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GEODESIC VECTORS OF THE SIX-DIMENSIONAL SPACES

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ABSTRACT. The result of this article is close to the article of O. Kowalski and S.Ž. Nikcevič: "On geodesic graphs of Riemannian g.o. spaces", but it works with an other method. There as problem arises the investigation of the structure of geodesic graphs in the framework of the general theory of g.o. spaces by Carolyn Gordon. We investigate now the geodesic graphs of the six-dimensional spaces according to the theory of g.o. spaces by Gordon.

1. INTRODUCTION

Let M be a connected homogeneous Riemannian manifold. If G is any connected transitive group of isometries of M and H is the isotropy subgroup at a point , then M is naturally identified with the coset space G/H with a left-invariant Riemannian metric.

Remark 1.1. The Lie algebra $\mathfrak{h}$ of H is compactly embedded in $\mathfrak{g}$; i.e., g admits an inner product relative to which $\mathrm{ad}\,(X)$ is skew-symmetric for all $X \in \mathfrak{h}$. In particular, we can choose a complement q of $\mathfrak{h}$ in $\mathfrak{g}$ with $\mathrm{ad}\,(H)q \subseteq q$. The space q is identified with the tangent space via the mapping $X \longrightarrow \frac{d}{dt}|_{t=0}\exp\,(tX)\cdot p$ and the Riemannian structure defines an $\mathrm{ad}\,(H)$-invariant inner product $\langle , \rangle$ on q.

Definition 1.2. (M, g) is said to be *naturally reductive* if for some transitive connected group G of isometries and decomposition $\mathfrak{g} = \mathfrak{h} + q$ as above, $\mathrm{ad}\,(X)$ is skew-symmetric for all $X \in q$. I.e., $\langle [X, Y]_q, Z\rangle = -\langle Y, [X, Z]_q\rangle$ for all $X, Y, Z \in q$, where the subscript q indicates the corresponding projection.

Remark 1.3. It is well known that a Riemannian homogeneous space $(M, g) = G/H$ with origin $p = \{H\}$ and with an $\mathrm{ad}\,(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + q$ is naturally reductive (with respect to this decomposition) if and only if the following holds: For any vector $X \in q\backslash\{0\}$,the curve $\gamma(t) = \exp\,(tX)p$ is a geodesic with respect to the Riemannian connection.

Definition 1.4. M is said to be *geodesic orbit (g. o.) space* if every geodesic in M is an orbit of a one-parameter group of isometries. I.e., there exists a transitive group G of isometries such that every geodesic in M is of the form $\exp\,(tX)\cdot p$ with $X \in \mathfrak{g}$, $p \in M$.

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Remark 1.5. The naturally reductive homogeneous spaces form a subclass of the g.o. manifolds, because the g.o. condition is considerably weaker than natural reductivity. There are Riemannian g.o. spaces which are never naturally reductive in any group extension. One class of such spaces consists of two-step nilpotent Lie-groups with two-dimensional centers, equipped with special left invariant metrics. Our paper is devoted to the investigation of these spaces.

All the calculations for a g.o. space G/H can be reduced to algebraic computations using geodesic vectors, which are defined in the following way:

Definition 1.6. Fix a choice of base point $p \in M$. For G a transitive group of isometries of M, we will say a nonzero element X of g is a *geodesic vector* if $\exp(tX) \cdot p$ is a geodesic.

The following criterion for geodesic vectors is important. See [3].

Lemma 1.7. *Let M be a connected homogeneous Riemannian manifold and G a transitive group of isometries. A nonzero element X of $\mathfrak{g}$ is a geodesic vector if and only if $\langle [X, Y]_q, X_q \rangle = 0$ for all $Y \in q$.*

The following proposition is consequence of Lemma 1.7. and the definitons above.

