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SUPERPOTENTIALS IN VARIATIONAL SEQUENCES

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ABSTRACT. In the Lagrangian approach to conservation laws of field theories one defines a suitable vector density which generates the *conserved Noether currents*. As it is known, in *natural* (and *gauge–natural*) field theories, along any section this density is the divergence of a skew–symmetric tensor density, which is called a *superpotential* for the conserved quantity.

Making use of some abstract results due to Horák and Kolář, in this paper we give a geometrical interpretation of superpotentials in the framework of variational sequences according to Krupka. We refer to our previous results on *variational Lie derivatives* concerning abstract versions of Noether's theorems, which are here interpreted in terms of conserved currents.

1. Introduction

As it is well known, the addition of a total divergence to the Lagrangian of any given Lagrangian field theory does not change the relevant field equations, however, in general, this will induce some change in the physical interpretation of the theory. In fact, by Stokes' theorem, the added divergence will contribute by an additional boundary term into the action, which in particular modifies the notion of energy.

As far as conservation laws are concerned, we refer here to the approach based on the Lagrangian formulation of natural field theories (see [2] and references quoted therein). This formulation amounts to define, in the variational framework, a suitable vector density which generates the conserved current; this density is the divergence of a skew–symmetric tensor density, which is called a *superpotential* for the conserved quantity. We stress that the energetic content of a geometric field theory along its critical sections is generated by its superpotentials.

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We consider the recent geometrical formulation of the Calculus of Variations on fibered manifolds developed by Krupka [10], which is stated on finite order jets of the fibering. As it is well known, in this formulation the *variational sequence* is defined as a quotient of the de Rham sequence on a finite order jet space with respect to an intrinsically defined subsequence, whose choice is inspired by the Calculus of Variations itself.

In this paper we provide a geometrical interpretation of superpotentials in the variational sequence. We make use of the representation of the quotient sheaves of the variational sequence as concrete sheaves of forms given in [14], as well as of our previous results concerning variational Lie derivatives [3]. Furthermore, we refer to some abstract results due to Horák and Kolář concerning decomposition formulae of morphisms, involved with the integration by parts procedure [6, 8, 9].

Manifolds and maps between manifolds are C^∞. All morphisms of fibered manifolds (and hence bundles) will be morphisms over the identity of the base manifold, unless otherwise specified. As for sheaves, we will use the definitions and the main results given in [15].

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2. Jet spaces and variational sequences

In this Section we recall some basic facts about jet spaces [1, 12, 13] and finite order variational sequences [3, 10, 14].

2.1. Jet spaces. Our framework is a fibered manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$.

For $r \geq 0$ we are concerned with the r–jet space $J_r\boldsymbol{Y}$; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fiberings $\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, $\pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X}$, and, among these, the *affine* fiberings π^r_{r-1}. We denote with $V\boldsymbol{Y}$ the vector subbundle of the tangent bundle $T\boldsymbol{Y}$ of vector fields on $\boldsymbol{Y}$ which are vertical with respect to the fibering π.

Charts on $\boldsymbol{Y}$ adapted to π are denoted by (x^λ, y^i). Greek indices $\lambda, \mu, \dots$ run from 1 to n and they label base coordinates, while Latin indices $i, j, \dots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote by $(\partial_\lambda, \partial_i)$ and (d^λ, d^i) the local bases of vector fields and 1–forms on $\boldsymbol{Y}$ induced by an adapted chart, respectively.

We denote multi–indices of dimension n by boldface Greek letters such as $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, with $0 \leq \alpha_\mu$, $\mu = 1, \dots, n$; by an abuse of notation, we denote with λ the multi–index such that $\alpha_\mu = 0$, if $\mu \neq \lambda$, $\alpha_\mu = 1$, if $\mu = \lambda$. We also set $|\boldsymbol{\alpha}| := \alpha_1 + \dots + \alpha_n$ and $\boldsymbol{\alpha}! := \alpha_1! \dots \alpha_n!$ The summation convention will be adopted also for multi–indices.

The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^\lambda, y^i_{\boldsymbol{\alpha}})$, with $0 \leq |\boldsymbol{\alpha}| \leq r$; in particular, we set $y^i_0 \equiv y^i$. The local vector fields and forms of $J_r\boldsymbol{Y}$ induced by the above coordinates are denoted by $(\partial^{\boldsymbol{\alpha}}_i)$ and $(d^i_{\boldsymbol{\alpha}})$, respectively.

