\le|\beta|\le r$. Then we have coordinate forms of λ^β_j and $\widetilde\lambda^\beta_j$, of the same form as those of Λ^β_j and $\widetilde\Lambda^\beta_j$. We have

$$\lambda^\beta_j = \frac{(\alpha+\beta)!}{\alpha!}x^i_\alpha\frac{\partial}{\partial x^i_{\alpha+\beta-\{j\}}}, \qquad \widetilde\lambda^\beta_j = \frac{(\alpha+\beta)!}{\alpha!}x^i_\alpha p_i^{\alpha+\beta-\{j\}} \tag{3}$$

Let k be the width of a monomial Weil algebra A. For $m\ge k$, define an immersion element $i\in T^A_0\mathbb{R}^m$ by $x^i_\alpha=0$ whenever $|\alpha|\ge 2$ and $x^i_j=\delta^i_j$ for $j\in\{1,\dots,k\}$. For general r, k, remind the jet group $G^r_k=\operatorname{inv}J^r_0(\mathbb{R}^k,\mathbb{R}^k)_0$, where inv indicates the invertibility of maps in question. The multiplication in G^r_k is defined by the jet composition. We give the coordinate form of the action of this group on T^*T^A. Let $a^i_{l_1,\dots,l_q}$ denote the canonical coordinates on G^s_m and $\tilde a^i_{l_1,\dots,l_q}$ indicate the inverse. Then the transformation law of the action of G^s_m on $T^A_0\mathbb{R}^m$ is of the form

$$\bar x^i_\alpha = a^i_{l_1\dots l_q}x^{l_1}_{\alpha_1}\dots x^{l_q}_{\alpha_q} \tag{4}$$

for all admissible multiindices α and their decompositions $\alpha_1,\dots,\alpha_q$.

The jet group G^r_k is identified with $\operatorname{Aut}\mathbb{D}^r_k$, the group of automorphisms of the algebra $\mathbb{D}^r_k$, as follows. For $j^r_0g\in G^r_k$ and $j^r_0\varphi\in\mathbb{D}^r_k$ define

$$j^r_0g(j^r_0\varphi) = j^r_0\varphi\circ(j^r_0g)^{-1} \tag{5}$$

Let A be a monomial Weil algebra of width k and height r and $p:\mathbb{D}^r_k\to A$ be the projection homomorphism.

In what follows, we shall consider A as $\mathbb{D}^s_m/(I\cup\{\tau_{k+1},\dots,\tau_m\})$ for $s\ge r$, $m\ge k$ with the properly modified projection $p:\mathbb{D}^s_m\to A$. Consider a group

$G_A = \{j_0^s g \in G_m^s; p \circ j_0^s g = p\}$, [1]. The following lemma characterizes G_A as the stability subgroup of the immersion element i.

Lemma 2. *Let $A = \mathbb{D}_m^s/I$ be a monomial Weil algebra of width k, height r and $\mathrm{St}(i) \subseteq G_m^s$ be the stability subgroup of the immersion element $i \in T_0^A\mathbb{R}^m$ under the canonical left action of G_m^s on $T_0^A\mathbb{R}^m$. Then it holds $G_A = \mathrm{St}(i)$.*

Proof. The formula (4) implies that every element of G_m^s stabilizes i if and only if $a_j^i = \delta_j^i$ for $j \in \{1, \dots, k\}$ and $a_\alpha^i = 0$ whenever $|\alpha| \geq 2$, $\tau^\alpha \notin I$ and $\tau^\alpha \in <\tau_1, \dots, \tau_k>$.

