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GLOBAL GEODESIC MAPPINGS AND THEIR GENERALIZATIONS FOR COMPACT RIEMANNIAN SPACES

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Abstract. This paper is devoted to the study of global geodesic mappings and their generalizations for compact Riemannian spaces both with boundary and without boundary.
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1. Introduction

General questions of global geodesic mappings have been studied by G. Vranceanu [16] and many other authors [2], [6], [7], [9], [14], ...

Conditions on compact Riemannian spaces V_n without boundary which do not admit global nontrivial geodesic mappings can be found in [2], [6], [14], [16]. We have succeeded in showing the existence of the compact Riemannian spaces V_n without boundary, admitting global nontrivial geodesic mapping [7], [9].

This paper is devoted to the study of global geodesic mappings and their generalizations for compact Riemannian spaces both with boundary and without boundary.

Throughout the paper, all the objects are assumed to be sufficiently smooth, no conditions are laid down on the signature of the metric ds^2 of a Riemannian space V_n, otherwise they are explicitly indicated.

A diffeomorphism of a Riemannian space V_n onto $\bar{V}_n$ which maps all geodesics in V_n into geodesics in $\bar{V}_n$ is called a *geodesic mapping* [3]. A geodesic mapping which is not affine is said to be *nontrivial*.

A diffeomorphism $f\colon V_n \to \bar{V}_n$ is a geodesic mapping if and only if, for every vector field x on V_n, we have

$$\bar{\nabla}_x x - \nabla_x x = 2\psi(x)x \tag{1}$$

where ∇, or $\bar{\nabla}$ denotes the Riemannian connections of V_n, or the connection induced on V_n by the Riemannian connection of $\bar{V}_n$, $\psi(x)$ is a linear (differential) form on V_n.

In the next pages, we shall often identity the geometric objects on $\bar{V}_n$ with the corresponding objects on V_n induced by a fixed diffeomorphism $f\colon V_n \to \bar{V}_n$. If $\psi(x) = 0$, then the mapping is known to be affine, otherwise this geodesic mapping is nontrivial.

In [10] the concept of F-planar mapping was introduced. Consider a special F-planar mapping of a Riemannian space V_n into a Riemannian space $\bar{V}_n$, characterized by the

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following conditions:

$$\begin{aligned} &a) \quad \bar{\nabla}_x x - \nabla_x x = 2\psi(x)x + 2\varphi(x)F(x);\\ &b) \quad g(x, F(x)) = 0;\\ &c) \quad \bar{g}(x, F(x)) = 0 \end{aligned} \tag{2}$$

where ψ, φ are linear forms, F is a (1,1) tensor field determined on V_n, g and $\bar{g}$ are the metric forms of V_n and $\bar{V}_n$. Such a mapping will be called F^*-*planar*.

If $\varphi = 0$ then a F^*-planar mapping is a geodesic mapping. Special cases of F^*-planar mappings are known as holomorphically-projective mappings of Kählerian spaces [16]. We note that the conditions (2a,c) hold for quasigeodesic mappings, introduced by A.Z. Petrov [11].

2. Global geodesic and F^*-planar mappings V_n into $\bar{V}_n$ with boundary ∂V

In this part we suppose that the metric of the space V_n is positive definite.

Theorem 1. *Let an orientable compact Riemannian space V_n $(n \geq 2)$ admit a global geodesic mapping onto $\bar{V}_n$. If all sectional curvatures at each point $m \in V_n$ are nonpositive, and the metrics of V_n and $\bar{V}_n$ obey the relation $\bar{g}(x, y) = f(m)g(x, y)$ at each point $m \in \partial V$, then the geodesic mapping V_n onto $\bar{V}_n$ is trivial, i.e. it is affine.*

This theorem is generalized by means of

Theorem 2. *Let an orientable compact Riemannian space V_n $(n \geq 4)$ admit a global F^*-planar mapping onto $\bar{V}_n$ with $\psi(x)$ being a gradient. If all sectional curvatures at each point $m \in V_n$ are nonpositive, and the metrics of V_n and $\bar{V}_n$ obey the condition $\bar{g}(x, y) = f(m)g(x, y)$ at each point $m \in \partial V$, then the F^*-planar mapping of V_n onto $\bar{V}_n$ is affine.*

In particular, Theorem 2 is true for holomorphically-projective mappings of Kählerian spaces. In case of $\partial V = \emptyset$ the condition on ∂V is satisfied, and the signature of the metric $\bar{g}(x, y)$ may be arbitrary.

