Differential Geometry and Its Applications
Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992
Silesian University, Opava, 1993, 193–200

A REDUCTION PROPERTY FOR A CLASS OF MECHANICAL SYSTEMS WITH FRICTION

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Abstract. A reduction scheme is established for a special class of non-Hamiltonian systems, defined on a cotangent bundle, which are invariant under the Hamiltonian action of a compact Lie group and for which the corresponding momentum map yields a conformal invariant. The reduction is constructed in terms of a suitably normalized momentum map and the reduced spaces are canonical manifolds. The class of systems under consideration in particular encompasses the phase space description of certain types of mechanical systems with friction.

Keywords. Symmetry, reduction, canonical manifolds.

MS classification. 58F05, 70G99.

1. Introduction

In a recent paper by Landi et al. [?], a general reduction procedure for dynamical systems has been proposed. The key observation in their analysis is that, given a vector field X on a manifold M, a reduction of X essentially requires the following two ingredients: (i) a submanifold M' of M which is invariant for X (*i.e.* such that X is everywhere tangent to M'), (ii) an equivalence relation on M' which, in an appropriate sense, is compatible with the flow of X. In case this equivalence relation induces a regular foliation on M', the restriction of X to M' will then project onto the corresponding quotient manifold.

The most prominent example of such a reduction scheme in mechanics is undoubtedly provided by the reduction of Hamiltonian (and Lagrangian) systems with symmetry. In brief the situation can be described as follows. (For a detailed treatment, see *e.g.* [?, ?, ?]). Let (M,ω) be a symplectic manifold and X_H a global Hamiltonian vector field on M, with Hamiltonian $H \in C^\infty(M)$. Let there be given a symplectic action of a Lie group G on (M,ω) with an Ad^*- equivariant momentum map $J : M \longrightarrow \mathcal{G}^*$, where $\mathcal{G}^*$ is the linear dual space of the Lie algebra $\mathcal{G}$ of G, and Ad^* denotes the coadjoint representation of G on $\mathcal{G}^*$. Suppose this action leaves H invariant and, hence, defines a symmetry of X_H. Then, J is a $\mathcal{G}^*$-valued first integral of X_H. This is in fact a geometrical transcription of Noether's theorem. Let μ be a weakly regular value of J and let G_μ denote the isotropy subgroup of G at μ under the coadjoint action. According to the Marsden-Weinstein reduction theorem [?], if G_μ acts freely and properly on the level set $J^{-1}(\mu)$, there exists a uniquely determined symplectic manifold structure on the orbit

This paper is in final form and no version of it will be submitted for publication elsewhere.

space $\mathcal{P}_\mu = J^{-1}(\mu)/G_\mu$. Exploiting the symmetry property of the given system again, one can then prove that $X_H|_{J^{-1}(\mu)}$ projects onto $\mathcal{P}_\mu$ and the projected vector field is also Hamiltonian. Referring to the general reduction scheme mentioned above, we thus have here that the invariant submanifolds of the given dynamical system X_H are the (regular) level sets of the momentum map, and the appropriate equivalence relation on such a level set is determined by the action of the symmetry group.

The purpose of the present paper now is to study the reduction of a particular class of non-Hamiltonian systems with symmetry defined on a cotangent bundle T^*Q. The systems we have in mind may serve as a model for the phase space description of mechanical systems with certain types of non-conservative forces (such as, for instance, frictional forces proportional to the velocity). The symmetry is again defined in terms of a strongly Hamiltonian group action (*i.e.* a symplectic action with an Ad^*-equivariant momentum map), which arises from the cotangent lift of an action on the base manifold Q. For the systems under consideration it turns out that the momentum map J, rather than being an invariant, yields a conformal invariant. (Noether's theorem does not apply to non-Hamiltonian systems!) In case the symmetry goup is compact, it will be shown that one can still establish a reduction scheme, similar to the one obtained for Hamiltonian systems with symmetry. The invariant submanifolds of the given system are now provided by the level sets of the normalized momentum map, where the normalization of J is defined with respect to a G-invariant norm on $\mathcal{G}^*$. The equivalence relation on such an invariant submanifold, needed to perform a reduction, is again determined by the symmetry group action. According to a result described elsewhere (see [?]), the resulting quotient spaces turn out to be Poisson manifolds or, more precisely, canonical manifolds.