Proposition 1.8. *a.) M is naturally reductive with respective to the transitive group G of isometries and decomposition $\mathfrak{g} = \mathfrak{h} + q$ if and only if every nonzero element of q is a geodesic vector.b.) Every geodesic in M is an orbit of a one-parameter group of isometries of G if and if and only if for each $X \in q$, there exists $A \in \mathfrak{h}$ such that $\langle [X + A, Y]_q, X \rangle = 0$ for all $Y \in q$. (I.e., $X + A$ is a geodesic vector.) In particular, M is a g.o. manifold if and only if this condition holds for $G = I_0(M)$, where $I_0(M)$ is the identity component of the full isometry group of M.*

2. G.o. nilmanifolds

A connected Riemannian manifold which admits a transitive nilpotent group N of isometries is called a *nilmanifold.* The action of N is a neccesarily simply transitive [1], thus the manifold may be identified with the group N endowed with a left-invariant metric. We say N is a *g.o. nilmanifold* if every geodesic is an orbit of a one-parameter subgroup of G.

We may restrict our attention to two-step nilmanifolds.

All two-step homogeneous nilmanifolds can be constructed in the following way:

Let $(\mathfrak{a}, \langle .,. \rangle_{\mathfrak{a}})$ and $(\mathfrak{z}, \langle .,. \rangle_{\mathfrak{z}})$ be inner product spaces and $j : \mathfrak{z} \longrightarrow \mathrm{so}(\mathfrak{a})$ an injective linear map. Let $(\mathfrak{n}, \langle .,. \rangle)$ be the direct sum of $(\mathfrak{a}, \langle .,. \rangle_{\mathfrak{a}})$ and $(\mathfrak{z}, \langle .,. \rangle_{\mathfrak{z}})$. The skew-symmetric bilinear map (Lie-bracket): $[.,.] : \mathfrak{n} \times \mathfrak{n} \longrightarrow \mathfrak{z}$ defined by $\langle j(Z)X, Y \rangle = \langle Z, [X, Y] \rangle$ for all $X, Y \in \mathfrak{a}$ and $Z \in \mathfrak{z}$; with the conditions

(i) $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{z}$

(ii) $[\mathfrak{n}, \mathfrak{z}] = 0,$

defines a Lie algebra structure on $\mathfrak{n}$. Then let N be the associated simply-connected Lie group with the left-invariant Riemannian metric defined by the inner product $\langle , \rangle$. We will say $(\mathfrak{a}, \mathfrak{z}, j)$ is the data triple associated with the simply-connected Riemannian nilmanifold N.

Since $[\mathfrak{n}\ , \mathfrak{n}\ ] = [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{z}$ and $[\mathfrak{n}\ , \mathfrak{z}] = 0$ N is a Riemannian two-step nilmanifold.

Proposition 2.1. [1] *Let N be a simply-connected Riemannian nilmanifold. The Lie algebra of the full isometry group is a semi-direct sum $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$ where the isotropy algebra $\mathfrak{h}$ consists of all derivations of $\mathfrak{n}$ which are skew-symmetric relative to the Riemannian inner product.*

To describe the full isometry group of N, we need only find the skew-symmetric derivations of $\mathfrak{n}$ and then apply Proposition 2.1.

Lemma 2.2. [1] *Let N be a simply-connected two-step nilmanifold and $(\mathfrak{a}, \mathfrak{z}, j)$ the associated data triple. Then a skew-symmetric linear map $D : \mathfrak{n} \longrightarrow \mathfrak{n}$ is a derivation if and only if the following conditions hold:*

(i) D leaves each of $\mathfrak{a}$ and $\mathfrak{z}$ invariant and

(ii)$j(D(Z)) = [D_{|\mathfrak{a}}, j(Z)]$ for all $Z \in \mathfrak{z}$ where the bracket is that of $so(\mathfrak{a})$.Thus the isotropy algebra of $\mathfrak{h}$ is isomorphic to the subalgebra of $so(\mathfrak{a})$ given by $\{\delta \in so(\mathfrak{a}) : [\delta, j(\mathfrak{z})] \subset j(\mathfrak{z})$ and $j^{-1} \circ ad(\delta) \circ j \in so(\mathfrak{z})\}$.