The proofs of the above mentioned theorems are based on the following assertion.

Theorem 3. *Let V_n be an orientable compact Riemannian space and $a(x, y)$ be a symmetric bilinear form on V_n, let a function J be determined in the following way*

$$J \equiv g^{ij} g^{kl} g^{mp} (a_{ik,l} a_{jm,p} - a_{ik,m} a_{jp,l}) \tag{3}$$

where a_{ij} are local coordinates of $a(x, y)$, $\|g^{ij}\| = \|g_{ij}\|^{-1}$, g_{ij} are local components of the form $g(x, y)$, and comma denotes the covariant derivative with respect to the connection on V_n. Then, if

1) all sectional curvatures at each point $m \in V_n$ are nonpositive,
2) the function $J \geq 0$ at each point of V_n,
3) there exists a function f: $\partial V \to R$ such that $a(x, y) = f(m)g(x, y)$ at each point $m \in \partial V$,

then $J = 0$ at each point $m \in V_n$.

In case that of all the sectional curvatures of V_n are negative, then under the conditions 2) and 3) there exists a function f such that $a(x,y) = f(m)g(x,y)$ at each point $m \in V_n$.

Theorem 3 is analogous to that of A. Švec, and it can be verified following [1,p. 12].

Let us restrict our attention to the proof of Theorem 2. Suppose that the form ψ is the differential of a function Ψ, that is $\psi(x) = \nabla_x \Psi$. Let us put

$$a^{ij} \equiv \exp(2\Psi)\,\bar{g}^{ij} \tag{4}$$

where $\|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}$, and $\bar{g}_{ij}$ are local components of the form $\bar{g}(x,y)$.

It is easy to show that (2a) leads to the following relations

$$a^{ij}_{\ ,k} = \lambda^{(i}\delta^{j)}_k + \Theta^{(i}F^{j)}_k$$

where

$$\lambda^i = -\exp(2\Psi)\,\bar{g}^{i\alpha}\psi_\alpha; \quad \Theta^i = -\exp(2\Psi)\,\bar{g}^{i\alpha}\varphi_\alpha,$$

ψ_i, φ_i, F^j_k are local coordinates of $\psi(x)$, $\varphi(x)$ and $F(x)$ respectively, δ^h_i is the Kronecker symbol, (i,j) denotes symmetrization with respect to i and j.

Putting

$$a_{ij} \equiv a^{\alpha\beta} g_{i\alpha} g_{j\beta} \tag{5}$$

we can verify that the invariant J may be written in the following way:

$$J = -\|\Theta^h\|^2 F^\alpha_\beta F^\beta_\alpha + \tfrac{1}{2}\|2\Theta^\alpha F^h_\alpha + (n+2)\,\lambda^h\|^2 + \tfrac{1}{2}(n^2 - 2n - 8)\,\|\lambda^h\|^2$$

where $\|\xi\|^2 \equiv \xi^\alpha \xi^\beta g_{\alpha\beta}$ is the square of the norm of the vector ξ^h.

It is obvious that for $n \geq 4$ it holds $J \geq 0$.

The conditions at the boundary of ∂V: $\bar{g}(x,y) = f(m)g(x,y)$ in virtue of (4) and (5) generate the condition

$$a(x,y) = f(m)\,\exp(2\Psi)\,g(x,y).$$

Hence, all the conditions of Theorem 3 are satisfied and $J = 0$. By a direct analysis we can check that $J = 0$ if and only if $\lambda^h = \Theta^h = 0$. The latter is equivalent to $\psi(x) = \varphi(x) = 0$. Hence, the F^*-planar mapping is affine, and Theorem 2 has been proved.

Theorem 4. *Let an orientable compact Riemannian space V_n admit a global affine mapping onto $\bar{V}_n$. If all the sectional curvatures at each point $m \in V_n$ are positive (or negative), and the metrics of V_n and $\bar{V}_n$ are connected by the relations $\bar{g}(x,y) =$ $= f(m)g(x,y)$ at each point $m \in \partial V$, then the mapping of V_n into $\bar{V}_n$ is a homothety.*

Proof. A Riemannian space V_n admits an affine mapping onto $\bar{V}_n$ if and only if $\nabla_z \bar{g}(x,y) = 0$. Putting $a(x,y) = \bar{g}(x,y)$ we assert that $J = 0$. Then on the basis of A. Švec's theorem [1, p. 12] and in virtue of Theorem 3 it follows $\bar{g}(x,y) = f(m)g(x,y)$ at each point $m \in V_n$. Obviously, the function $f = const$, and the proof is finished.