In the next section we will first investigate the behaviour of the systems under consideration in the presence of an arbitrary (*i.e.* not necessarily compact) symmetry group. Conformal invariance of the momentum map will then only result in a reduction of the system with respect to the zero level set of the momentum map. Section 3 then deals with the case of a compact symmetry group and describes the reduction procedure in terms of the normalized momentum map.

We close this introduction by fixing some notation. Let G be a Lie group, with Lie algebra $\mathcal{G}$, and suppose there is given a smooth (left) action of G on a manifold M, $\phi : G \times M \longrightarrow M$. For any $g \in G$ we put $\phi_g = \phi(g, .)$. The isotropy subgroup of G at a point $m \in M$ is denoted by G_m, *i.e.* $G_m = \{g \in G | \, \phi(g, m) = m\}$. The fundamental vector field of the given action, corresponding to $\xi \in \mathcal{G}$, is the vector field ξ_M on M defined by

$$\xi_M(m) = \frac{d}{dt}\phi(exp(-t\xi), m)|_{t=0}.$$

Note that in defining ξ_M we follow the conventions of [?]. In particular, the assignment $\xi \longmapsto \xi_M$ then yields a Lie algebra homomorphism from $\mathcal{G}$ into the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on M. The Lie algebra of the isotropy subgroup G_m is the subalgebra $\mathcal{G}_m$ of $\mathcal{G}$ given by $\mathcal{G}_m = \{\xi \in \mathcal{G} | \, \xi_M(m) = 0\}$. Finally, the natural pairing between $\mathcal{G}$ and $\mathcal{G}^*$ and, more generally, the pairing between any linear space and its dual, will be denoted by $\langle \, , \rangle$.

2. Mechanical systems with friction and symmetry

Let Q be a smooth, finite dimensional, connected manifold and T^*Q its cotangent bundle, with projection $\pi_Q : T^*Q \longrightarrow Q$. On T^*Q we have the canonical Liouville 1-form θ_Q and the corresponding symplectic form $\omega_Q = d\theta_Q$ which, in canonical coordinates (q^i, p_i), read $\theta_Q = p_i dq^i$ and $\omega_Q = dp_i \wedge dq^i$. We now consider a dynamical system, described by a vector field X on T^*Q which is determined by

$$i_X \omega_Q = -dH - f\theta_Q, \tag{1}$$

for some functions $H, f \in C^\infty(T^*Q)$. In canonical coordinates, the corresponding equations of motion read

$$\dot{q}^i = \frac{\partial H}{\partial p_i}(q,p), \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}(q,p) - f(q,p)p_i. \tag{2}$$

These equations can be thought of as representing the phase space description of a mechanical system with Hamiltonian $H = T + V$ (kinetic + potential energy) and $F_i = -fp_i$ the i^{th} 'generalized' component of a nonconservative force. In particular, this nonconservative force can be a frictional or resistive force, in which case f will be strictly positive. An example of such a system is provided by a model for the Kepler problem with drag, with vector equation of motion with respect to a Cartesian reference frame (in the appropriate units):

$$\ddot{\mathbf{r}} + f(\mathbf{r}, \dot{\mathbf{r}})\dot{\mathbf{r}} + k\frac{\mathbf{r}}{r^3} = \mathbf{0}$$

where k is a real constant (see *e.g.* [**?**]). When passing to the phase space description we indeed obtain a system of the form (2). Although we will mainly think of (1) as representing a mechanical system with friction, for the subsequent analysis we do not make any specific assumption about the sign of the function f.