Theorem 2.3. [1] *Let N be a simply-connected 2-step nilmanifold. In the notation as above, N is a g.o. manifold if and only if for each $X \in \mathfrak{a}$ and $Z \in \mathfrak{z}$, there exists a skew-symmetric derivation D of $\mathfrak{n}$ such that $D(Z) = 0$ and $D(X) = j(Z)X$.*

Proof. Let $D \in \mathfrak{h}$ and $U \in \mathfrak{n}$. Depending on whether we are viewing D as an element of the Lie-algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$ or as a derivation of $\mathfrak{n}$, we will write, respectively $[D, U]$ or $D(U)$ for the action of D on $U \in \mathfrak{n}$.Given $X \in \mathfrak{a}$, $Z \in \mathfrak{z}$, and D $\in \mathfrak{h}$, then by Lemma 1.7., $X + Z + D$ is a geodesic vector if and only if the following holds:
$\langle [X + Z + D, Y]_{\mathfrak{n}}, X_{\mathfrak{n}} \rangle = 0$ for all Y $\in \mathfrak{n} = \mathfrak{a} + \mathfrak{z}$

(i) for all U $\in \mathfrak{a}$, we have

$$0 = \langle [X + Z + D, U], X + Z\rangle = \langle [X, U], X + Z\rangle + \langle [Z, U], X + Z\rangle + \langle [D, U], X + Z\rangle =$$

$$= \langle [X, U], Z\rangle + \langle D(U), X\rangle = \langle j(Z)X - D(X), U\rangle \Longrightarrow D(X) = j(Z)X$$

(ii) for all W $\in \mathfrak{z}$, we have

$$0 = \langle [X + Z + D, W], X + Z\rangle = \langle [D, W], Z\rangle = -\langle D(Z), W\rangle \Longrightarrow D(Z) = 0$$

□

3. Szenthe-construction

Kowalski and Vanhecke [3] classified the g.o. nilmanifolds using the results of J.Szenthe [4]. In the following we give an analog characterisation for g.o. nilmanifolds, because we use this method by the case if we have $X + Z \in \mathfrak{n}$, so that $X = 0 \in \mathfrak{a}$ and $Z \in \mathfrak{z}$.

Let $N = G/H$ be a reductive homogeneous space with an ad(H)- invariant decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$. For any $X \in \mathfrak{n} \setminus \{0\}$, put

$q_x = \{A \in \mathfrak{h} : [A, X] = \lambda X$ for some $\lambda \in \mathbf{R}\}$,

$N_x = \{B \in \mathfrak{h} : [B, A] \in q_x$ all $A \in q_x\}$,

$c'_x = \{B \in \mathfrak{h} : [B, A] = 0$ for all $A \in q_x\}$.

Thus N_x is the normalizer of q_x in $\mathfrak{h}$ and $c\prime_x$ is the centralizer of q_x in $\mathfrak{h}$. Obviously $q_x \subset N_x$ and $c'_x \subset N_x$. In the Riemannian situation the following holds.

Proposition 3.1. [3] *If N is the Riemannian connection of $(N, g) = G/H$, then for any $X \in \mathfrak{n} \setminus \{0\}, q_x$ is a subalgebra of $\mathfrak{h}$ given by $q_x = \{A \in \mathfrak{h} : [A, X] = 0\}$*

Proposition 3.2. [3] *Let (N, g) be a Riemannian g.o. space. Then for each $X \in \mathfrak{n}\setminus\{0\}$, there is at least one element $A \in \mathfrak{h}$ such that $X + A$ is a geodesic vector. We always have $A \in N_x$.*

Corollary 3.3. [3] *Let $X \in \mathfrak{g}\setminus\{0\}$ be a geodesic vector and $A \in \mathfrak{h}$. Then the vector $X + A$ is geodesic if and only if $[A, X_n] = 0$*

Proposition 3.4. [3] *Let G/H be a Riemannian g.o. space. Then for each $X \in \mathfrak{n}\setminus\{0\}$, there is an element $B \in c'_x$ such that $X + B$ is a geodesic vector.*

Proof. (from [3], based on an idea from [4]) Consider an ad(H) - invariant scalar product $\langle,\rangle$on $\mathfrak{h}$. Put $N_x = q_x + c_x$ (the orthogonal decomposition with respect to $\langle,\rangle$). We first prove that $c_x \subset c'_x$.