3. Global geodesic mappings of $V_n(B)$-spaces

A Riemannian space V_n is called a $V_n(B)$–space if it satisfies the following conditions

$$\begin{aligned} &a)\quad \nabla_z a(x,y) = \lambda(x)g(y,z) + \lambda(y)g(x,z); \\ &b)\quad \nabla_y \lambda(x) = \mu\, g(x,y) + B\, a(x,y) \end{aligned} \tag{6}$$

where $a(x,y)$ is a symmetric regular quadratic form, $\lambda(x)$ is a nonvanishing linear form, and μ, B are functions.

If $B = const$, then the conditions $\nabla_x \mu = 2B\,\lambda(x)$ are satisfied. If the relations (6) are true in all points, then $V_n(B)$ is globally determined. By the correctness of the equations (6a) under the condition $\lambda(x) \not\equiv 0$ the existence of a nontrivial geodesic mapping is guaranteed [13].

Theorem 5. *There exists no compact space $V_n(B)$, for $B = 0$.*

The proof of this theorem follows from the fact that the relation (6b) may be written in the form

$$\nabla_x \nabla_y \lambda = \mu\, g(x,y) \tag{7}$$

where $\mu = const$, $\lambda(x) = \nabla_x \lambda$.

Vector fields, satisfying locally (7) were studied by P.A. Shirokov [12]. We and S.G. Leiko have shown that the equations (7) on a compact V_n have only a trivial solution $\lambda(x) = \nabla_x \lambda = 0$.

Note. It is shown in [6] that in a compact pseudo Riemannian space V_n the equation $\nabla_x \nabla_y \lambda = \mu\, g(x,y)$ with μ being a function has the solutions $\lambda = const$ and $\mu = 0$.

Theorem 6. *Let a compact $V_n(B)$, $B = const$, admit a global geodesic mapping onto $\bar{V}_n$. Then $\bar{V}_n$ is $\bar{V}_n(\bar{B})$ with $\bar{B} = const$, and the signatures of the metrics of $V_n(B)$ and $\bar{V}_n(\bar{B})$ coincide.*

In case $V_n(B)$ is a Riemannian space with a positive signature of the metric, Theorem 6 has been proved by V.A. Kiosak [2].

It has been shown that the spaces of constant curvature, the Einstein spaces, the generalized semisymmetric (pseudosymmetric), generalized Ricci-semisymmetric (Ricci-pseudosymmetric) spaces, admitting nontrivial geodesic mapping may be, in particular, the spaces $V_n(B)$ [4], [5]. Hence, for all the spaces mentioned above Theorems 5 and 6 can be applied.

4. Geodesic mapping of generalized T-semisymmetric spaces

For generalized T-semisymmetric spaces determined in [5] the following assertion may be proved.

Theorem 7. *Let a tensor field T of $(0,m)$ – type exist in a Riemannian space V_n $(n > 2(m-1))$ and let the following condition be satisfied*

$$\sum_{r=1}^{m} T_{i_1 i_2 \dots i_{r-1} \alpha i_{r+1} \dots i_m}\, Z^{\alpha}_{i_r k l} = 0 \tag{8}$$

where $Z^h_{ijk} \equiv R^h_{ijk} + B(\delta^h_k g_{ij} - \delta^h_j g_{ik})$, R^h_{ijk} is the Riemann tensor. If V_n admits a geodesic mapping, then either V_n is $V_n(B)$ or the tensor T may be represented as the tensor sum of products of invariants and the metric tensor.

Proof. Let in V_n the pointed tensor T exist, satisfying the conditions (8). Further, we shall suppose V_n to admit a geodesic mapping. Hence by [1],[5] in V_n the equations (6a) hold. The conditions of their integrability are

$$a_{\alpha(i} R^{\alpha}_{j)kl} = \lambda_{l,(i} g_{j)k} - \lambda_{k,(i} g_{j)l}. \tag{9}$$

where a_{ij} and λ_i are local coordinates of the forms $a(x,y)$ and $\lambda(x)$ respectively. In [5] the theorem is proved for the case when $m = 1, 2$, and a special case when $m = 4$ and T is the Riemann tensor. By induction we shall prove the theorem for an arbitrary $m > 2$ assuming that it is correct for 1,2,...,m–1. Let us suppose the converse, i.e. that for a geodesic mapping the conditions (6) do not hold. Evidently, any convolution of T with g^{ij} satisfies the relations (8) and by our assumption these convolutions are a tensorial sum of products of the invariants and the metric tensor. For the m odd the pointed convolutions are equal to zero. For m even and $n \geq 2m - 1$ one easily can construct a tensor T^*, as a sum of the tensor T and the tensorial sums of products of the definite invariants and the metric tensor, such that any convolution of T^* with g^{ij} is zero. The tensor T^* satisfies the conditions (8), therefore without loss of generality, later on we will consider the tensor T^* instead of T.