In the theory of variational sequences a fundamental role is played by the *contact maps* on jet spaces (see [1, 10, 11, 12]). Namely, for $r \geq 1$, we consider the natural complementary fibered morphisms over $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$

$$д : J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{r-1}\boldsymbol{Y}\,, \qquad \vartheta : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} TJ_{r-1}\boldsymbol{Y} \to VJ_{r-1}\boldsymbol{Y}\,,$$

with coordinate expressions, for $0 \leq |\boldsymbol{\alpha}| \leq r-1$, given by

$$д \;= d^\lambda \otimes д_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\boldsymbol{\alpha}+\lambda}\partial^{\boldsymbol{\alpha}}_j)\,, \vartheta \;= \vartheta^j_{\boldsymbol{\alpha}} \otimes \partial^{\boldsymbol{\alpha}}_j = (d^j_{\boldsymbol{\alpha}} - y^j_{\boldsymbol{\alpha}+\lambda} d^\lambda) \otimes \partial^{\boldsymbol{\alpha}}_j\,.$$

Let us denote $\overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}] := \operatorname{im}\vartheta^*_r$, where $\vartheta^*_r : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} V^*J_{r-1}\boldsymbol{Y} \to J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y}$ and $\operatorname{im}\vartheta^*_r \subset J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} \subset T^*J_r\boldsymbol{Y}$. We have

$$(1) \qquad J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*J_{r-1}\boldsymbol{Y} = \Big(J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X}\Big) \oplus \overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}]\,.$$

We have the isomorphism $\overset{*}{\mathcal{C}}_{r-1}[\boldsymbol{Y}] \simeq J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} V^*J_{r-1}\boldsymbol{Y}$.

If $f : J_r\boldsymbol{Y} \to I\!R$ is a function, then we set $D_\lambda f \;:= (д)_\lambda f$, $D_{\boldsymbol{\alpha}+\lambda} f \;:= D_\lambda D_{\boldsymbol{\alpha}} f$, where we have set $D_{\boldsymbol{\alpha}} f \circ j_{r+|\boldsymbol{\alpha}|}\sigma = \partial_{\boldsymbol{\alpha}}(f \circ j_r\sigma)$ for any section σ. The operator D_λ is called the *formal derivative*. A Leibniz's rule holds for $D_{\boldsymbol{\alpha}}$ (see [13]). Given a vector field $Z : J_r\boldsymbol{Y} \to TJ_r\boldsymbol{Y}$, the splitting (1) yields $Z \circ \pi^{r+1}_r = Z_H + Z_V$ where, if $Z = Z^\gamma \partial_\gamma + Z^i_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}_i$, then we have $Z_H = Z^\gamma D_\gamma$ and $Z_V = (Z^i_{\boldsymbol{\alpha}} - y^i_{\boldsymbol{\alpha}+\gamma} Z^\gamma)\partial^{\boldsymbol{\alpha}}_i$.

The splitting (1) induces also a decomposition of the exterior differential on $\boldsymbol{Y}$, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$, where we define the *horizontal* and *vertical differential* to be the sheaf morphisms:

$$d_H := д \rfloor d - d\, д \rfloor : \overset{p}{\Lambda}_r \to \overset{p}{\Lambda}_{r+1}\,, \quad d_V := \vartheta \rfloor d - d\, \vartheta \rfloor : \overset{p}{\Lambda}_r \to \overset{p}{\Lambda}_{r+1}\,,$$

and $\rfloor$ is the interior product (see [13]). The action of d_H and d_V on functions and 1–forms on $J_r\boldsymbol{Y}$ uniquely characterizes d_H and d_V (see, *e.g.*, [14] for more details). A *projectable vector field* on $\boldsymbol{Y}$ is defined to be a pair (Ξ, ξ) where:

i. $\Xi : \boldsymbol{Y} \to T\boldsymbol{Y}$ and $\xi : \boldsymbol{X} \to T\boldsymbol{X}$ are vector fields;

ii. $\Xi : \boldsymbol{Y} \to T\boldsymbol{Y}$ is a fibered morphism over $\xi : \boldsymbol{X} \to T\boldsymbol{X}$.

See [3] for coordinate expressions of (Ξ, ξ) and its jet prolongation $(j_r\Xi, \xi)$.