Suppose we are given a smooth (left) action $\phi : G \times Q \longrightarrow Q$ of a connected Lie group G on Q. For simplicity we henceforth assume this to be a *free action*. Let $\Phi : G \times T^*Q \longrightarrow T^*Q$ be the canonically lifted action of G on T^*Q, *i.e.* Φ is completely determined by

$$\langle v, \Phi_g(\alpha) \rangle = \langle T\phi_{g^{-1}}(v), \alpha \rangle$$

for each $\alpha \in T^*Q$ and $v \in T_{\phi_g(m)}Q$, with $m = \pi_Q(\alpha)$. It is well-known that Φ is a *strongly Hamiltonian action* on (T^*Q, ω_Q), *i.e.* a symplectic action with an Ad^*-equivariant momentum map $J : T^*Q \longrightarrow \mathcal{G}^*$ (*cf.* [**?**, **?**]). In fact, we have for each $g \in G$,

$$\Phi_g^* \theta_Q = \theta_Q \tag{3}$$

and

$$i_{\xi_{T^*Q}} \omega_Q = -dJ_\xi, \tag{4}$$

where $J_\xi = \langle \xi_{T^*Q}, \theta_Q \rangle \in C^\infty(T^*Q)$ is a global Hamiltonian for ξ_{T^*Q}. The momentum map J is then determined by

$$\langle \xi, J(\alpha) \rangle = J_\xi(\alpha)$$

for each $\alpha \in T^*Q$ and each $\xi \in \mathcal{G}$. From the general properties of a momentum map we recall, in particular, that the image (Im) of the tangent map of J at a point $\alpha \in T^*Q$ is given by

$$Im(T_\alpha J) = \mathcal{G}_\alpha^0 \tag{5}$$

where $\mathcal{G}_\alpha^0 \subset \mathcal{G}^*$ is the annihilator of the isotropy subalgebra $\mathcal{G}_\alpha$ (see *e.g.* [**?**, **?**, **?**]). Since ϕ was assumed to be a free action, the lifted action Φ is also free and, hence, $\mathcal{G}_\alpha = \{0\}$. From (5) it then follows that J is a submersion and thus, for each $\nu \in \mathcal{G}^*$, $J^{-1}(\nu)$ is a closed regular submanifold of T^*Q (or the empty set).

Returning to the dynamical system X defined by (1), we first derive the following property:

Proposition 1. *If Φ leaves H invariant, i.e. $H \circ \Phi_g = H$ for all $g \in G$, then*

$$X(J) = -fJ.$$

PROOF: Taking the contraction of both sides of (1) with ξ_{T^*Q}, for $\xi \in \mathcal{G}$, we find

$$-i_X(i_{\xi_{T^*Q}}\omega_Q) = -\xi_{T^*Q}(H) - f\langle \xi_{T^*Q}, \theta_Q \rangle.$$

Since H is invariant under Φ it follows that $\xi_{T^*Q}(H) = 0$ and thus, with (4), we obtain that

$$X(J_\xi) = -fJ_\xi.$$

This holds for every $\xi \in \mathcal{G}$, from which the result follows. □

Under the conditions of the proposition, we thus have that the momentum map of the given action Φ is a conformal invariant of X. Moreover, J being a submersion, $0 \in \mathcal{G}^*$ is a regular value of J and so we readily have:

Corollary 1. *If Φ leaves H invariant, then $J^{-1}(0)$ is an invariant submanifold of X.* □

Note that for the previous properties to hold, it suffices that H is invariant under the given group action, *i.e.* G need not be a symmetry group for the dynamical system X. If we now also require invariance of f, then we can easily derive the following result. (Note that, in view of the equivariance of J and the fact that $G_0 = G$, the action Φ restricts to an action of G on $J^{-1}(0)$).