Let $B \in c_x$ and $A \in q_x$. First, $[B, A] \in q_x$ because $B \in N_x$. Because the scalar product $\langle,\rangle$ is ad(H) - invariant, we get $\langle[B, A], [B, A]\rangle = \langle B, [A, [B, A]]\rangle$.

The last expression is zero because $B \in c_x$ and $[A, [B, A]] \in q_x$; hence $[B, A] = 0$. Because $A \in q_x$ was arbitrary, we obtain $B \in c'_x$ as required.Now if $X + A$ is a geodesic vector, $A \in N_x$, we can write $A = A_1 + A_2$, where $A_1 \in q_x$, $A_2 \in c_x$. According to Proposition 3.1. and Corollary 3.3., $X + A_2$ is a geodesic vector, as well. Now $A_2 \in c'_x$. ∎

Proposition 3.5. [4] *There exists an ad(H)-invariant map $\xi : \mathfrak{n} \longrightarrow \mathfrak{h}$ such that for any $X \in \mathfrak{n}\setminus\{0\}$ the vector $X + \xi(X)$ is geodesic vector.*

4. The six-dimensional case

By six-dimensional spaces we give an explicit expression for the ad(H)-invariant nonlinear map ξ: $\mathfrak{n} \longrightarrow \mathfrak{h}$ by which is described in the Szenthe-theorem. We begin the investigation by a special case of the six-dimensional spaces, and then we extend it to the general case.

The first counter-example of a g.o. space, which is in no way naturally reductive comes from A. Kaplan [2].This is a six-dimensional Riemannian nilmanifold with a two-dimensional center, one of the so called H-type groups. An H-type Lie algebra is a 2-step nilpotent Lie algebra $\mathfrak{n}$ endowed with an inner product $\langle.,.\rangle$ such that the following property holds: if Z is any element in the center $\mathfrak{z}$ of $\mathfrak{n}$, and $\mathfrak{a}$ is the orthogonal complement of $\mathfrak{z}$, then the linear operator $j_z : \mathfrak{a} \longrightarrow \mathfrak{a}$ defined by $\langle j(Z)X, Y\rangle = \langle Z, [X, Y]\rangle$ for $X, Y \in \mathfrak{a}$ satisfies the identity $[j(Z)]^2 = -||Z||^2 id_a$.

A generalized Heisenberg group (or an H-type group or the six-dimensional Kaplan space) is a connected Lie group whose Lie algebra is an H-type algebra.It is endowed with a left-invariant Riemannian metric.

Let be $\mathfrak{a} = \mathbf{R}^4 = H$, $\mathfrak{z} = V^2 \subset \mathrm{so}\,(4) = so^{(1)}(3) + so^{(2)}(3)$ and $j : \mathfrak{z} \longrightarrow \mathrm{so}\,(\mathfrak{a})$

One can prove on the basis of [1], the only proper two-dimensional subspaces of so (4) can lie in one of the ideals so (3),such that the associated nilmanifold is g.o. space,but not naturally reductive.

In fact,$\mathbf{R}^4$ can be viewed as the quaternions and the two so (3) factors act as left,respectively right, multiplication by pure quaternions.