2.2. Variational sequences. We shall be here concerned with some distinguished sheaves of forms on jet spaces [1, 10, 11, 13, 14].

i. For $r \geq 0$, we consider the standard sheaves $\overset{p}{\Lambda}_r$ of p–forms on $J_r\boldsymbol{Y}$.

ii. For $0 \leq s \leq r$, we consider the sheaves $\overset{p}{\mathcal{H}}_{(r,s)}$ and $\overset{p}{\mathcal{H}}_r$ of *horizontal forms*, i.e. of local fibered morphisms over π^r_s and π^r of the type $\alpha : J_r\boldsymbol{Y} \to \overset{p}{\wedge} T^*J_s\boldsymbol{Y}$ and $\beta : J_r\boldsymbol{Y} \to \overset{p}{\wedge} T^*\boldsymbol{X}$, respectively.

iii. For $0 \leq s < r$, we consider the subsheaf $\overset{p}{\mathcal{C}}_{(r,s)} \subset \overset{p}{\mathcal{H}}_{(r,s)}$ of *contact forms*, i.e. of sections $\alpha \in \overset{p}{\mathcal{H}}_{(r,s)}$ with values into $\overset{p}{\wedge}(\operatorname{im}\vartheta^*_{s+1})$. We have the distinguished subsheaf $\overset{p}{\mathcal{C}}_r \subset \overset{p}{\mathcal{C}}_{(r+1,r)}$ of local fibered morphisms $\alpha \in \overset{p}{\mathcal{C}}_{(r+1,r)}$ such that $\alpha = \overset{p}{\wedge}\vartheta^*_{r+1} \circ \tilde{\alpha}$, where $\tilde{\alpha}$ is a section of the fibration $J_{r+1}\boldsymbol{Y} \underset{J_r\boldsymbol{Y}}{\times} \overset{p}{\wedge} V^* J_r\boldsymbol{Y} \to J_{r+1}\boldsymbol{Y}$ which projects down onto $J_r\boldsymbol{Y}$.

The fibered splitting (1) yields the sheaf splitting $\overset{p}{\mathcal{H}}_{(r+1,r)} = \bigoplus_{t=0}^{p} \overset{p-t}{\mathcal{C}}_{(r+1,r)} \wedge \overset{t}{\mathcal{H}}_{r+1}$, which restricts to the inclusion $\overset{p}{\Lambda}_r \subset \bigoplus_{t=0}^{p} \overset{p-t}{\mathcal{C}}_r \wedge \overset{t}{\mathcal{H}}{}^h_{r+1}$, where $\overset{p}{\mathcal{H}}{}^h_{r+1} := h(\overset{p}{\Lambda}_r)$ for $0 < p \leq n$ and h is defined to be the restriction to $\overset{p}{\Lambda}_r$ of the projection of the above splitting onto the non–trivial summand with the highest value of t. We define also the map $v := \mathrm{id} - h$.

We recall now the theory of variational sequences on finite order jet spaces, as it was developed by Krupka in [10].

By an abuse of notation, let us denote by $d\ker h$ the sheaf generated by the presheaf $d\ker h$ (see [15]). We set $\overset{*}{\Theta}_r := \ker h + d\ker h$; we have the following natural 'contact' subsequence of the de Rham sequence on $J_r\boldsymbol{Y}$:

$$0 \longrightarrow \overset{1}{\Theta}_r \xrightarrow{d} \overset{2}{\Theta}_r \xrightarrow{d} \dots \xrightarrow{d} \overset{I}{\Theta}_r \xrightarrow{d} 0$$

where, in general, the highest integer I depends on the dimension of the fibers of $J_r\boldsymbol{Y} \to \boldsymbol{X}$. The above sequence is exact [10].

Standard arguments of homological algebra prove that the following diagram is commutative and that its rows and columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \overset{1}{\Theta}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Theta}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{I}{\Theta}_r \arrow[r, "d"] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \\
0 \arrow[r] & I\!\!R \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{2}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & \overset{I}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & 0 \\
0 \arrow[r] & I\!\!R \arrow[r] \arrow[d] & \overset{0}{\Lambda}_r \arrow[r, "\mathcal{E}_0"] \arrow[d] & \overset{1}{\Lambda}_r/\overset{1}{\Theta}_r \arrow[r, "\mathcal{E}_1"] \arrow[d] & \overset{2}{\Lambda}_r/\overset{2}{\Theta}_r \arrow[r] \arrow[d] & \dots \arrow[r] & \overset{I}{\Lambda}_r/\overset{I}{\Theta}_r \arrow[r, "\mathcal{E}_I"] \arrow[d] & \overset{I+1}{\Lambda}_r \arrow[r, "d"] \arrow[d] & \dots \arrow[r, "d"] & 0 \\
 & 0 & 0 & 0 & 0 & & 0 & 0 & &
\end{tikzcd}$$

Definition 2.1. The bottom row of the above diagram, obtained as the quotient of the de Rham sequence on $J_r\boldsymbol{Y}$ with respect to the 'contact' subsequence, is said to be the r–th order *variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$ [10].