On the other hand, $G_A = \{j_0^s g \in G_m^s; p \circ j_0^s \varphi \circ (j_0^s g)^{-1} = p \circ j_0^s \varphi \ \forall j_0^s \varphi \in \mathbb{D}_m^s\}$. In coordinates, we have

$$\bar{x}_\alpha = x_{l_1, \dots l_q} \tilde{a}_{\alpha_1}^{l_1} \dots \tilde{a}_{\alpha_q}^{l_q} \tag{6}$$

where $\bar{x}_\alpha$ indicates the transformed value of $j_0^s\varphi$ (in coordinates x_α) under an automorphism $j_0^s g$ (with coordinates a_α^i. Substituting an i-th projection pr_i for φ, we obtain $\bar{x}_\alpha = \tilde{a}_\alpha^i$ and consequently $\tilde{a}_j^i = a_j^i = \delta_j^i$ for $j \in \{1, \dots, k\}$ and $\tilde{a}_\alpha^i = a_\alpha^i = 0$ for $|\alpha| \geq 2$, $\tau^\alpha \notin I$ and $\tau^\alpha \in <\tau_1 \dots, \tau_k>$. Thus we have $G_A \subseteq \mathrm{St}(i)$. The converse inclusion is immediately obtained from (6), taking into account the coordinate form of i. It proves our claim. $\square$

We remind the concept of a regular A-point of a Weil bundle M_A. An element $\varphi \in M_A$ is said to be regular (a regular A-point) if and only if its image coincides with A, [1]. Taking into account the identification (1), such a concept can be extended to an A-velocity $j^A\varphi \in T^A M$. Clearly, it is regular if and only if φ is an immersion in $0 \in \mathbb{R}^k$, where k is the width of A. Further, it must hold $\dim M \geq k$. In the case $m = k$ the concept of regularity coincides with that of invertibility. The map $\tilde{\iota}$ from Proposition 1 preserves regularity and thus $\tilde{\iota} : A^k \to \mathbb{R}^k$ can be restricted to $\mathrm{reg}(N^k) \to G_k^r$, where N denotes the nilpotent ideal of A.

Alonso in [1] proved that there is a structure of a fiber bundle on $\mathrm{reg}\, T^A M$ with the standard fiber G_k^r/G_A over a k-dimensional manifold M and therefore $\mathrm{reg}\, T_0^A\mathbb{R}^k$ is identified with G_k^r/G_A. The elements of $\mathrm{reg}(T^A)_0\mathbb{R}^k$ are left classes $j_0^r g G_A$. We extend this assertion of his to m-dimensional manifolds for $m \geq k$. For $\tilde{\iota} : A^m \to (\mathbb{D}_k^r)^m$ corresponding to a Weil algebra of width k we define a map $\tilde{\iota}^* : A^m \to (\mathbb{D}_k^r)^m$ by

$$\tilde{\iota}^*(x_\alpha^i \tau^\alpha) = x_\alpha^i \tau^\alpha + \delta_i^p \tau_p \qquad p \geq k+1 \tag{7}$$

Then we have a lemma, giving the decomposition of any $j_0^r g \in G_m^r$ onto its projection from $\tilde{\iota}^* \circ \tilde{p}(G_m^r)$ and the component in G_A.

Lemma 3. *Let $A = \mathbb{D}_k^r/I$ be a Weil algebra of width k and $j_0^r g \in G_m^r$, $m \geq k$. There is an element $j_0^r h \in G_A$ such that*

$$j_0^r g = \tilde{\iota}^* \circ \tilde{p}(j_0^r g) \circ j_0^r h \tag{8}$$

Proof. The proof of the assertion is done in coordinates and it is based on the iterated application of (4). We do it for only for k, since for $m \geq k$ it is almost the same. Let c^i_γ denote the coordinates of $j^r_0 g$, a^i_γ the coordinates of $\tilde{\imath} \circ \tilde{p}(j^r_0 g)$ and b^i_γ the coordinates of $j^r_0 h$ to be found. Clearly, $a^i_\gamma = c^i_\gamma$ whenever $\tau^\gamma \notin I$. In the first step suppose that α is a minimal multiindex such that $\tau^\alpha \in I$. It follows from (4), that $c^i_\alpha = a^i_l b^l_\alpha$, if we consider the conditions for $j^r_0 h$. The unique solution is given by the invertibility of $j^r_0 g$. Suppose the assertion being proved for $|\alpha| \leq p$. We prove it for $|\alpha| = p+1$. By (4) we have $c^i_\alpha = a^i_{l_1\dots l_s} b^{l_1}_{\alpha_1} \dots b^{l_s}_{\alpha_s} + a^i_l b^l_\alpha$, $s \geq 2$. From the regularity of $j^r_0 g$ we obtain again the unique solution b^l_α, which proves our claim. □

