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GEOMETRY OF A CLASS OF NON–SYMMETRIC HARMONIC MANIFOLDS

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Abstract. We discuss the geometric and algebraic structure of the non-symmetric examples of harmonic manifolds given by E. Damek and F. Ricci. In addition we focus on some aspects of their geometry in relation with Osserman's conjecture and some open problems about k-harmonicity. This leads to a new conjecture about harmonic spaces for which we provide partial answers.

Keywords. Lie algebras of Heisenberg type and generalized Heisenberg groups, solvable extensions, homogeneous Hadamard manifolds, harmonic and k-harmonic spaces, Osserman's conjecture, Jacobi operator.

MS classification. 53C25, 53C30, 53C35.

1. Introduction

A Riemannian manifold (M, g) is said to be (locally) harmonic if all sufficiently small geodesic spheres are hypersurfaces with constant mean curvature. All (M, g) which are locally isometric to a two-point homogeneous space are trivial examples and Lichnerowicz's fundamental conjecture states that these examples are the only ones. In 1990 Z. Szabó gave a positive answer to this longstanding conjecture for compact manifolds with finite fundamental group [25]. We refer to [4], [24], [25], [28], [29] for more details about harmonic spaces.

The big surprize came in 1991-92 when E. Damek and F. Ricci announced their discovery of counterexamples for the non-compact case [14], [15]. In their research about harmonic analysis on homogeneous spaces they discovered some solvable extensions of generalized Heisenberg groups which are non-symmetric but (locally) harmonic. These examples are closely related to non-compact rank one symmetric spaces but although the analysis on the examples seems to be very similar to that on the symmetric spaces, their geometry shows some remarkable different features.

The main purpose of this note is to discuss the geometry of these counterexamples, which from now on will be called *DR-spaces*, without pretending to give an exhaustive survey. We refer to [14], [15], [26] for further information and to a forthcoming paper [27] for more details about the proofs and for further results.

We start the paper by focusing in Section 2 on the algebraic structure of the DR-spaces and on the non-positivity of their curvature. Then, in Section 3, we show how to characterize them as a very special subclass of homogeneous Hadamard manifolds

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and we derive some properties which focus on the contrasting features of their geometry when compared with that of the symmetric examples. Section 4 is devoted to a brief discussion about harmonic spaces and in Section 5 we show how the DR-spaces also provide counterexamples for some open problems or conjectures about the so-called k-harmonic spaces. This leads to the following conjecture :

> *A Riemannian manifold is locally isometric to a two-point homogeneous space if and only if all sufficiently small geodesic spheres are isoparametric hypersurfaces, that is, have constant principal curvatures.*

We provide a partial positive answer, namely for all odd-dimensional manifolds and for dimension ≤ 4.

Finally, we note that the study of harmonicity may be based on an extensive use of the theory of Jacobi vector fields and hence of the Jacobi operator. The Osserman conjecture states [23] that two-point homogeneous spaces may be characterized locally by the global constancy of the eigenvalues of the Jacobi operators (for all unit vectors). This conjecture is very similar to the one stated above. In Section 6 we give a rather short proof of the fact that the DR-spaces do not provide counterexamples for it. We refer to [26] for an alternative but longer proof based on the explicit computation of the Jacobi operators.

The authors are grateful to F. Ricci for giving them the opportunity to know his and Damek's result already in 1991 and for several useful discussions.

2. DR-spaces : algebraic structure and curvature

We start with a description of the DR-spaces and some aspects of their curvature. The DR-spaces are simply connected solvable Lie groups of a very special kind, endowed with a suitable left invariant Riemannian metric $\langle , \rangle$. The Lie algebra $\mathfrak{g}$ of G splits as an orthogonal sum $\mathfrak{a} + \mathfrak{n}$ of its derived subalgebra $\mathfrak{n} = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and a one-dimensional subalgebra $\mathfrak{a}$. The nilpotent subalgebra $\mathfrak{n}$, endowed with the inner product induced by $\langle , \rangle$, is a *Lie algebra of Heisenberg type*. This means that

(i) $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement;
(ii) $[\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{z}$, that is, $\mathfrak{n}$ is a two-step nilpotent Lie algebra;
(iii) the operators J_Z defined for each $Z \in \mathfrak{z}$ by

$$\langle J_Z V, V' \rangle = \langle [V, V'], Z \rangle, \quad V, V' \in \mathfrak{v},$$

satisfy

$$J_Z^2 = -\|Z\|^2 id_{\mathfrak{v}}.$$

The action of $\mathfrak{a}$ on $\mathfrak{n}$ is defined as follows. There exists a *unit* vector A of $\mathfrak{a}$ such that

$$ad_A(V) = \frac{1}{2}V, \quad ad_A(Z) = Z \tag{2.1}$$

for all $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$.