We can consider the 'short' variational sequence [14]:

$$0 \longrightarrow I\!R \longrightarrow \overset{0}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_0} \dots \xrightarrow{\mathcal{E}_n} \overset{n+1}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_{n+1}} \overset{n+2}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_{n+2}} 0\,,$$

where $\overset{0}{\mathcal{V}}_r = \overset{0}{\Lambda}_r$, $\overset{p}{\mathcal{V}}_r = \overset{p}{\mathcal{H}}{}^h_r$, for $0 < p \leq n$, $\overset{n+1}{\mathcal{V}}_r$ is the sheaf generated by the presheaf $\left(\overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1} + d_H(\overset{1}{\mathcal{C}}_{(2r,r-1)} \wedge \overset{n-1}{\mathcal{H}}_{2r})\right) \cap \left(\overset{1}{\mathcal{C}}_{(2r+1,0)} \wedge \overset{n}{\mathcal{H}}_{2r+1}\right)$, and $\overset{n+2}{\mathcal{V}}_r$ is isomorphic to $\mathcal{E}_{n+1}\left((\overset{p-n}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1})/h(d\ker h)\right)$.

Remark 2.2. A section $\lambda \in \overset{n}{\mathcal{V}}_r$ is just a Lagrangian of order $(r+1)$ of the standard literature.

Let $\alpha \in \overset{n+1}{\Lambda}_r$, i.e. $h(\alpha) \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$. We say $E_{h(\alpha)} \in \overset{n+1}{\mathcal{V}}_r$ to be an *Euler–Lagrange type morphism*. $\mathcal{E}_n(\lambda) \in \overset{n+1}{\mathcal{V}}_r$ coincides with the standard higher order Euler–Lagrange morphism associated with λ.

Let $\eta \in \overset{n+1}{\mathcal{V}}_r$ be an Euler–Lagrange morphism and $\sigma : \boldsymbol{X} \to \boldsymbol{Y}$ be a section. We recall that σ is said to be *critical* with respect to η if $(j_{2r+1}\sigma)^*\eta = 0$.

3. Natural Lagrangian theories

We recall that in *natural* physical theories fields may be Lie–dragged along the flow of any tangent vector field in space–time. This happens only if the fields are fields of geometric objects, i.e. if changes of coordinates in space–time define uniquely the transformation laws of the object themselves (see *e.g.* [2, 4, 5, 7]). This is the natural framework for defining and investigating the physically fundamental concept of conserved quantity.

3.1. Natural lift. Denote by $\mathcal{T}_{\boldsymbol{X}}$ and $\mathcal{T}_{\boldsymbol{Y}}$ the sheaf of vector fields on $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. From now on, we assume a functorial mapping is defined which lifts any local diffeomorphism $\phi_{\boldsymbol{X}}$ of the basis $\boldsymbol{X}$ into a unique local automorphism $\phi_{\boldsymbol{Y}} = \hat{\mathcal{N}}(\phi_{\boldsymbol{X}})$ (over $\phi_{\boldsymbol{X}}$) of the bundle $\boldsymbol{Y}$. This lifting depends on derivatives of $\phi_{\boldsymbol{X}}$ of order k. Its infinitesimal version associate to $\xi \in \mathcal{T}_{\boldsymbol{X}}$ a unique *projectable* vector field $\Xi_\xi := \mathcal{N}(\xi)$ in the following way:

$$\mathcal{N} : \boldsymbol{Y} \underset{\boldsymbol{X}}{\times} J_k T\boldsymbol{X} \to T\boldsymbol{Y} \,:(\boldsymbol{y}, j_k\xi) \mapsto \Xi_\xi(\boldsymbol{y})\,, \tag{2}$$

where, for any $\boldsymbol{y} \in \boldsymbol{Y}$, one sets: $\Xi_\xi(\boldsymbol{y}) = \frac{d}{dt}[(\phi_t)_{\boldsymbol{Y}}(\boldsymbol{y})]_{t=0}$, and $(\phi_t)_{\boldsymbol{X}}$ denotes the (local) flow in $\boldsymbol{X}$ generated by ξ, while $(\phi_t)_{\boldsymbol{Y}}$ is its natural lift.

Remark 3.1. The natural lift fulfills the following properties:

(1) $\mathcal{N}$ is linear over $id_{\boldsymbol{Y}}$;

(2) we have $T\pi \circ \mathcal{N} = id_{T\boldsymbol{X}} \circ \bar{\pi}^k_0$, where $\bar{\pi}^k_0$ is the projection $\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} J_k T\boldsymbol{X} \to T\boldsymbol{X}$;

(3) for any pair (η, ξ) of vector fields on $\boldsymbol{X}$, we have

$$\mathcal{N}([\eta, \xi]) = [\mathcal{N}(\eta), \mathcal{N}(\xi)]\,;$$

(4) we have the coordinate expression of $\mathcal{N}$:

$$\mathcal{N} = d^{\lambda}\otimes\partial_{\lambda} + d^{\lambda}_{\boldsymbol{\alpha}}\otimes(\mathcal{Z}^{i\boldsymbol{\alpha}}_{\lambda}\,\partial_i)\,,$$

where $|\boldsymbol{\alpha}| = k$ and $Z^{i\boldsymbol{\alpha}}_{\lambda} \in C^{\infty}(\boldsymbol{Y})$ are suitable functions which depend on the bundle.