In the proof of the assertion giving the main result, we need to describe the stability group of $j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})$. The transformation laws for the action of G^{r+2}_{m+1} on $(J^{r+1}T)_0\mathbb{R}^m$ has the coordinate expression

$$\bar{X}^i_\alpha = a^i_{l\gamma_1} X^l_{\gamma_2} \tilde{a}^\gamma_\alpha, \tag{9}$$

where X^i_α, $|\alpha| \leq r+1$ denote the canonical coordinates of $j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})$. Further, any multiindex γ including the empty one is decomposed into γ_1, γ_2 and the notation $\tilde{a}^\gamma_\alpha$ denotes the system of all $\tilde{a}^{l_1}_{\alpha_1} \dots \tilde{a}^{l_s}_{\alpha_s}$ for $l_1, \dots, l_s$ forming the multiindex γ and decompositions $\alpha_1, \dots, \alpha_s$ forming α. It follows, that in coordinates any element of G^{r+2}_{m+1} must satisfy $a^i_j = \delta^i_{m+1}$ and $a^i_\alpha = 0$ whenever the multiindex α formed by all $1, \dots, m+1$ contains any $m+1$ for $|\alpha| \geq 2$. To describe the stability group of $j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})$ by terms of Lemma 2 and Lemma 3, denote A^s_{m+1} the Weil algebra of $\mathbb{D}^s_{m+1}/I$ for $I = <\tau_{m+1}\tau^\alpha>$, $|\alpha| \geq 1$. Thus we have proved the following lemma

Lemma 4. *The stability group of $j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})$ in G^{r+2}_{m+1} is of the form $\tilde{\imath}((A^{r+2}_{m+1})^{m+1})$ $\cap G^{r+2}_{m+1}$. Moreover, the stability group of $j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})$ and the immersion element $i \in T^A_0\mathbb{R}^{m+1}$ is of the form $G_{A;m+1} = G_A \cap \tilde{\imath}((A^{r+2}_{m+1})^{m+1})$.*

Let us consider the base $\widetilde{\mathcal{B}}$ of all T-functions $\widetilde{\Lambda}$ defined on T^*T^A (not natural in general), constructed from the non-absolute natural operators $L(\tau^\alpha)\mathcal{T}^A$ and from the absolute operators Λ^β_j with the coordinate expression given by (3). Let $\widetilde{\mathcal{B}}_1$ denote the subbasis of $\widetilde{\mathcal{B}}$ formed by natural operators $T \to TT^A$. It follows from Lemma 3, that any element $j^A g \in \operatorname{reg} T^A M$ is identified with $\tilde{\imath}^*(j^A g) \in G^{r+1}_{m+1}$, the only representative of the left class $j^A g G_A$ in the sense of Lemma 3. Therefore we have

$$i = l((\tilde{\imath}^*(j^A g))^{-1}, j^A g) \tag{10}$$

where l is the symbol for the left action of G^{r+1}_{m+1} on $T^A_0\mathbb{R}^{m+1}$ to be used also for the action of this group on $(J^{r+1}T)_0\mathbb{R}^{m+1} \times (T^*T^A)_0\mathbb{R}^{m+1}$. Let us define a map $\operatorname{Imm} : T^*(\operatorname{reg} T^A)_0\mathbb{R}^{m+1} \to (T^*_i T^A)_0\mathbb{R}^{m+1}$ as follows

$$\operatorname{Imm}(w) = l((\tilde{\imath}^*(q(w)))^{-1}, w), \tag{11}$$

$w \in T^*(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$.