The Riemannian invariants of $(G, \langle , \rangle)$ such as the Levi Civita connection and the Riemannian curvature tensor may be computed via standard methods. This has been done independently by E. Damek and J. Boggino (see [6], [11], [12], [13]). In particular, the *sectional curvature* K is given by

$$\begin{aligned} 4K(\mathrm{span}(X,Y)) = & - a^2\|aZ' + [V,V']\|^2 - 3\langle Z,Z'\rangle^2 \\ & - (3\|Z\|^2\|Z'\|^2 + 6\langle J_Z V, J_{Z'}V'\rangle + 1) \end{aligned} \tag{2.2}$$

where

$$X = aA + V + Z, \quad Y = V' + Z' \tag{2.3}$$

are orthogonal unit vectors. From the expression (2.2) one easily deduces

Proposition 2.1. [6], [12] *The Riemannian manifold* $(G, \langle,\rangle)$ *has non-positive sectional curvature.*

Moreover, J. Boggino proved

Proposition 2.2. [6] *The Riemannian manifold* $(G, \langle,\rangle)$ *has negative sectional curvature if and only if it is a symmetric space.*

Note that a simply connected symmetric space of strictly negative curvature must be a rank one symmetric space of non-compact type and hence, it is isometric to the hyperbolic real space $H\mathbb{R}^n$, the hyperbolic complex space $H\mathbb{C}^n$, the hyperbolic quaternionic space $H\mathbb{H}^n$ or the exotic hyperbolic Cayley plane $H\mathbb{C}ay^2$.

A DR-space with one-dimensional center is always symmetric and hence isometric to $H\mathbb{C}^n$. Those with $\dim \mathfrak{z} \neq 1, 3, 7$ are never symmetric but there are non-symmetric DR-spaces even when $\dim \mathfrak{z} = 3$ or 7. We will call the non-symmetric examples *non-classical DR-spaces* in contrast with the classical ones, namely $H\mathbb{H}^n$ and $H\mathbb{C}ay^2$. We refer to [10] for their explicit construction.

3. DR-spaces as homogeneous Hadamard manifolds

The result of Damek and Boggino shows that the DR-spaces are homogeneous manifolds of non-positive curvature, that is, *homogeneous Hadamard manifolds* (briefly denoted by HHM in what follows). The structure of an HHM is well-known. First of all, it can be realized as a *solvable* Lie group G endowed with a left invariant Riemannian metric (see [1], [2], [3], [18]). Moreover, the orthogonal complement $\mathfrak{a}$ of the derived subalgebra $\mathfrak{n} = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ of the Lie algebra $\mathfrak{g}$ of G is *Abelian* ([2], [3]). But this is not enough. In fact, a left invariant Riemannian metric on G has non-positive sectional curvature if and only if $\mathfrak{g}$ is an NC-*algebra*. This notion has been introduced by H. Azencott and E. Wilson to whose papers [2], [3] we refer for all the details. Here we shall restrict to the subclass of *Lie algebras of Iwasawa type*.

A solvable Lie algebra $\mathfrak{g}$ is said to be of *Iwasawa type* if

(i) $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ where $\mathfrak{n} = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{a}$ is Abelian;
(ii) all operators ad_A, $A \in \mathfrak{a}$, are semisimple (that is, diagonalizable over $\mathbb{C}$);
(iii) there is an $A_0 \in \mathfrak{a}$ such that $ad_{A_0}|_{\mathfrak{n}}$ has eigenvalues with strictly positive real parts.