Definition 3.2. The map $\hat{\mathcal{N}}$ is called the *natural lifting functor*. The vector field $\mathcal{N}(\xi)$ is called the *natural lift* of ξ to the bundle $\boldsymbol{Y}$.

Remark 3.3. We recall that $(\boldsymbol{Y}, \boldsymbol{X}, \pi)$ provided with a natural lifting functor can be considered as a bundle of geometric objects of finite order k, i.e. associated to the bundle $L^k(\boldsymbol{X})$ of k–frames in X (see [4] for the definition).

Let γ be a (local) section of $\boldsymbol{Y}$ and $\xi \in \mathcal{T}_{\boldsymbol{X}}$. We define a (local) section $\pounds_{\xi}\gamma : \boldsymbol{X} \to V\boldsymbol{Y}$, by setting: $\pounds_{\xi}\gamma = T\gamma \circ \xi - \Xi \circ \gamma$.

Definition 3.4. The (local) section $\pounds_{\xi}\gamma$ is called the *Lie derivative of* γ *along the vector field* ξ (see [7]).

Remark 3.5. This section is a vertical prolongation of γ, i.e. it satisfies the property: $\nu_{\boldsymbol{Y}} \circ \pounds_{\xi}\gamma = \gamma$, where $\nu_{\boldsymbol{Y}}$ is the projection $\nu_{\boldsymbol{Y}} : V\boldsymbol{Y} \to \boldsymbol{Y}$. Its coordinate expression is given by $(\pounds_{\xi}\gamma)^i = \xi^{\lambda}D_{\lambda}\gamma^i - \mathcal{Z}^{i\boldsymbol{\beta}}_{\lambda}(\gamma^j)\xi^{\lambda}_{\boldsymbol{\beta}}$.

Furthermore, we can write $\pounds : J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} J_kT\boldsymbol{X} \to V\boldsymbol{Y}$.

Remark 3.6. The Lie derivative $\pounds_{\xi}\gamma$ satisfies the following properties:

(1) for any vector field ξ over $\boldsymbol{X}$, the mapping $\gamma \mapsto \pounds_{\xi}\gamma$ is a first–order quasilinear differential operator;

(2) for any local section γ of $\boldsymbol{Y}$, the mapping $\xi \mapsto \pounds_{\xi}\gamma$ is a linear differential operator of order k;

(3) by using the canonical isomorphism between $VJ_r\boldsymbol{Y}$ and $J_rV\boldsymbol{Y}$, we have $\pounds_{\xi}[j_r\gamma] = j_r[\pounds_{\xi}\gamma]$, for any (local) section γ of $\boldsymbol{Y}$ and for any (local) vector field ξ over $\boldsymbol{X}$.

3.2. Natural Lagrangians and their symmetries. We consider now a projectable vector field (Ξ, ξ) on $\boldsymbol{Y}$ and take into account the Lie derivative with respect to the prolongation $j_r\Xi$ of Ξ. Such a prolonged vector field preserves the fiberings π^s_r, π^r; hence it preserves the splitting (1).

It is known [3] that the standard Lie derivative operator with respect the r-th prolongation $j_r\Xi$ of a projectable vector field (Ξ, ξ) on $\boldsymbol{Y}$ passes on the quotient spaces $\overset{p}{\Lambda}_r/\overset{p}{\Theta}_r$ and that it can be represented by an operator (the *variational Lie derivative*) on the sheaves of the short variational sequence. In the case $p = n$, by using a pull–back from $J_r\boldsymbol{Y}$ to $J_{2r}\boldsymbol{Y}$ we have

$$\mathcal{L}_{j_r\Xi} : \overset{n}{\mathcal{V}}_r \to \overset{n}{\mathcal{V}}_{2r+1} : \lambda \mapsto \Xi_V \rfloor \mathcal{E}(\lambda) + d_H(j_{2r}\Xi_V \rfloor p_{d_V\lambda} + \xi\rfloor\lambda)\,, \tag{3}$$

where $p_{d_V\lambda}$ is a momentum associated to λ (see [1, 3, 14]).