Proposition 5. *Let A be a monomial Weil algebra and $(T^*(\operatorname{reg} T^A))_0\mathbb{R}^{m+1} \to (\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$ be the restriction of the natural bundle $T^*T^A\mathbb{R}^{m+1} \to T^A\mathbb{R}^{m+1}$ to the opened submanifold $(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$. Then all operators from $\widetilde{\mathcal{B}} - \widetilde{\mathcal{B}}_0$ are G^{r+2}_{m+1}-invariant in respect to the map* Imm.

Proof. We prove the assertion from the transformation laws of the action of G^{r+2}_{m+1} on $(J^{r+1}T)_0\mathbb{R}^{m+1} \times (T^*T^A)_0\mathbb{R}^{m+1}$. We complete them for p^α_j. Denote $\gamma = \alpha - \{j\}$ the multiindex from (3). Then we have

$$\bar p^\beta_j = \frac{(\beta+\gamma)!}{\beta!\gamma_1!\dots\gamma_s!}\tilde a^l_{jl_1\dots l_s}\bar x^{l_1}_{\gamma_1}\dots \bar x^{l_s}_{\gamma_s}p^{\beta\gamma}_j \tag{12}$$

when the sum is made for all decompositions $\gamma_1,\dots,\gamma_s$ of multiindices γ. The formula is obtained from (3) and the standard combinatorics. To accent $\operatorname{Imm}(w)$ as a transformed value for any $w \in T^*(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$, use $\bar p^\alpha_i$ for the additional coordinates (obviously, the coordinates $\bar x^i_\alpha$ coincide with those of i). Then we have $\widetilde\Lambda^\beta_j(\operatorname{Imm}(w)) = \widetilde\Lambda^\beta_j(\bar x^i_\alpha)\bar p_i^{\alpha+\beta-\{j\}} = \beta!\bar p^\beta_j = \beta!\frac{(\beta+\gamma)!}{\beta!\gamma!}\tilde a^i_{j\gamma}p^{\beta\gamma}_i$ if we put $\gamma = \alpha - \{j\}$, which follows from (12). If we consider the coordinate expression of $\tilde\imath(A^{m+1})$ and the formula (3), we obtain that the last expression coincides with $\frac{(\beta+\gamma)!}{\gamma!}x^i_{j\gamma}p^{\beta\gamma}_i = \frac{\alpha_j}{\alpha_j+\beta_j}\frac{(\alpha+\beta)!}{\alpha!}x^i_\alpha p_i^{\alpha+\beta-j} = \widetilde\Lambda(x^i_\alpha, p^\alpha_i) = \widetilde\Lambda(w)$. It proves our claim. $\square$

The following lemma specifies a certain class of functions, among which all investigated ones must be contained.

Lemma 6. *Let $m \geq k$. Then every natural T-function $f : T^*T^A\mathbb{R}^{m+1} \to \mathbb{R}$ is of the form $h(\widetilde{L(\tau^\alpha)}\mathcal{T}^A, \widetilde\Lambda^\beta_j)$ for some smooth function h of the suitable type.*