$so^{(1)}(3) = \lambda_i\mathbf{R} + \lambda_j\mathbf{R} + \lambda_k\mathbf{R}$ and

$so^{(2)}(3) = \rho_i\mathbf{R} + \rho_j\mathbf{R} + \rho_k\mathbf{R}$, where

$$\lambda_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \lambda_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \lambda_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \rho_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \rho_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Let be $V^2 = \lambda_i\mathbf{R} + \lambda_j\mathbf{R} \subset so^{(1)}(3)$, $\mathfrak{n} = \mathfrak{a} + \mathfrak{z} = \mathbf{R}^4 + V^2$. According to Proposition 2.1 $\mathfrak{h} = Der(\mathfrak{n}) \cap \mathrm{so}\,(\mathfrak{n})$, so

$$\delta = \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & \lambda & \mu \\ -\beta & -\lambda & 0 & \nu \\ -\gamma & -\mu & -\nu & 0 \end{pmatrix}$$

Theorem 4.1. *For the six-dimensional space there exists an $ad\,(H)$-invariant map $\xi : \mathfrak{n} \longrightarrow \mathfrak{h}$ such that for any $X \in \mathfrak{n}\backslash\{0\} X + \xi(X)$ is geodesic vector, i. e. the curve $exp\,(t(X + \xi(X))) \cdot p$ is a geodesic.*

Proof. (a) If $X \neq 0 \in \mathfrak{a}$, $Z \in \mathfrak{z}$

According to Lemma 2.2 the isotropy algebra of $\mathfrak{h}$ is isomorphic to the subalgebra of so (4) given by

$$\{\delta \in \mathrm{so}\,(\mathfrak{a}) : [\delta, j(\mathfrak{z})] \subset j(\mathfrak{z}) \text{ and } j^{-1} \circ \mathrm{ad}\,(\delta) \circ j \in \mathrm{so}\,(\mathfrak{z})\}$$

$$\Longrightarrow [\delta, \lambda_i] \in \lambda_i\mathbf{R} + \lambda_j\mathbf{R} \text{ and } [\delta, \lambda_j] \in \lambda_i\mathbf{R} + \lambda_j\mathbf{R}$$

On the basis of these conditions $\mu = \beta$ and $\nu = -\alpha \Longrightarrow \delta = \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & \lambda & \beta \\ -\beta & -\lambda & 0 & -\alpha \\ -\gamma & -\beta & \alpha & 0 \end{pmatrix}$

$$\Longrightarrow \widetilde{\mathfrak{h}} = \left\{ \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ -\alpha & 0 & \lambda & \beta \\ -\beta & -\lambda & 0 & -\alpha \\ -\gamma & -\beta & \alpha & 0 \end{pmatrix}, (\alpha,\beta,\gamma,\lambda) \in R^4 \right\}$$

i) The six-dimensional case: Since $[\delta, \lambda_i] = -(\lambda+\gamma)\lambda_j$ and $[\delta, \lambda_j] = (\lambda+\gamma)\lambda_i$

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \alpha & \beta & \gamma & 0 & 0 \\ -\alpha & 0 & \lambda & \beta & 0 & 0 \\ -\beta & -\lambda & 0 & -\alpha & 0 & 0 \\ -\gamma & -\beta & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma+\lambda \\ 0 & 0 & 0 & 0 & -(\gamma+\lambda) & 0 \end{pmatrix}, (\alpha,\beta,\gamma,\lambda) \in \mathbf{R}^4 \right\}$$

ii) The six-dimensional case in general:

According to the framework of the general theory of g.o. spaces by C. Gordon [G] each scalar product is admissible. We suppose that λ_i and λ_j are not orthonormal basis with respect to the admissible scalar product $\langle .,. \rangle$.

Let $E = \langle \lambda_i, \lambda_i \rangle$ $F = \langle \lambda_i, \lambda_j \rangle$ $G = \langle \lambda_j, \lambda_j \rangle$ and the orthonormal basis $e_1 = x\lambda_i + y\lambda_j$ and $e_2 = u\lambda_i + v\lambda_j \Longrightarrow \langle e_1, e_2 \rangle = Exu + F(yu + xv) + Gyv$

Since $\mathrm{ad}\,(\delta)_{|V}$ is skew-symmetric with respect to the scalar product $\langle .,. \rangle \Longrightarrow$ $\langle \delta(x\lambda_i + y\lambda_j, u\lambda_i + v\lambda_j y \rangle = -\langle x\lambda_i + y\lambda_j, \delta(u\lambda_i + v\lambda_j) \rangle$

Hence we get the following solutions:

(1) $(\lambda+\gamma)E = (\lambda+\gamma)G$

(2) $(\lambda+\gamma)F = 0$

Then $\lambda + \gamma = 0 \Longrightarrow \lambda = -\gamma \Longrightarrow$

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \alpha & \beta & \gamma & 0 & 0 \\ -\alpha & 0 & -\gamma & \beta & 0 & 0 \\ -\beta & \gamma & 0 & -\alpha & 0 & 0 \\ -\gamma & -\beta & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, (\alpha,\beta,\gamma) \in \mathbf{R}^3 \right\}.$$

This is a subalgebra of the isotropy algebra of the Kaplan space or $E = G$ and $F = 0$,i.e. λ_i and λ_j are orthogonal basis, so $\langle .,. \rangle = c \cdot \langle .,. \rangle^*$, where c is a constant and $\langle .,. \rangle^*$is the scalar product of the Kaplan-space. $\Longrightarrow$

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \alpha & \beta & \gamma & 0 & 0 \\ -\alpha & 0 & \lambda & \beta & 0 & 0 \\ -\beta & -\lambda & 0 & -\alpha & 0 & 0 \\ -\gamma & -\beta & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma+\lambda \\ 0 & 0 & 0 & 0 & -(\gamma+\lambda) & 0 \end{pmatrix}, (\alpha,\beta,\gamma,\lambda) \in \mathbf{R}^4 \right\}$$

is the isotropy algebra of all six-dimensional g.o. spaces.

According to Theorem 2.3 N is a g.o. manifold if and only if for each $X \in \mathfrak{a}$ and $Z \in \mathfrak{z}$, there exists a skew-symmetric derivation D of $\mathfrak{n}$ such that $D(Z) = 0$ and $D(X) = j(Z)X$.

From the condition $D(Z) = 0$ follows that $\lambda = -\gamma$, if $Z \neq 0$.

From the second condition $D(X) = Z(X)$ we have an inhomogeneous equation–system :

$$\begin{aligned} \alpha x_1 + \beta x_2 + \gamma x_3 &= -z_1 x_1 - z_2 x_2 \\ -\alpha x_0 - \gamma x_2 + \beta x_3 &= z_1 x_0 + z_2 x_3 \\ -\beta x_0 + \gamma x_1 - \alpha x_3 &= z_2 x_0 - z_1 x_3 \\ -\gamma x_0 - \beta x_1 + \alpha x_2 &= -z_2 x_1 + z_1 x_2 \end{aligned}$$

Since we have supposed that $X \neq 0 \in \mathfrak{a}$, this system has the following unique solution :

$$\begin{aligned} \alpha &= \frac{-z_1(x_0^2 + x_1^2 - x_2^2 - x_3^2) - 2z_2(x_1 x_2 + x_0 x_3)}{x_0^2 + x_1^2 + x_2^2 + x_3^2} \\ \beta &= \frac{z_2(x_3^2 + x_1^2 - x_2^2 - x_0^2) - 2z_1(x_1 x_2 - x_0 x_3)}{x_0^2 + x_1^2 + x_2^2 + x_3^2} \\ \gamma &= \frac{2z_2(x_0 x_1 - x_2 x_3) - 2z_1(x_0 x_2 + x_1 x_3)}{x_0^2 + x_1^2 + x_2^2 + x_3^2} \end{aligned}$$

(ii) If $X = 0$ and $Z \in \mathfrak{z}$ using concepts of Szenthe, Kowalski and Vanhecke

$$c'_z = \left\{ \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & 0 \\ 0 & -\lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \end{pmatrix}, \ \lambda \in \ \mathbf{R} \right\}$$

According to Proposition 3.3 there exists a $C \in c'_z$ such that $Z + C$ is geodesic vector.

For $\delta = 0$ $(\lambda = 0)$ $Z + \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & 0 \\ 0 & -\lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \end{pmatrix} + q_z$ leaves invariant $\exp(Z + \delta)$, in this case the geodesic will be an orbit of a one–parameter group. $\square$

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