Theorem 1. *If both H and f are invariant under Φ, then G is a symmetry group of X. Moreover, if G acts properly on $J^{-1}(0)$, $X|_{J^{-1}(0)}$ projects onto the reduced Marsden-Weinstein symplectic manifold $\mathcal{P}_0 = J^{-1}(0)/G$.*

PROOF: Using (3) and the invariance of H and f, it immediately follows from (1) that

$$\begin{aligned} i_{\Phi_g^* X}\omega_Q &= -dH - f\theta_Q \\ &= i_X\omega_Q. \end{aligned}$$

Hence, $\Phi_g^* X = X$ for each $g \in G$, which proves that G is a symmetry group of X.
From the previous corollary we already know that X is everywhere tangent to $J^{-1}(0)$. The action of G induced on $J^{-1}(0)$ is free (since Φ is free) and proper (by assumption). The Marsden-Weinstein reduction theorem then tells us that the orbit space $\mathcal{P}_0 = J^{-1}(0)/G$ is a smooth symplectic manifold. Since G is a symmetry group of X, the flow of $X|_{J^{-1}(0)}$ commutes with the group action and, therefore, induces a flow on $\mathcal{P}_0$. □

Let us denote the symplectic form on $\mathcal{P}_0$ by ω_0. In case the actio ϕ of G on Q, which we already assumed to be free, is also proper, one can prove that $(\mathcal{P}_0, \omega_0)$ is symplectomorphic to $(T^*(Q/G), \omega_{Q/G}$ (*cf.* [?]). One can then easily verify that the reduced vector field X_0, *i.e.* the projection of $X|_{J^{-1}(0)}$ onto $\mathcal{P}_0$, satisfies the equation

$$i_{X_0}\omega_{Q/G} = -dH_0 - f_0\theta_{Q/G},$$

where H_0 and f_0 are the functions induced by H and f, respectively, on $\mathcal{P}_0 \cong T^*(Q/G)$.

In the next section we continue the analysis of the dynamical system X under the additional hypothesis that the symmetry group G is compact.

3. Compact symmetry groups and reduction

Henceforth we assume that G is a *compact* connected Lie group, acting freely on the connected manifold Q. Let Φ again denote the corresponding lifted Hamiltonian action on T^*Q with associated Ad^*- equivariant momentum map $J : T^*Q \longrightarrow \mathcal{G}^*$.

G being compact we know that its Lie algebra $\mathcal{G}$ admits a G- invariant positive definite metric (see *e.g.* [?]). Fixing such a metric, let (,) denote the corresponding induced metric on $\mathcal{G}^*$. In particular, we then have for any $\nu_1, \nu_2 \in \mathcal{G}^*$ and for each $g \in G$, that

$$(Ad_g^*\nu_1, Ad_g^*\nu_2) = (\nu_1, \nu_2). \tag{6}$$

This metric further determines a G-invariant norm on $\mathcal{G}^*$ according to $\|\nu\| = (\nu, \nu)^{1/2}$. We can now introduce the normalized momentum map associated with the given Hamiltonian action Φ, namely

$$\hat{J} : T^*Q \backslash J^{-1}(0) \longrightarrow \mathcal{G}^*, \ \alpha \longmapsto \hat{J}(\alpha) = \frac{J(\alpha)}{\|J(\alpha)\|}.$$

Next, let $S_{\mathcal{G}^*}$ denote the unit sphere in $\mathcal{G}^*$ with respect to the given metric, *i.e.* $S_{\mathcal{G}^*} = \{\mu \in \mathcal{G}^* | \, \|\mu\| = 1\}$, and consider the natural projection

$$\rho : \mathcal{G}^* \backslash \{0\} \longrightarrow S_{\mathcal{G}^*}, \; \nu \longmapsto \rho(\nu) = \frac{\nu}{\|\nu\|}.$$

For simplicity of notation, we put $T^*Q \backslash J^{-1}(0) = T^*Q_0$. Clearly, T^*Q_0 is an open symplectic submanifold of T^*Q, with symplectic form (the restriction of) ω_Q. One easily verifies that $\hat{J}$ can be regarded as a smooth map from T^*Q_0 into $S_{\mathcal{G}^*}$ and we then have

$$\hat{J} = \rho \circ J|_{T^*Q_0}. \tag{7}$$

Moreover, taking into account (6) and the Ad^*-equivariance of J, it follows that

$$\hat{J}(\Phi_g(\alpha)) = Ad^*_g(\hat{J}(\alpha))$$

for each $g \in G$ and each $\alpha \in T^*Q_0$, *i.e.* $\hat{J}$ is equivariant with respect to the actions of G induced on T^*Q_0 and $S_{\mathcal{G}^*}$, respectively (see also [**?**]).

