THE FIRST BOUNDARY PROBLEM FOR A DEGENERATE ELLIPTIC SYSTEM[1]

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In 1955 I.N. Vekua (see [1,2]) raised the problem of investigation of such cusped plates, the thickness of which vanishes on a part of the plate boundary or on the whole boundary. The problem mathematically leads to the question of setting and solving boundary value problems for even order elliptic type system with order degeneration.

Here we'll consider the first approximation of Vekua's version of the prismatic shell theory. The zero approximation was inversigated earlier (see [4,5]).

The compression tension system to be considered has the form

$$\left.\begin{aligned}
(\lambda+2\mu)\frac{\partial}{\partial x}\left(h\frac{\partial u_1}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(h\frac{\partial u_1}{\partial y}\right)+\lambda\frac{\partial}{\partial x}\left(h\frac{\partial u_2}{\partial y}\right) &\quad + \\
+\mu\frac{\partial}{\partial y}\left(h\frac{\partial u_2}{\partial x}\right)+3\lambda\frac{\partial}{\partial x}(hv_3)+X_1 &= 0, \\
\mu\frac{\partial}{\partial x}\left(h\frac{\partial u_2}{\partial x}\right)+(\lambda+2\mu)\frac{\partial}{\partial y}\left(h\frac{\partial u_2}{\partial y}\right)+\mu\frac{\partial}{\partial y}\left(h\frac{\partial u_1}{\partial y}\right) &\quad + \\
+\lambda\frac{\partial}{\partial y}\left(h\frac{\partial u_1}{\partial x}\right)+3\lambda\frac{\partial}{\partial y}(hv_3)+X_2 &= 0, \\
\mu\frac{\partial}{\partial x}\left(h^3\frac{\partial v_3}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(h^3\frac{\partial v_3}{\partial y}\right)-\lambda h\frac{\partial u_1}{\partial x}-\lambda h\frac{\partial u_2}{\partial y} &\quad - \\
-3(\lambda+2\mu)hv_3+X_3 &= 0
\end{aligned}\right\} \tag{1}$$

(see [2], p.75), where, according to I.N. Vekua, u_1, u_2 are the zero moments and u_3 is the first moment of displacement vector components, $2h$ is the thickness of the plate, $\lambda, \mu > 0$ are Lamé constants, X_1, X_2, and X_3 are the zero and first moments of volume force components. The system (1) is considered on the middle surface of the plate Ω, which is a bounded, simply connected plane domain with the boundary $\partial\Omega \in C^1$. Furthermore, $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$,

$$\begin{aligned}
\Gamma_0 &= \{(x,0): \ a < x_1 < b, \ a,b \in R, \ a < b\}, \\
\Gamma_1 &= \{(x,y): \ (x,y) \in \partial\Omega, \ y > 0\},
\end{aligned}$$

[1]These results were included in the talk of G.Jaiani, Application of Vekua's Dimension Reduction Method to Cusped Plates and Bars

$2h = y^m, 0 < m = const < +\infty$.

Our aim is to investigate first boundary value problem for the system (1).

Let $D(\Omega)$ be a set of infinitely differentiable compactly supported functions on Ω. We define a scalar product and a norm on $D(\Omega)$ according to the formulas:

$$(u,v)_m \equiv \int\limits_{\Omega} y^m[\nabla u \nabla v + uv]d\tau, \tag{2}$$

$$\|u\|_m \equiv \left(\int\limits_{\Omega} y^m[(\nabla u)^2 + u^2]d\tau\right)^{\frac{1}{2}}, \tag{3}$$

where $u, v \in D(\Omega)$.

We complete $D(\Omega)$ by the norm (3) to obtain the Hilbert space $H_m^0(\Omega)$, where a scalar product and a norm are defined by the formulas (2) and (3).