Proof. By general theory, we are searching for all G^{r+2}_{m+1}-invariant functions defined on $(J^{r+1}T)_0\mathbb{R}^{m+1} \times (T^*T^A)_0\mathbb{R}^{m+1}$. Let $w \in (T^*T^A)_0\mathbb{R}^{m+1}$ and x^i_α denote the coordinates of $q(w)$, $q : T^*T^A \to T^A$ being the cotangent bundle projection. By a general lemma from [5], Chapter VI, the natural T-function must satisfy $f(j^{r+1}_0X, w) = h(X^i_\gamma p^\beta_i, x^i_\alpha p^\beta_i)$ for any non-zero j^{r+1}_0X of a vector field X on $\mathbb{R}^{m+1}$. The coordinates used in the recent identity coincide with those defined before Lemma 2. The last expression can be considered in the form $h(\widetilde{L(\tau^\alpha)}\mathcal{T}^A, X^i_\gamma p^\beta_i, \widetilde\Lambda^\beta_j,\ x^i_\delta p^\beta_i)$ for $|\beta| \geq 0$, $|\gamma| \geq 1$ and $|\delta| \geq 2$. Identify $q(w)$ with j^Ag for any $w \in T^*(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$, i.e. $q(w) = l(\tilde\imath(j^Ag), i)$ and put $j^{r+1}_0Y = l(\tilde\imath(j^Ag)^{-1}, j^{r+1}_0Y)$. Then $f(j^{r+1}_0X, w) = h(\widetilde{L(\tau^\alpha)}\mathcal{T}^A, Y^i_\gamma\bar p^\beta_i, \widetilde\Lambda^\beta_j, 0, p^0_i)$ for $|\gamma| \geq 1$ and $i \in \{1,\dots,k\}$. Here $\bar p^\beta_i$ indicate the transformed values of p^β_i under the map Imm. The last identity follows from Proposition 5. Further, there is $j^{r+2}_0g \in G_A \cap G_{A^{r+2}_{m+1}}$ such that $l(j^{r+1}_0g, j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})) = j^{r+1}_0Y$. Then we have $f(j^{r+1}_0X, w) =$

$h(\widetilde{L(\tau^\alpha)\mathcal{T}^A}, 0, \widetilde{\Lambda}^\beta_j, 0, p^0_i)$ for $i \in \{1, \dots, k\}$. The excessive coordinates p^0_i are annihilated by an element of $\operatorname{Ker} \pi^{r+1}_r \cap \tilde{\iota}((A^{r+2}_{m+1})^{m+1})$, namely by an element satisfying in coordinates $a^i_\alpha = 0$ except of $\alpha = \underbrace{(i, \dots, i)}_{(r+1)-\text{times}}$. Such an element stabilizes $j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})$ as well as i, which completes the proof. □

Searching for all natural T-functions $T^*T^A\mathbb{R}^{m+1} \to \mathbb{R}$ among those from Lemma 6, we state the basis $\mathcal{B}$ of functions defined on $T^*_i T^A \mathbb{R}^{m+1}$ and identify it with $\widetilde{\mathcal{B}}$. By general theory, [5], every natural T-function in question is determined by its value over $j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})$ on $(T^*T^A)_0\mathbb{R}^{m+1}$. Further, it follows from Lemma 4 and the formula (11) that the map Imm stabilizes $j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})$ in the following sense. For any $w \in T^*(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$ the action of $\tilde{\iota}(q(w))$ on $(J^{r+1}T)_0\mathbb{R}^{m+1}$ stabilizes $j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})$.

Set $\mathcal{B}$ the basis of functions defined on $T^*_i T^A\mathbb{R}^{m+1}$ obtained by the restriction of $\widetilde{\mathcal{B}}$ over $j^{r+1}_0(\frac{\partial}{\partial x^{m+1}})$ onto $T^*_i T^A\mathbb{R}^{m+1}$. Conversely, $\mathcal{B}$ determines $\widetilde{\mathcal{B}}$ by

$$\widetilde{\mathcal{B}}(j^{r+1}_0(\frac{\partial}{\partial x^{m+1}}), w) = \mathcal{B} \circ \operatorname{Imm}(w) \tag{13}$$