Variational Lie derivatives allow us to compute 'variationally relevant' infinitesimal symmetries of Lagrangians in the variational sequence.

Definition 3.7. Let (Ξ, ξ) be a projectable vector field on $\boldsymbol{Y}$. Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a Lagrangian. We say Ξ to be a *symmetry* of λ if $\mathcal{L}_{j_{r+1}\Xi}\,\lambda = 0$.

Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a Lagrangian. We say λ to be a *natural Lagrangian* if the lift $\mathcal{N}(\xi)$ of any vector field ξ on $\boldsymbol{X}$ is a symmetry for λ, *i.e.* if $\mathcal{L}_{j_{r+1}\xi}\,\lambda = 0$. In this case the projectable vector field $(\mathcal{N}(\xi), \xi)$ is called a *natural symmetry* of λ.

Remark 3.8. We can regard $\pounds_\xi : J_r\boldsymbol{Y} \to V\boldsymbol{Y}$ as a morphism over the basis $\boldsymbol{X}$. In this case it is meaningful to consider the (standard) prolongation of $\pounds_\xi$, denoted by $j_r\pounds_\xi : J_{2r}\boldsymbol{Y} \to VJ_r\boldsymbol{Y}$, where we made use of the isomorphism (iii) recalled in Remark 3.6.

Symmetries of a Lagrangian λ are calculated by means of Noether's Theorem, which takes a particularly interesting form in the case of natural Lagrangians.

Theorem 3.9. *(Noether's Theorem for natural Lagrangians) Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a natural Lagrangian and (Ξ, ξ) a natural symmetry of λ. Then by (3) we have*

$$0 = -\pounds_\xi \rfloor \mathcal{E}(\lambda) + d_H(-j_{2r}\pounds_\xi \rfloor p_{d_V\lambda} + \xi \rfloor \lambda)\ . \tag{4}$$

Suppose that the section $\sigma : \boldsymbol{X} \to \boldsymbol{Y}$ fulfills the criticality condition

$$(j_{2r+1}\sigma)^*(-\pounds_\xi \rfloor \mathcal{E}(\lambda)) = 0\,. \tag{5}$$

Then, the $(n-1)$-form $\epsilon = -j_{2r}\pounds_\xi \rfloor p_{d_V\lambda} + \xi \rfloor \lambda$ fulfills the equation $d((j_{2r}\sigma)^*(\epsilon)) = 0$.

Remark 3.10. If σ is a critical section for $\mathcal{E}(\lambda)$ the above equation admits a physical interpretation as a *weak conservation law* for the density associated with ϵ.

Definition 3.11. Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a natural Lagrangian and $\xi \in \mathcal{T}_{\boldsymbol{X}}$. Then the sheaf morphism ϵ is said to be a *natural conserved current*.

Remark 3.12. In general, this conserved current is not uniquely defined. In fact, it depends on the choice of $p_{d_V\lambda}$, which is not unique (see [14] and references quoted therein). Moreover, we could add to the conserved current any form $\beta \in \overset{n-1}{\mathcal{V}}_{2r}$ which is variationally closed, i.e. such that $\mathcal{E}_{n-1}(\beta) = 0$ holds. The form β is locally of the type $\beta = d_H\gamma$, where $\gamma \in \overset{n-2}{\mathcal{V}}_{2r+1}$.

Lemma 3.13. *(Fundamental Lemma) Let $\alpha : J_k(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}) \to \overset{p}{\wedge} T^*\boldsymbol{X}$ be linear with respect to the fibering $J_k\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} J_kT\boldsymbol{X} \to J_k\boldsymbol{Y}$ and let D_H be the horizontal differential on $\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}$. We can uniquely write α as*

$$\overline{\alpha} : J_k\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_k[T\boldsymbol{X}] \wedge \overset{p}{\wedge} T^*\boldsymbol{X}\,,$$

then

$$\overline{D_H \alpha} = D_H \overline{\alpha}\,.$$

Proof. This is a naturality property of D_H which follows from linearity of $D_H \alpha$ with respect to the fibering $J_{k+1}\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} J_{k+1}T\boldsymbol{X} \to J_{k+1}\boldsymbol{Y}$, the isomorphism $J_kT\boldsymbol{X} \underset{\boldsymbol{X}}{\times} (J_kT\boldsymbol{X})^* \simeq V^*J_kT\boldsymbol{X}$ and the isomorphism $\overset{*}{\mathcal{C}}_{k+1}[T\boldsymbol{X}] \wedge \overset{p+1}{\wedge} T^*\boldsymbol{X} \simeq V^*J_{k+1}T\boldsymbol{X} \otimes \overset{p+1}{\wedge} T^*\boldsymbol{X}$. □

Let $\epsilon : J_{2r}\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} J_kT\boldsymbol{X} \to \overset{n-1}{\wedge} T^*\boldsymbol{X}$ be a conserved current. We can regard ϵ as the equivalent morphism $\overline{\epsilon} : J_{2r}\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_k[T\boldsymbol{X}] \wedge \overset{n-1}{\wedge} T^*\boldsymbol{X}$.