All these algebras are NC-algebras and therefore, the corresponding simply connected solvable Lie groups carry a left invariant metric of non-positive sectional curvature. Moreover, the Lie algebras of Iwasawa type are natural generalizations of the Lie algebras attached to the symmetric spaces of non-compact type.

Next, following T.H. Wolter [32] we say that the Lie algebra of Iwasawa type $\mathfrak{g}$ is *metric* if $\mathfrak{g}$ admits an inner product $\langle , \rangle$ such that $\mathfrak{a}$ and $\mathfrak{n}$ are *orthogonal* and all the operators ad_A are *symmetric*.

Now, let $\mathfrak{g}$ be such a Lie algebra and put $G = \exp \mathfrak{g}$. Denote by $\langle , \rangle$ the left invariant metric induced by the inner product of $\mathfrak{g}$. Let $\mathfrak{G}$ be the class of metric Lie groups of Iwasawa type such that $\mathfrak{n}$ (with the induced scalar product) is a two-step nilpotent Lie algebra of Heisenberg type and $\dim \mathfrak{a} = 1$. Then the DR-spaces can be characterized as the groups of $\mathfrak{G}$ which are Einsteinian. In fact, it is sufficient to compare the definition of a DR-space with the following

Proposition 3.1. *Let $(G, \langle , \rangle)$ be in $\mathfrak{G}$. Then it is an Einstein manifold if and only if there exists a unit vector $A \in \mathfrak{a}$ such that*

$$2ad_A|_{\mathfrak{v}} = id_v, \quad ad_A|_{\mathfrak{z}} = id_{\mathfrak{z}}$$

where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement in $\mathfrak{n}$.

Proof. From [32, Proposition 1.5] we get that there exists a unit vector $A \in \mathfrak{a}$ such that ad_A and the Ricci operator $\mathrm{ric}^{\mathfrak{n}}$ of the induced metric on $N = \exp \mathfrak{n}$ have common eigenvectors $X_1, ..., X_n$ with eigenvalues $\lambda_1, ..., \lambda_n$ and $\nu_1, ..., \nu_n$, respectively, where $n = \dim \mathfrak{n}$. Moreover, the metric $\langle , \rangle$ on G is Einsteinian, with scalar curvature $\tau = -c/n$, if and only if

i) $\lambda_i = \frac{1}{d}(\nu_i + c), \quad d^2 = \sum_j \frac{1}{c}(\nu_j + c)^2$;

ii) $\sum_i \nu_i(\nu_i + c) = 0$;

iii) $\sum_j \lambda_j^2 = c$.

On the other hand, the Ricci operator $\mathrm{ric}^{\mathfrak{n}}$ has eigenvalues $\nu_1 = -\frac{1}{2}k$ with multiplicity $m = \dim \mathfrak{v}$, and $\nu_2 = \frac{1}{4}m$ with multiplicity $k = \dim \mathfrak{z}$ (see [19, p. 132]). Hence (ii) yields

$$4c = m + 4k.$$

It follows that

$$4d^2 = (m + 2k)^2$$

and therefore

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = 1$$

which proves the required result.

Among the metric Lie groups of Iwasawa type the DR-spaces have another interesting geometric property : *they carry always a left invariant metric of (strictly) negative curvature.* In fact, put

$$\begin{aligned} \langle X, Y \rangle_t &= \langle aA + V + Z, \quad a'A + V' + Z' \rangle \\ &= t^2 aa' + \langle V, V' \rangle + \langle Z, Z' \rangle. \end{aligned} \tag{3.1}$$

Then a result of E. Heintze [18, Theorem 2] implies easily that there exists a $t_0 > 0$ such that for all $t \geq t_0, \langle , \rangle_t$ has strictly negative sectional curvature. Of course, in view of Proposition 3.1, $\langle , \rangle_t$ is an Einstein metric if and only if $t = 1$. Therefore, the DR-spaces can also be characterized as the only Einstein metrics in the one-parameter family (3.1).