Lemma 1. *The trace of a function $u \in H_m^0(\Omega), 0 \le m \le 1$, on $\partial\Omega$ is zero.*

Proof. Let

$$\rho(x,y) = dist\{(x,y),\ \partial\Omega\}.$$

Complete $D(\Omega)$ by the norm

$$\|u\|_{\rho^m} = \left(\int\limits_{\Omega} \rho^m(x,y)[(\nabla u)^2 + u^2]d\tau\right)^{\frac{1}{2}}. \tag{4}$$

We obtain the Hilbert space $\overset{0}{W}{}^{1}_{2,\frac{m}{2}}(\Omega)$ the properties of which are well known (see [6]). In particular, the trace of every function $u \in \overset{0}{W}{}^{1}_{2,\frac{m}{2}}(\Omega),\ \ 0 < m < 1$, on $\partial\Omega$ is zero (see [6], p.393).

It's easy to show that

$$\|u\|_m \ge \|u\|_{\rho^m}$$

for every $u \in D(\Omega)$. Therefore

$$\overset{0}{H}{}_{1,m}(\Omega) \subset \overset{0}{W}{}_{2,\frac{m}{2}}(\Omega),$$

and if $u \in \overset{0}{H}{}_{1,m}(\Omega)$, then $u\,|_{\partial\Omega} = 0$.

Let's define the vector space

$$\overset{\to}{\overset{0}{H}}{}_{m_1,m_2}(\Omega) = \overset{0}{H}{}_{m_1}(\Omega) \times \overset{0}{H}{}_{m_2}(\Omega)$$

whith the norm: if $\vec{u} = (u_1,u_2) \in \overset{\to}{\overset{0}{H}}{}_{m,m}(\Omega)$, then $\|\,\vec{u}\,\|^2_{m_1,m_2} = \|u_1\|^2_{m_1} + \|u_2\|^2_{m_2}$.

$\overset{\to}{\overset{0}{H}}{}_{m_1,m_2}(\Omega)$ is a Hilbert space.

Lemma 2 (Korn's weighted inequality). *Let* $\vec{u} = (u_1, u_2) \in \overset{\rightarrow}{\overset{0}{H}}_{m,m}(\Omega)$ *and* $m \neq 1$. *Then the following inequality is valid*

$$\int\limits_{\Omega} y^m[(\nabla u)^2 + u^2]d\tau \leq c \int\limits_{\Omega} y^m \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 + \left(\frac{\partial u_1}{\partial y} + + \frac{\partial u_2}{\partial x}\right)^2 \right] d\tau,$$

where c is independent of $\vec{u}$.

Lemma 2 is proved similarly to [4].

Definition. We say, that $\vec{u} = (u_1, u_2, u_3) \in \overset{\rightarrow}{\overset{0}{H}}_{m_1,m_2,m_3}(\Omega) \equiv \overset{0}{H}_m(\Omega) \times \overset{0}{H}_m(\Omega) \times \overset{0}{H}_{3m}(\Omega)$ is a generalised solution of the system (1) if

$$\begin{aligned} B(\vec{u}, \vec{w}) &= \int\limits_{\Omega} y^m \left[(\lambda + 2\mu)\frac{\partial u_1}{\partial x}\frac{\partial w_1}{\partial x} + \mu\frac{\partial u_1}{\partial y}\frac{\partial w_1}{\partial y} + \lambda\frac{\partial u_2}{\partial y}\frac{\partial w_1}{\partial x} + \right. \\ &+ \mu\frac{\partial u_2}{\partial x}\mu\frac{\partial w_1}{\partial y} + 3\lambda v_3\frac{\partial w_1}{\partial x} + \mu\frac{\partial u_2}{\partial x}\frac{\partial w_2}{\partial x} + (\lambda + 2\mu)\frac{\partial u_2}{\partial y}\frac{\partial w_2}{\partial y} + \\ &+ \mu\frac{\partial u_1}{\partial y}\frac{\partial w_2}{\partial x} + \lambda\frac{\partial u_1}{\partial x}\frac{\partial w_2}{\partial y} + 3\lambda v_3\frac{\partial w_2}{\partial y} + \\ &+ y^{2m}3\mu\left(\frac{\partial u_3}{\partial x}\frac{\partial w_3}{\partial x} + \frac{\partial v_3}{\partial y}\frac{\partial w_3}{\partial y}\right) + 3\lambda\frac{\partial u_1}{\partial x}w_3 + 3\lambda\frac{\partial u_2}{\partial y}w_3 + \\ &+ \left. 9(\lambda + 2\mu)v_3 w_3 \right] d\tau = \int\limits_{\Omega} (X_1 w_1 + X_2 w_2 + 3X_3 w_3)\, d\tau, \end{aligned}$$

$\forall \vec{w} = (w_1, w_2, w_3), \quad \vec{w} \in [D(\Omega)]^3$.