Lemma 3.14. *We have*

$$d_H(j_k\xi \rfloor \epsilon) = j_{k+1}\xi \rfloor D_H \overline{\epsilon}\,.$$

Proof. We used standard formulae which define the horizontal differentials D_H and d_H given in Section 2.1. □

4. Superpotentials

As it is well known [5, 2], performing a covariant integration by parts enables us to decompose the current ϵ into the sum of the so–called *reduced current* and the formal divergence of a skew–symmetric tensor density called *superpotential* (which is defined modulo a divergence). It is also well known that all conservation laws which occur in natural theories are *strong laws*, i.e. they hold along any section of the bundle. Along critical sections the reduced current vanishes so that the current ϵ is not only closed, but it is also exact 'on shell'. We stress that the whole energetic content of a geometric (free) field theory along its critical sections is generated by its superpotentials, a fact which has important applications in Mathematical Physics (see, *e.g.*, [2]).

Remark 4.1. Let λ be a natural Lagrangian. By the linearity of £ with respect to the vector bundle structure $J_kT\boldsymbol{X} \to \boldsymbol{X}$ we have

$$\mu \equiv \mu(\lambda) = £ \rfloor \mathcal{E}(\lambda) : J_{2r}\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_k[T\boldsymbol{X}] \wedge \overset{n}{\wedge} T^*\boldsymbol{X}\,.$$

In the following we shall give the main result which enables us to describe superpotentials in the short variational sequence. We shall apply to the 'total' space $\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}$ a standard result concerning the integration by parts procedure involved in variational formulae (see *e.g.* [3, 14]).

The following Lemma is an application of an abstract result due to Kolář and Horák [6, 9] concerning a decomposition formula for vertical morphisms.

Lemma 4.2. *Let* $\mu : J_{2r}\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_k[T\boldsymbol{X}] \wedge \overset{p}{\wedge} T^*\boldsymbol{X}$, *with* $0 \leq p \leq n$ *and let* $D_H\mu = 0$. *We regard* μ *as the extended morphism* $\hat{\mu} : J_{2r}(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}) \to \overset{*}{\mathcal{C}}_k[\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}] \wedge \overset{p}{\wedge} T^*\boldsymbol{X}$. *Then we have*

$$\hat{\mu} = E_{\hat{\mu}} + F_{\hat{\mu}} \,,$$

where

$$E_{\hat{\mu}} : J_{2r+k}(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}) \to \overset{*}{\mathcal{C}} * 0[\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}] \wedge \overset{p}{\wedge} T^*\boldsymbol{X} \,,$$

locally, $F_{\hat{\mu}} = D_H M_{\hat{\mu}}$, *with*

$$M_{\hat{\mu}} : J_{2r+k-1}(\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}) \to \overset{*}{\mathcal{C}}_{k-1}[\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}] \wedge \overset{p-1}{\wedge} T^*\boldsymbol{X} \,.$$

Proof. We evaluate $E_{\hat{\mu}}$ and $D_H M_{\hat{\mu}}$ by means of a backwards procedure. (see *e.g.* [1, 8]). □

Remark 4.3. In general there is no uniquely determined $M_{\hat{\mu}}$. In fact it can be proved that a linear symmetric connection on $\boldsymbol{X}$ yields a distinguished choice of $M_{\hat{\mu}}$ in analogy to [8], Prop. 1, see also [1, 5].

Theorem 4.4. *Let* $\hat{\mu}$ *is of the type* $\mu : J_{2r}\boldsymbol{Y} \to \overset{*}{\mathcal{C}}_k[T\boldsymbol{X}] \wedge \overset{n}{\wedge} T^*\boldsymbol{X}$, *then the following decomposition formula holds*

$$\mu = \tilde{\mu} + D_H \phi_\mu \,, \tag{6}$$

where

$$\tilde{\mu} := E_\mu : J_{2r+k}(\boldsymbol{Y}) \to \overset{*}{\mathcal{C}}_k[T\boldsymbol{X}] \wedge \overset{n}{\wedge} T^*\boldsymbol{X} \,,$$

and

$$\phi_\mu := M_\mu : J_{2r+k-1}(\boldsymbol{Y}) \to \overset{*}{\mathcal{C}}_{k-1}[T\boldsymbol{X}] \wedge \overset{n-1}{\wedge} T^*\boldsymbol{X} \,.$$