Analogously, we construct $\mathcal{B}_1$ from $\widetilde{\mathcal{B}_1}$. Moreover, for any $w \in T^*_i(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$, the values formed by $\mathcal{B}(w)$ coincide with the coordinates p^β_j of w defined before (2) for $j = 1, \dots, k$ except that of p^0_j for the absolute functions and p^β_{m+1} for the non-absolute ones. Thus any base T-function of $\mathcal{B}$ defined on $T^*_i(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$ corresponds to some projection $\operatorname{pr}^\beta_j : T^*_i(\operatorname{reg} T^A)_0\mathbb{R}^{m+1} \to \mathbb{R}$. It follows from Lemma 4 and the fact that $\widetilde{L(\tau^\alpha)\mathcal{T}^A}$ are natural that all natural T-functions $(T^*T^A)\mathbb{R}^{m+1} \to \mathbb{R}$ from Lemma 6 are in the canonical bijection with G_A-invariant functions defined on $T^*_i T^A\mathbb{R}^{m+1}$ which are of the form $h(\widetilde{L(\tau^\alpha)\mathcal{T}^A})(\widetilde{\Lambda}^\beta_j)$ for $\widetilde{\Lambda}^\beta_j : T^*_i T^A\mathbb{R}^{m+1} \to \mathbb{R}$. Using coordinates, we find all G_A-invariants of p^β_j, $j \in \{1, \dots, k\}$, $|\beta| \geq 1$. Then we identify the functions $h(\widetilde{L(\tau^\alpha)\mathcal{T}^A})(p^\beta_j)$ with $h(\widetilde{L(\tau^\alpha)\mathcal{T}^A})(\widetilde{\Lambda}^\beta_j)$ and by (12), we obtain all natural T-functions on $T^*T^A\mathbb{R}^{m+1}$.

This way we have deduced that our problem can be reduced to the problem of searching for all G_A-invariant functions defined on $T^*_i T^A\mathbb{R}^{m+1}$ which can be identified with a smooth function $h : \mathbb{R}^N \to \mathbb{R}$ for a suitable integer N. The coordinate expression of the action of G_A on $T^*_i T^A\mathbb{R}^{m+1}$ is induced by (12) and it is of the form

$$\bar{p}^\beta_j = p^\beta_j - C(\beta+\gamma, \beta) a^l_{j\gamma} p^{\beta\gamma}_l \quad \text{for} \quad \tau_j\tau^\gamma \in I \quad \text{and} \quad \tau^\beta\tau^\gamma \notin I \tag{14}$$

where C indicates the multicombinatorial number. Clearly, $T^*_i T^A\mathbb{R}^{m+1}$ is identified with the space R^N endowed with the action (14) of G_A. We are going to investigate $G_A \cap G^r_{m+1}$-orbits on R^N, since only p^0_j depend on B^{r+1}_{m+1} and they can be annihilated by this subgroup. For those orbits, we construct all functions distinguishing them and then we express the corresponding invariants by terms of elements from $\widetilde{\mathcal{B}}$.

The following assertion describes an important property of $(G_A \cap \operatorname{Ker} \pi_s^r)$-orbits to bee necessary in the proof of the main result. Denote by $\mathcal{B}_s \subseteq \mathcal{B}$ the set of all $(G_A \cap \operatorname{Ker} \pi_s^r)$-invariants selected from $\mathcal{B}$ and denote by N_s the number of elements in $\mathcal{B}_s$. Clearly, $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots \subseteq \mathcal{B}_{r-1} \subseteq \mathcal{B}_r$. Further, denote $\mathcal{B}_t^s = \mathcal{B}_s - \mathcal{B}_t$ and $N_t^s = N_s - N_t$. Then we have

Proposition 7. *Let $w \in \mathbb{R}^N$ and $\mathcal{O}rb_s(w)$ be its $(G_A \cap \operatorname{Ker} \pi_s^r)$-orbit. Then $\mathcal{B}_s^{s+1}(\mathcal{O}rb_s(w))$ has the structure of an affine subspace of $R^{N_s^{s+1}}$, the modelling vector space of which being determined by the formula* (14) *restricted to $B_{m+1}^{s+1} \cap G_A$.*

Proof. is done directly applying the formula (14). Let w_1 and w_2 be elements of $\mathcal{B}_s^{s+1}(\mathcal{O}rb(w))$. Then w_1 can be achieved from w by the action of an element of $B_{m+1}^{s+1} \cap G_A$. The coordinate expression of such a transformation is given by $\bar{p}_j^\beta = p_j^\beta - C(\beta+\gamma,\beta) a_{j\gamma}^l p_l^{\beta\gamma}$. Analogously for w_1 and w_2, we have $\bar{\bar{p}}_j^\beta = \bar{p}_j^\beta - C(\beta+\gamma,\beta) b_{j\gamma}^l \bar{p}_l^{\beta\gamma}$. Then $\bar{\bar{p}}_j^\beta = p_j^\beta - (a_{j\gamma}^l + b_{j\gamma}^l) p_l^{\beta\gamma}$, which follows $\overrightarrow{ww_2} = \overrightarrow{ww_1} + \overrightarrow{w_1w_2}$. It proves our claim. ☐