We continue this section with some additional results concerning DR-spaces. As we mentioned already, *DR*-spaces are harmonic. Now, every non-flat harmonic space is irreducible (see for example [7]) and so *all DR-spaces are irreducible.* In particular, they do not have a Euclidean de Rham factor. (Note that this also follows from considerations about the rank and the visibility axiom for an HHM [16].) So, we get

Theorem 3.2. *Let $(G, \langle , \rangle)$ be a DR-space. Then the following statements are equivalent :*

(i) $(G, \langle , \rangle)$ *is symmetric;*

(ii) the geodesics of $(G, \langle , \rangle)$ *are orbits of one-parameter subgroups of* $\mathrm{Isom}_o(G, \langle , \rangle)$ *(in other words,* $(G, \langle , \rangle)$ *is a g.o. space* [21]*);*

(iii) $(G, \langle , \rangle)$ *admits a quotient of finite volume;*

(iv) $(G, \langle , \rangle)$ *is a commutative space (that is, the algebra of invariant differential operators is commutative with respect to* $\mathrm{Isom}_o(G, \langle , \rangle)$*).*

Proof. The equivalence of (i), (ii) follows from a general result proved in [31]. On the other hand, the equivalence of (i) and (iii) is proved in [3]. Note that these equivalences hold for an arbitrary HHM. The equivalence of (i) and (iv) is stated in [32]. (In [6] it is proved that a naturally reductive DR-space is symmetric.)

Finally, we note that each g.o. space has volume-preserving geodesic symmetries [20]. The converse does not hold. In fact, since any harmonic space has volume-preserving geodesic symmetries, Theorem 3.2 yields that *all the non-classical* DR-*spaces provide examples of manifolds with volume-preserving geodesic symmetries which are neither g.o. spaces nor commutative spaces.*

4. Harmonic spaces

In view of the most surprizing aspect of the DR-spaces, namely their harmonicity, we concentrate in this section on some well-known facts about harmonic manifolds. For more details we refer to [4], [25], [28], [29].

Let (M, g) be a (connected) n-dimensional Riemannian manifold and $m \in M$. Further, let $(x^1, ..., x^n)$ be a system of normal coordinates and g_{ij} the components of g with respect to these coordinates. Then (M, g) is said to be *(locally) harmonic* if and only if the *volume density function*

$$\theta_m(r, u) = (\det g_{ij})^{1/2}(\exp_m(ru)) \tag{4.1}$$

is a *radial function*, that is, if it does not depend on the unit tangent vector u, for all $m \in M$ and all sufficiently small $r > 0$.

For this volume density function we have, in general,

$$\theta_m(r, u) = r^{1-n} \det A(r) \tag{4.2}$$

where $A(r)$ is an $(n-1)\times(n-1)$ non-singular matrix which satisfies the *Jacobi equation*

$$A'' + R \circ A = 0 \tag{4.3}$$

along the geodesic $\gamma : r \mapsto \exp_m(ru)$ through $m = \gamma(0)$ and with initial conditions

$$A(0) = 0, \quad A'(0) = I. \tag{4.4}$$

Here $R = R_{\gamma'}.\gamma'$ denotes the Jacobi operator field with respect to γ. The elements $R^a_b(r)$ of the matrix $R(r)$ are the components with respect to the basis obtained by parallel translating the orthogonal basis $\{\frac{\partial}{\partial x^i}(m), i = 1, ..., n\}$ along γ and where $\frac{\partial}{\partial x^1}(m) = u$.

Derivation of (4.2) yields

$$(\theta'_m\theta_m^{-1})(r,u) = -(n-1)r^{-1} + \operatorname{tr}(A'A^{-1})(r). \tag{4.5}$$

The trace of $A'A^{-1}$ has two important features. From the *analytical* point of view we have

$$\Delta r = -\operatorname{tr}(A'A^{-1})(r) \tag{4.6}$$

where Δr is the Laplacian on (M,g) of the distance function. From the *geometrical* point of view we have

$$h_m(r,u) = \operatorname{tr}(A'A^{-1})(r) \tag{4.7}$$

where $h_m(r,u)$ denotes the *mean curvature* of the geodesic sphere $G_m(r)$ with center r and small radius r, at $p = \exp_m(ru)$. In fact, the *shape operator* $S_m(r,u)$ at p of $G_m(r)$ is given by

$$S_m(r,u) = (A'A^{-1})(r). \tag{4.8}$$

Note that here again we have identified both members in (4.8) via the parallel translation mentioned above. So, (4.5), (4.6) and (4.7) yield the fundamental formula