Proof. We take into account that $D_H\mu$ is obviously vanishing, then the result is a straightforward consequence of Lemma 4.2 with $p = n$. □

Remark 4.5. If the coordinate expression of μ is given by

$$\mu = \mu_i^{\boldsymbol{\alpha}} \vartheta^i_{\boldsymbol{\alpha}} \wedge \omega \,,$$

where $\vartheta^i_{\boldsymbol{\alpha}}$ are contact forms on $J_kT\boldsymbol{X}$ coordinate expressions of E_μ and M_μ are given by

$$E_\mu = E_i \vartheta^i \wedge \omega \,,$$
$$M_\mu = M_i^{\boldsymbol{\alpha}+\lambda} \vartheta^i_{\boldsymbol{\alpha}} \wedge \omega_\lambda \,,$$

being ϑ^i contact forms on $\boldsymbol{Y}$. We have, in particular

$$E_\mu = (-1)^{|\boldsymbol{\beta}|} D_{\boldsymbol{\beta}} \mu_i^{\boldsymbol{\beta}} \vartheta^i \wedge \omega \,,$$

with $0 \leq |\boldsymbol{\beta}| \leq k$.

Corollary 4.6. *We have*

$$\mu = D_H \phi_\mu \,.$$

Proof. For any ξ, $\tilde{\mu} \equiv \mathcal{E}(\mu(\xi))$ is identically vanishing being the Euler–Lagrange morphism of a contraction with another Euler–Lagrange morphism. We stress that these are just the *generalized Bianchi identities.* □

Definition 4.7. The form ϕ_μ is said to be a *reduced current.*

It can be proved that a linear symmetric connection on $\boldsymbol{X}$ yields a distinguished choice of ϕ_μ in analogy to [8], Prop. 1, see also [5].

Corollary 4.8. *Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a natural Lagrangian and (Ξ, ξ) a natural symmetry of λ. Then, being $\mu = D_H \epsilon$, the following holds:*

$$D_H(\epsilon - \phi_\mu) = 0 \,. \tag{7}$$

Eq. (7) is referred as a 'strong conservation law' for the density $\epsilon - \phi_\mu$.

We can now reformulate the main result about the existence of superpotentials in the framework of variational sequences.

Theorem 4.9. *Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a natural Lagrangian and (Ξ, ξ) a natural symmetry of λ. Then there exists a (global) sheaf morphism $\mathcal{U} \in \left(\overset{n-2}{\mathcal{V}}{}_{2r-1}\right)_{\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}}$ such that*

$$D_H \mathcal{U} = \epsilon - \phi_\mu \,.$$

Proof.

(1) (local existence) By applying Lemma 3.13, we can consider

$$\epsilon - \phi_\mu : J_{2r+k-1}\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} J_{r+k-2}T\boldsymbol{X} \to \overset{n-1}{\wedge} T^*\boldsymbol{X} \,,$$

then we take eq. (7) into account and we integrate over the variational sequence associated with $\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X}$.

(2) (global existence) Eq. (7) assures us that the hypotheses of Lemma 4.2 are satisfied, so we have

$$\epsilon - \phi_\mu = \overline{\epsilon - \phi_\mu} + D_H \mathcal{U} \,,$$

where $\overline{\epsilon - \phi_\mu}$ is vanishing because of an uniqueness argument. Globality follows from Lemma 4.2. □

Definition 4.10. We define the sheaf morphism $\mathcal{U}$ to be a *superpotential* of λ.

Remark 4.11. As a consequence of Remarks 3.12 and 4.3, superpotentials are not defined uniquely. In fact generally the choice of linear symmetric connections over $\boldsymbol{X}$ yields distinguished superpotentials.

Remark 4.12. We stress that the current ϵ is conserved along critical sections of $\boldsymbol{Y}$, while the quantity $\epsilon - \phi_\mu$ is conserved along any section of $\boldsymbol{Y}$. As usual, we say that ϵ is *weakly conserved*, while $\epsilon - \phi_\mu$ is *strongly conserved.*

5. Conclusions

In this paper we have given a suitable geometric description of the natural lift of vector fields on bundles of geometric objects. Applying to natural Lagrangians previous results of ours about the variational Lie derivative and the symmetries in variational sequences, we have provided a geometrical setting for the description of superpotentials in variational sequences. We stress that given a linear symmetric connection on $\boldsymbol{X}$ we are able to determine ϵ, ϕ_μ and $\mathcal{U}$. We call this connection the *background connection.* The extension of this setting in the gauge–natural theories case and explicit examples of applications to physical theories will be considered elsewhere.

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