In what follows, we construct a basis $\widetilde{\mathcal{D}}$ of natural functions from $\widetilde{\mathcal{B}}$. The construction is given by a procedure, generating step by step a base of G_A-invariants determining the base of natural functions. We start the procedure selecting elements of $\mathcal{B}_1$ and put $\widetilde{\mathcal{D}}_1 = \widetilde{\mathcal{B}}_1$. For any $w \in T_i^* T^A \mathbb{R}^{m+1}$, consider its orbit $\mathcal{O}rb(w) = \mathcal{O}rb_1(w)$.

In the second step, consider $\mathcal{B}_1^2(\mathcal{O}rb_1(w))$, which is by Proposition 7 a k_2-dimensional affine subspace of the affine space $\mathbb{R}^{N_1^2}$ for some $k_2 \le N_1^2$. For almost every G_A-orbit in the sense of density, such an affine subspace contains a unique point I^{C_2} satisfying $\operatorname{pr}_j(I^{C_2}) = 0$ for $j \in C_2$. The remaining components of I^{C_2} determine G_A-invariants $I_1^{C_2}, \dots, I_{N_1^2-k_2}^{C_2}$ identified with natural functions $\widetilde{I}_1^{C_2}, \dots, \widetilde{I}_{N_1^2-k_2}^{C_2}$.

In order to express them in formulas, we notice the following property of $\mathcal{B}_s^{s+1}(\mathcal{O}rb_s(w))$ for any $s = 1, \dots, r-1$. Proposition 7 and its proof imply that if an element of $\mathcal{B}_s^{s+1}(\mathcal{O}rb_s(w))$ is stabilized by $j_0^{s+1} g \in B_{m+1}^{s+1}$ under the canonical left action then the whole $\mathcal{B}_s^{s+1}(\mathcal{O}rb_s(w))$ is stabilized. Denote $\mathrm{St}_{s;m+1}^{s+1} \subseteq G_A \cap B_{m+1}^{s+1}$ the stability group of $\mathcal{B}_s^{s+1}(\mathcal{O}rb_s(w))$. One can easily deduce that $\mathrm{St}_{s;m+1}^{s+1}$ satisfies the stability property of this kind for almost every $w \in \mathbb{R}^N$. Clearly, $\mathrm{St}_{s;m+1}^{s+1}$ is a closed and normal subgroup of $G_A \cap B_{m+1}^{s+1}$ and thus $H_{s;m+1}^{s+1} = G_A \cap B_{m+1}^{s+1} / \mathrm{St}_{s;m+1}^{s+1}$ is a Lie group. It follows the existence of a section $\sigma_{s+1;m+1} : H_{s;m+1}^{s+1} \to G_A \cap B_{m+1}^{s+1}$.

Hence for any $w \in \mathbb{R}^N$ we have a unique $j_0^2 h \in \sigma_{2;m+1}(H_{1;m+1}^2) \simeq H_{1;m+1}^2$ such that $\mathcal{B}_1^2(l(j_0^2 h, w)) = I^{C_2}(w)$. Thus we have a map $\alpha_{C_2} : \mathbb{R}^N \to H_{1;m+1}^2$. Therefore, any $w \in T_i^* T^A \mathbb{R}^{m+1}$ is transformed onto

$$l(\alpha_{C_2}(w), w) = l_{\alpha_{C_2}}(w) = (I_{j_s}^{C_2}(w)), \qquad s = 1, \dots N_1^2 - k_2, j_s \notin C_s \tag{15}$$

Applying the identification (13), we obtain $\widetilde{I}_1^{C_2}, \dots, \widetilde{I}_{N_1^2-k_2}^{C_2}$ and put $\widetilde{\mathcal{D}}_2 = \widetilde{\mathcal{D}}_1 \cup \{\widetilde{I}_1^{C_2}, \dots, \widetilde{I}_{N_1^2-k_2}^{C_2}\}$.