$$h_m(r,u) = (n-1)r^{-1} + (\theta'_m\theta_m^{-1})(r,u). \tag{4.9}$$

Using (4.6) one obtains easily that (M,g) *is harmonic if and only if* Δr *is a radial function*, that is,

$$\Delta r = F(r) \tag{4.10}$$

or, *if and only if the small geodesic spheres have constant mean curvature.*

Further, using (4.3) and (4.8) one gets at once that the shape operator $S = (S_m)$ satisfies a first order non-linear differential equation of Riccati type, namely

$$S' + S^2 + R = 0. \tag{4.11}$$

From this equation one obtains

$$rS(r) = I - \frac{1}{3}R(0)r^2 + O(r^3). \tag{4.12}$$

Therefore, if (M,g) is harmonic,

$$\operatorname{tr} R(0) = \sum_a (R_m)(u,e_a,u,e_a) = (\rho_m)(u,u),$$

where $\{e_a, a = 1,\ldots,n\}$ is an arbitrary orthonormal basis and ρ the Ricci tensor of type $(0,2)$, is independent of u and hence, (M,g) is an *Einstein space.*

Here we note that each Einstein manifold is analytic in harmonic coordinates and hence in normal coordinates (see for example [5]). Hence, from (4.11) one may derive the Taylor expansion of $rS(r)$ in function of the curvature tensor R and its covariant derivatives. Also, (4.11) yields the so-called Ledger recursion formulas (see [4], [7], [24], [28], [29]). The Taylor expansion and the recursion formulas play an important role in the study of the geometry in a normal neighborhood and in the study of harmonic spaces.

Finally, for a harmonic space, the scalar curvature is constant. Since, as a consequence of Szabó's result, a complete harmonic space with $\tau > 0$ is locally isometric to a two-point homogeneous space, we are left with the following

Problem 1. *Classify the complete harmonic spaces with non-positive scalar curvature.*

In particular, we state

Problem 2. *Do there exist examples of harmonic spaces which are not (locally) homogeneous ?*

Problem 3. *Do there exist non-flat Ricci-flat harmonic spaces ?*

Problem 4. *Is it true that a harmonic space with negative sectional curvature is locally symmetric ?*

It could be worthwhile to start here first with the homogeneous case and use the results of [18] for an HHM.

5. k-HARMONIC SPACES

The examples of Damek and Ricci also provide counterexamples for several open problems and conjectures about the so-called k-harmonic spaces. There are several notions of k-harmonicity. We refer to [28], [29] for more details and for a description of some of the open problems. In this section we shall concentrate on one of these notions which we introduce now by using the shape operator of a small geodesic sphere. For the other notions and the other results see [27].

A Riemannian manifold is said to be *k-harmonic* if the k-th elementary symmetric function of the eigenvalues of the shape operators of all small geodesic spheres are radial functions. Hence, 1-harmonic means harmonic in the usual sense. Using the results of [7] one has

Proposition 5.1. *(M, g) is 2-harmonic if and only if it is harmonic.*

Up to now it was an open problem if k-harmonic for one k implies k-harmonic for all k or if k-harmonic implies ℓ-harmonic for some $\ell \neq k$. The DR-spaces provide negative answers. Indeed, we have

Proposition 5.2. *A DR-space (M, g) is 3-harmonic if and only if it is locally symmetric.*

Proof. First, let (M, g) be locally symmetric. Then it is locally isometric to a two-point homogeneous space and the result follows at once since all the eigenvalues of the shape operator or radial functions.

Conversely let (M, g) be a DR-space. Then we use

Lemma 5.3. *Let (M, g) be a harmonic space which is also 3-harmonic. Then*

$$\operatorname{tr} R'(0)R'(0) = \sum_{a,b} (\nabla_u R)(u, e_a, u, e_b)(\nabla_u R)(u, e_a, u, e_b)$$

is independent of the unit vector $u \in T_m M$ for all $m \in M$.

(The proof will be given in [27]. We just mention here that it follows from an extensive use of the power series expansion for the shape operator mentioned above.)