With X again being the dynamical system defined by (1), suppose Φ leaves H invariant. From corollary 1 we then know that $J^{-1}(0)$ is an invariant submanifold for X. We now consider the dynamical system obtained by taking the restriction of X to T^*Q_0.

Proposition 2. *If the action Φ of the compact connected Lie group G leaves H invariant, then $\hat{J}$ is constant along the integral curves of $X|_{T^*Q_0}$.*

PROOF: Choose an orthonormal basis (e^i) of $\mathcal{G}^*$ with respect to the given metric, *i.e.* such that $(e^i, e^j) = \delta^{ij}$, and let (e_i) denote the dual basis of $\mathcal{G}$. The momentum map J can then be written as $J = J_i e^i$, with $J_i = J_{e_i} \in C^\infty(T^*Q)$. Furthermore, we then have $\|J\| = (\sum_i J_i^2)^{1/2}$. From proposition 1 one can infer that $X(J_i) = -fJ_i$ and a straightforward computation then reveals that

$$X(\frac{J_i}{\|J\|}) = -f\frac{J_i}{\|J\|} - \frac{J_i}{\|J\|^2}X(\|J\|) = 0$$

and since $\hat{J} = (J_i/\|J\|)e^i$, this indeed proves that $\hat{J}$ is a conserved quantity of $X|_{T^*Q_0}$. □

As mentioned already in the previous section, the momentum map J is a submersion since the given Hamiltonian action Φ is assumed to be free. ¿From (7) it is then seen that $\hat{J}$ is also a submersion from T^*Q_0 into $S_{\mathcal{G}^*}$. For each $\mu \in Im(\hat{J})$, the corresponding fibre $\hat{J}^{-1}(\mu)$ therefore determines a closed submanifold of T^*Q_0. From the previous proposition we then immediately deduce:

Corollary 2. *If Φ leaves H invariant, then for each $\mu \in S_{\mathcal{G}^*}$, $\hat{J}^{-1}(\mu)$ is an invariant submanifold of $X|_{T^*Q_0}$.* □

We now add the condition that f is also invariant under the given action, such that Φ defines a symmetry of X. In view of the Ad^*- equivariance of $\hat{J}$, it follows that G_μ, the isotropy subgroup of G at $\mu \in S_{\mathcal{G}^*}$, induces an action on $\hat{J}^{-1}(\mu)$. This action is free (since Φ is free) and proper (since G_μ, being a closed subgroup of a compact group, is also compact). Consequently, the orbit space $\mathcal{Q}_\mu = \hat{J}^{-1}(\mu)/G_\mu$ admits a smooth manifold structure such that the natural projection $\hat{\pi}_\mu : \hat{J}^{-1}(\mu) \longrightarrow \mathcal{Q}_\mu$ becomes a submersion. According to the previous corollary, X is everywhere tangent to $\hat{J}^{-1}(\mu)$ and, since Φ is a symmetry of X, the flow of $X|_{\hat{J}^{-1}(\mu)}$ commutes with the action of G_μ. Hence, $X|_{\hat{J}^{-1}(\mu)}$ projects onto $\mathcal{Q}_\mu$ and we denote the projected vector field by $\hat{X}_\mu$. In particular, we thus have

$$\hat{X}_\mu \circ \hat{\pi}_\mu = T\hat{\pi}_\mu \circ X|_{\hat{J}^{-1}(\mu)} .$$

In [?] we have shown that under some appropriate regularity conditions, which are in particular verified in the present situation, the reduced space $\mathcal{Q}_\mu$ admits the structure of a canonical manifold. Recall that, following Lichnerowicz [?], a canonical manifold is a triple $(P, \{\ ,\ \}, \tau)$, consisting of a Poisson manifold $(P, \{\ ,\ \})$ of constant rank (*i.e.* the corresponding Poisson tensor field has constant rank) and a regular map $\tau : P \longrightarrow \mathbb{R}$, such that the connected components of the fibres of τ are the symplectic leaves of the Poisson structure.