In the $(s+1)$-th step of the procedure we come out from the basis $\widetilde{\mathcal{D}}_s$ of natural functions and an element $w_s = l_{\alpha_{C_s}} \circ \dots \circ l_{\alpha_{C_2}}(w) \in \mathcal{O}rb_1(w)$ instead w from the second step. By Proposition 7, $\mathcal{B}_s^{s+1}(\mathcal{O}rb_s(w_s))$ is an affine subspace of dimension k_{s+1} of $\mathbb{R}^{N_s^{s+1}}$ for some k_{s+1}. Select $C_{s+1} \subseteq \{1, \dots, N_s^{s+1}\}$. For almost every $w_s \in T_i^* T^A \mathbb{R}^{m+1}$ there is a unique point $I^{C_{s+1}}(w_s) = I^{C_{s+1}}(\mathcal{B}_s^{s+1}(\mathcal{O}rb_s(w_s))$ such that $\mathrm{pr}_j \circ I^{C_{s+1}} = 0$ for $j \in C_{s+1}$. The remaining components of $I^{C_{s+1}}$ determine analogously to the second step of the procedure G_A-invariants and by (13) natural functions $\widetilde{I}_{l_{s+1}}^{C_{s+1}, \dots, C_2}$ for $l_{s+1} \notin C_{s+1}$. Analogously to the second step, for any w_s under discussion there is a unique element $j_0^{s+1} h \in \sigma_{s+1;m+1}(H_{s;m+1}^{s+1})$ such that $l(j_0^{s+1} h, \mathcal{B}_s^{s+1}(w_s)) = I^{C_{s+1}}(w_s)$. Hence we have a map $\alpha_{C_s} : \mathbb{R}^N \to \sigma_{s+1;m+1}(H_{s;m+1}^{s+1})$ such that $l(\alpha_{C_{s+1}}(w_s), w_s) = I^{C_{s+1}}(w_s) = l_{\alpha_{C_{s+1}}} \circ \dots \circ l_{\alpha_{C_2}}(w) = \widetilde{I}^{C_{s+1}, \dots, C_2}(w)$ taking into account the identification (13). Hence we obtained the basis $\widetilde{\mathcal{D}}_{s+1} = \widetilde{\mathcal{D}}_s \cup \{\widetilde{I}_{l_{s+1}}^{C_{s+1}, \dots, C_2}; l_{s+1} \notin C_{s+1}\}$. We proved the main result, given by the following Proposition

Proposition 8. *Let $A = \mathbb{D}_k^r / I$ be a monomial Weil algebra of width k, $\dim M = m \geq k+1$. Let $\tilde{\imath} : T^A M \to T_k^r M$ be an embedding described in Proposition 1. Consider a basis C of A and a basis $\mathcal{B}_0$ of $\mathrm{Der}(\mathbb{D}_k^r)$. Further, let $\widetilde{\mathcal{B}}$ be a basis of functions defined on $T^* T^A M$ constructed from operators $T\tilde{p} \circ \lambda_D \circ \tilde{\imath}$ by the operation $\widetilde{\ }$ defined in the very end of Section 1, $D \in \mathcal{B}_0$. Then all natural T-functions $f_M : T^* T^A M \to \mathbb{R}$ are of the form*

$$h(\widetilde{L_M(c)} \mathcal{T}_M^A, \widetilde{I}_{l_1}, \widetilde{I}_{l_2}^{C_2}, \dots, \widetilde{I}_{l_s}^{C_s, \dots, C_2})$$

where h is any smooth function of a suitable type, $\widetilde{I}_{l_1}$ are natural functions selected directly from $\widetilde{\mathcal{B}}$ and $\widetilde{I}_{l_s}^{C_s, \dots, C_2}$ $(l_s \notin C_s)$ are obtained by the procedure.

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