Let $L_{2,m}(\Omega)$ be a Hilbert space of measurable functions u such, that the norm

$$\|u\|_{L_{2,m}} = \left(\int\limits_{\Omega} y^m u^2(x_1, x_2) d\tau \right)^{\frac{1}{2}}$$

is finite, and

$$\vec{L}_{2,m_1,m_2,m_3}(\Omega) \equiv L_{2,m_1}(\Omega) \times L_{2,m_2}(\Omega) \times L_{2,m_3}(\Omega)$$

with the norm

$$\| \vec{u} \|^2_{L_{2,m_1,m_2,m_3}} = \|u_1\|^2_{L_{2,m_1}} + \|u_2\|^2_{L_{2,m_2}} + \|u_3\|^2_{L_{2,m_3}},$$

where $\vec{u} = (u_1, u_2, u_3)$.

The foolowing theorem is valid.

Theorem. *If* $\vec{X} = (X_1, X_2, X_3) \in \vec{L}_{2,m_1,m_2,m_3}(\Omega)$ *and* $m < \frac{1}{3}$, *then the system (1) has as a unique generalized solution* $\vec{u} = (u_1, u_2, u_3) \in \overset{\to}{\overset{0}{H}}_{m_1,m_2,m_3}(\Omega)$ *and we have the following estimate*

$$\| \vec{u} \|_{H_{m,m,3m}} \leq c \| \vec{X} \|_{L_{2,m_1,m_2,m_3}}.$$

Proof. According to the Lemma 2, it isn't difficult to show, that the form $B(\vec{u}, \vec{w})$ is coercive on the space $\overset{\to}{\overset{0}{H}}_{m_1,m_2,3m}(\Omega)$, i.e.

$$\begin{aligned} | B(\vec{u}, \vec{w}) | &\leq k_1 \| \vec{u} \|_{m,m,3m} \| \vec{w} \|_{m,m,3m}, \\ | B(\vec{u}, \vec{w}) | &\geq k_2 \| \vec{u} \|_{m,m,3m} \end{aligned}$$

where $k_1, k_2 > 0$; $\vec{u}, \vec{w} \in \overset{\to}{\overset{0}{H}}_{m,m,3m}(\Omega)$.

Furthemore the linear functional

$$\int_\Omega \vec{X}\, \vec{u}\, d\tau,$$

where $\vec{X} \in \vec{L}_{2,m,m,3m}(\Omega)$ is continuous on $\overset{\to}{\overset{0}{H}}_{m,m,3m}(\Omega)$.

Thus, the Lax-Milgram Lemma completes the proof.

References

[1] Vekua, I.N. On a Way of Calculating of Prismatic Shells. Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences, 21, 1955, 191-259 (Russian)

[2] Vekua, I.N. The Theory of Thin Shallow Shells of Variable Thickness. Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences, 30, 1965, 5-103 (Russian)

[3] Devdariani, G. On a System of Equations of Elliptic Type Degenerating on the Whole Boundary of a Smooth Domain. Dokl. Acad. Nauk, 353, 1991, 5, 1-4 (Russian).

[4] Jaiani, G.V. The First Boundary Value Problem of Cusped Prismatic Shell Theory in Zero Approximation of Vekua Theory. Proceedings of I.Vekua Institute of Applied Mathematics, 29, 1988, 5-38 (Russian, Georgian and English summaries)

[5] Devdariani, G. On a System of Equations of Elliptic Type Degenerating on the Rectangle. Diff. Urav., 33, 1997, 89-95 (Russian).

[6] Nikolskii, S.M. Approximation of Functions of Several Variables and Embedding Theorems. Moscow, "Nauka", 1971 (Russian).