Now by integrating the relation in Lemma 5.3 over the unit sphere in $T_m M$ (see [7], [17] for more details) we obtain

$$\sum_{a,b} \{(\nabla_u R)(u, e_a, u, e_b)\}^2 = \frac{3}{n(n+2)(n+4)} \|\nabla R\|^2. \tag{5.1}$$

But, for a DR-space $(G, \langle , \rangle)$, we always have [13]

$$(\nabla_A R)_{XY} Z = 0 \tag{5.2}$$

where $A \in \mathfrak{a}$ and X, Y, Z are arbitrary. Therefore, (5.1) and (5.2) imply $\nabla R = 0$ and this proves the required result.

As a consequence of this result a natural problem arises : *study the spaces which are k-harmonic for all $k \geq 1$*. This condition means that the shape operators of the geodesic spheres have radial eigenvalues with contant multiplicities, or otherwise, all the small geodesic spheres are *isoparametric hypersurfaces*. In this context we state the following

Conjecture. *A Riemannian manifold is locally isometric to a two-point homogeneous space if and only if all small geodesic spheres are isoparametric hypersurfaces.*

Note that not all the small geodesic spheres of a non-classical DR-space are isoparametric since such a DR-space is not 3-harmonic. Nevertheless, they are level surfaces of the distance function r which is an *isoparametric function*. In fact, the harmonicity yields

$$\Delta r = F(r) \tag{5.3}$$

and $\|\operatorname{grad} r\|^2 = 1$.

We finish this section with some partial results about the conjecture.

Theorem 5.4. *The conjecture is true for* $\dim M = 2, 3, 4$ *and* $2k + 1$.

Proof. First note that small geodesic spheres in a space which is locally isometric to a two-point homogeneous space are isoparametric hypersurfaces.

Conversely, suppose that all small geodesic spheres of (M, g) are isoparametric. Then (M, g) is harmonic and the result follows trivially for $\dim M = 2, 3, 4$ since then (M, g) is locally symmetric and hence locally isometric to a two-point homogeneous space. Therefore, we suppose from now on that $\dim M = 2k + 1$. The result for this case follows from the two following lemmas.

Lemma 5.5. [30] *A Riemannian manifold has constant curvature if and only if all sufficiently small geodesic spheres are umbilical hypersurfaces.*

Lemma 5.6. *Each isoparametric geodesic sphere is an umbilical hypersurface.*

Proof of Lemma 5.6. Let $N = G_m(r)$ be the small geodesic sphere with center m and radius r. The shape operator S of N may be considered as a map defined on the orthonormal frame bundle ON with values in $\operatorname{Hom}(V, V)$ where $V = \mathbb{R}^{2k}$. The map is defined by

$$S(u)\xi = (u^{-1} \circ S_{\pi(u)} \circ u)\xi \tag{5.4}$$

for each $u \in ON, \xi \in V$. Here π denotes the projection of ON onto N and u is considered as an isomorphism between $V = \mathbb{R}^{2k}$ and $T_{\pi(u)}N$. The orthogonal group $O(2k)$ acts on the left on $\operatorname{Hom}(V, V)$ and on the right on ON. Moreover S is *equivariant* with respect to these actions, namely

$$S(ua) = a^{-1} S(u) \tag{5.5}$$

for each $a \in O(2k)$.

On the other hand, N is isoparametric if and only if $S(ON)$ is contained in a single $O(2k)$-orbit of an element S_0 of $\mathrm{Hom}(V,V)$ for which we can suppose to have the diagonal form (the canonical form of a symmetric endomorphism). Then, it follows from a general reduction theorem (see for example [22]) that ON is reducible to the stability subgroup H of S_0 in $O(2k)$. If S_0 has h distinct eigenvalues with multiplicities $m_1, ..., m_h$, respectively, then

$$H = O(m_1) \times ... \times O(m_h)$$

where $\sum_{i=1}^{h} m_i = 2k$. This means that TN has h vector subbundles of rank $m_1, ..., m_h$, respectively. But N is diffeomorphic to an even-dimensional sphere and therefore TN does not have any non-trivial subbundle. Hence $H = O(2k)$ and S_0 is a multiple of the identity, that is, N is umbilical.