One can now indeed prove that the reduced space $\mathcal{Q}_\mu$, constructed above, admits a Poisson stucture $\{\ ,\ \}_\mu$ of constant rank, which is completely determined by (the restriction to $\hat{J}^{-1}(\mu)$ of) the Poisson structure on T^*Q. Moreover, there is a fibration $\tau_\mu : \mathcal{Q}_\mu \longrightarrow \mathbb{R}_0^+$, with $\mathbb{R}_0^+$ the set of stictly positive real numbers, which is determined by

$$\tau_\mu \circ \hat{\pi}_\mu = \|J\||_{\hat{J}^{-1}(\mu)}.$$

The symplectic leaves of the Poisson structure on $\mathcal{G}_\mu$ are the connected components of the reduced Marsden-Weinstein spaces $\mathcal{P}_\nu = J^{-1}(\nu)/G_\nu$, for $\nu \in J(\hat{J}^{-1}(\mu)) \subset \{k\mu |\, k \in \mathbb{R}_0^+\}$, and the latter precisely coincide with the fibres of τ_μ. One can furthermore easily verify that $\mathcal{Q}_\mu$ is odd dimensional with dim $\mathcal{Q}_\mu = (\dim M - \dim G - \dim G_\mu) + 1$. (See [?] for more details).
Summarizing the above discussion we thus have:

Theorem 2. *If both H and f are invariant under the (lifted) free Hamiltonian action of the compact connected Lie group G, then for each $\mu \in Im(\hat{J}) \subset S_{\mathcal{G}^*}$, the restriction of X to the invariant submanifold $\hat{J}^{-1}(\mu)$ projects onto the quotient space $\mathcal{Q}_\mu = \hat{J}^{-1}(\mu)/G_\mu$, which admits the structure of a canonical manifold.* $\square$

Let $j_\mu : \hat{J}^{-1}(\mu) \longrightarrow T^*Q_0$ be the natural inclusion and let $\hat{H}_\mu$ and $\hat{f}_\mu$ denote the functions induced on $\mathcal{Q}_\mu$ by H and f, respectively, *i.e.* $\hat{H}_\mu \circ \hat{\pi}_\mu = H \circ j_\mu$ and $\hat{f}_\mu \circ \hat{\pi}_\mu = f \circ j_\mu$. If Λ_μ denotes the Poisson tensor field on $\mathcal{Q}_\mu$, then one can check that the projected vector field $\hat{X}_\mu$ is of the form (see also [?])

$$\hat{X}_\mu = -\hat{f}_\mu \hat{\Delta}_\mu - \Lambda_\mu(d\hat{H}_\mu) \tag{8}$$

where, on the right-hand side, Λ_μ is regarded as a vector bundle homomorphism from $T^*\mathcal{Q}_\mu$ into $T\mathcal{Q}_\mu$ and $\hat{\Delta}_\mu$ denotes the reduction of (the restriction to $\hat{J}^{-1}(\mu)$ of) the canonical dilation vector field $\Delta = p_i\,\partial/\partial p_i$ on T^*Q. It is indeed easily verified that Δ is invariant under the lifted group action and, moreover, that $\hat{J}$ is a conserved quantity of Δ. From (1) it is seen that the given vector field X can still be written as $X = -f\Delta - \Lambda_Q(dH)$, with $\Lambda_Q = \omega_Q^{-1}$ the Poisson tensor field on T^*Q, from which (8) now readily follows. We finally note that $\hat{\Delta}_\mu(\tau_\mu) = \tau_\mu$ (cf.[**?**]), from which it can be deduced that the component of $\hat{X}_\mu$ transverse to the symplectic leaves of $\mathcal{Q}_\mu$ is given by $-\hat{f}_\mu\tau_\mu\,\partial/\partial\tau_\mu$. A more detailed analysis of this reduced system, as well as some examples, will be treated elsewhere.

Acknowledgements. The author is indebted to the Belgian National Fund for Scientific Research for a Research Grant.

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Received 14 October 1992