We shall convolve the relation (8) with $a_{k\alpha} g^{\alpha j}$, and then symmetrize with respect to l and k. Taking into account the structure of the tensor Z^h_{ijk}, the conditions (9) and the properties of the Riemann tensor, after a corresponding grouping we get

$$\sum_{r=1}^{m} \{\Lambda_{h_r(i} T_{|h_1 \dots h_{r-1}|j)h_{r+1}\dots h_m} - g_{h_r(i} T^r_{|h_1\dots h_{r-1}|j)h_{r+1}\dots h_m}\} = 0 \tag{10}$$

where

$$\Lambda_{ij} \equiv \lambda_{i,j} - \mu\, g_{ij} - B\, a_{ij}; \tag{11}$$

$$\mu \equiv (\lambda_{\alpha,\beta} - B a_{\alpha\beta}) g^{\alpha\beta}/n;$$

$$T^r_{h_1\dots h_m} \equiv \Lambda_{\alpha h_r} g^{\alpha\beta} T_{h_1\dots h_{r-1}\beta h_{r+1}\dots h_m}.$$

Convolving (10) in turn with $g^{ih_p}\, g^{jh_q}$ where $p, q = 1, 2, \dots, m\ (p \neq q)$, we shall obtain equations which for $n > 2(m-1)$ lead to

$$g^{\alpha\beta} T^r_{h_1\dots h_{p-1}\alpha h_{p+1}\dots h_{r-1}\beta h_{r+1}\dots h_m} = 0. \tag{12}$$

Then we convolve (10) with g^{ih_p}. From the obtained relations, taking into account (12) we shall convince, that $T^r_{h_1\dots h_m} = 0$. Then the formulas (10) are simplified. From them one can show that either $\Lambda_{ij} = 0$, or $T_{h_1\dots h_m} = 0$. Taking into account the formulas (11), the theorem is completely proved.

It is easy to check that the following assertion is true. By Theorem 7, and taking into account [8] the local result follows:

Theorem 8. *A Riemannian space with nonconstant curvature V_n and with one of the following relations being satisfied*

$$a) \quad R^h_{ijk,m_1m_2\dots m_p} = 0, \quad (n > 2(p+1))$$

or

$$b) \quad R_{i[j,k],m_1m_2\dots m_p} = 0, \quad R_{i[j,k]} \neq 0 \quad (n > 2p)$$

where R_{ij} is the Ricci tensor, $[j,k]$ denotes the alternation with respect to j and k, does not admit a local nontrivial geodesic mapping.

Theorem 9. *Let in a compact Riemannian space $V_n \in C^3$ $(n \geq 2)$ one of the following relation be true at each point:*

1) $T_{h_1h_2\dots h_p,[lm]} = 0$ and the tensor T cannot be represented as a sum of products of invariants and the metric tensor $(n > 2(p-1))$;

2) $R_{ij,[lm][pr]} = 0$.

If the condition 1) is satisfied at least at one point then V_n does not admit a global geodesic mapping onto $\bar{V}_n \in C^3$.

Theorem 10. *Suppose that in a compact Riemannian space $V_n \in C^k$ $(n \geq 2)$ one of the following conditions is satisfied:*

$$a) \quad R_{ij,[lm][pr]} = 0; \quad R_{ij,[lm]} \not\equiv 0; \quad (k = 6);$$

$$b) \quad R_{ij,[lm]} = 0; \quad R_{ij} \not\equiv \rho\, g_{ij}; \quad (k = 4);$$

$$c) \quad R^h_{ijk,[lm]} = 0; \quad R^h_{ijk} \not\equiv \rho(\delta^h_j g_{ik} - \delta^h_k g_{ij}); \quad (k = 4);$$

$$d) \quad R_{ij} = 0; \quad (k = 3).$$

Then the Riemannian space V_n does not admit a global nontrivial geodesic mapping onto $\bar{V}_n \in C^3$.

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