6. The Osserman conjecture and some more results

The conjecture about the extrinsic geometry of geodesic spheres stated in Section 5 has an interesting intrinsic analog, known as the Osserman conjecture. To state this conjecture we first note that (M, g) satisfies the *Osserman hypothesis* if the Jacobi operator

$$j_{m,u} : x \mapsto (R_m)_{ux}u, \quad \|u\| = 1,$$

has globally constant eigenvalues (counting multiplicities), that is, the eigenvalues are independent of u and m. Then we have the

Conjecture of Osserman. [23] *An irreducible Riemannian manifold satisfying the Osserman hypothesis is locally isometric to a rank one symmetric space.*

Q.-S. Chi proved this conjecture in several cases [8], [9]. Here we prove a result which is also stated in [26] but our proof is different.

Proposition 6.1. *A DR-space satisfies the Osserman hypothesis if and only if it is symmetric.*

Proof. The result follows at once for a symmetric DR-space.

Conversely, let (M, g) be a DR-space which satisfies the Osserman hypothesis. Then

$$\sum_{a,b,c} R(u, e_a, u, e_b)R(u, e_b, u, e_c)R(u, e_c, u, e_a) = \lambda \tag{6.1}$$

where λ is independent of the unit tangent vector $u \in T_mM$ and m. Here $\{e_a, a = 1, ..., \dim M\}$ is a arbitrary orthonormal basis of T_mM.

Using again the power series technique (see [27] for more details) we have

Lemma 6.2. *Let* (M, g) *be harmonic. Then*

$$\mathrm{tr}\, R(0)R'(0) = 0 \tag{6.2}$$

and

$$\mathrm{tr}(45R'(0)R'(0) + 48R''(0)R(0) + \frac{32}{3}R(0)R(0)R(0)) = \mu \tag{6.3}$$

where μ is independent of m and u.

From this lemma and (6.1) we obtain that $\operatorname{tr} R'(0)R'(0)$ is independent of u and hence the result follows as in the proof of Proposition 5.2.

Remarks. A. Sometimes the Osserman conjecture is stated by deleting the independence of the point m. (In this case we call it the *pointwise* Osserman conjecture.) But this does not change the result proved above.

B. For the shape operator S_m of a sufficiently small geodesic sphere $G_m(r)$ one has the expansion (4.12). Further, the operator

$$\bar{S}_m = \frac{1}{r}(S_m - \frac{1}{r}I)$$

is a smooth operator on $[0, \epsilon)$ for ϵ sufficiently small. If the eigenvalues of S_m are radial functions, then also those of $\bar{S}_m$ are radial. By taking the limit as $r \to 0$ and using (4.12) one gets that in this case also the eigenvalues of the Jacobi operator $R(0)$ are independent of the direction and so we have

Theorem 6.3. *A Riemannian manifold all of whose sufficiently small geodesic spheres are isoparametric satisfies the pointwise Osserman hypothesis.*

So, we may conclude that Theorem 5.4 extends to all the cases where the pointwise Osserman conjecture is true.

We finish this note with some comments about the behavior of the geometry of the classical DR-spaces compared with that of the non-classical ones. First, in contrast to the classical examples *the non-classical examples are not*

(i) *g.o. spaces*;

(ii) *commutative spaces*,

and do not

(iii) *satisfy the Osserman hypothesis*;

(iv) *admit quotients of finite volume (in particular, compact ones).*

On the other hand, *all DR-examples are harmonic and the corresponding classical and non-classical examples (with center of dimension* 3 *or* 7*) have the same*

(i) *volume density function θ*;

(ii) *linear and quadratic invariants $\tau, \|R\|^2, \|\rho\|^2$*,

and

(iii) *the geodesic spheres with the same radius have the same volume, mean curvature and scalar curvature.*

(i) follows from the explicit expression for θ given in [14], [15]. (ii) follows from the Taylor expansion for θ (see [7], [17]). Finally, (iii) follows from (i), (4.9), the Gauss equation for a hypersurface and the Riccati equation (4.11).

Note that (iii) implies that there exist non-symmetric Riemannian manifolds whose geodesic spheres have the same volume as in the hyperbolic quaternionic space $H\mathbb{H}^n$ without being isometric to this last model space. This shows that the hypothesis on the holonomy cannot be dropped in Theorem 9.2 of [17].

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