SPECTRAL AND SPECTRAL ELEMENT METHODS

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Spectral element techniques are high order methods which allow for either obtaining very accurate results or reducing the number of degrees of freedom for a fixed standard accuracy. We quote the meteorology forecast and the direct simulation of turbulence among the most complex applications. As firstly explained by D. Gottlieb and S.A. Orszag, they rely on high degree piecewise polynomial approximation, and on the use of tensorized bases of polynomials. For these reasons, the basic geometries for these methods are tensorized, *i.e.* they are square or cubes. The discrete spaces for spectral methods are simply polynomial spaces, and the discretization parameter is the maximal degree of these polynomials. The discretization is of Galerkin type, i.e. it relies on the variational formulation of the continuous problem: the discrete problem is built by replacing the functional spaces by discrete spaces and the integrals by appropriate tensorized quadrature formulas. Its numerical analysis mainly relies on results of best polynomial approximation and also of polynomial interpolation at the nodes of the formula. We firstly describe and analyze the discretization of the Laplace equation, and we present its implementation. Next we extend these results to the Stokes and Navier–Stokes equations, the main difficulty being the compatibility of the discrete spaces of velocity and pressure. Some further arguments are necessary for handling nonlinear hyperbolic equations: indeed, the approximation of discontinuous functions by polynomials leads to spurious oscillations (Gibbs phenomenon). We present these arguments and apply them to the Bürgers' equation. Handling complex geometries by spectral type methods mainly relies on a domain decomposition, most often without overlapping,each element of the decomposition being either a rectangle or a rectangular parallelepiped (or smooth deformations of them, however we do not treat this last case to avoid too technical proofs). The basic spectral element method, due to A.T. Patera, relies on conforming discretizations, so that the decomposition of the domain must also satisfy some properties of conformity that are standard in finite elements (the intersection of two different elements, if not empty, is a corner, an edge or a face of both elements). However this conformity restriction can be avoided when using the mortar element method, so we have rather only consider this more general technique. The mortar element method, due to C. Bernardi, Y. Maday and A.T. Patera, is a rather general way of handling nonconforming decompositions, since several types of variational discretizations can be coupled on these decompositions. More specifically, it allows for coupling spectral methods of different degrees on each element, finite elements with incompatible meshes or also finite elements on a part of the domain with spectral methods on the remaining part. We present and analyze it for a model equation in the spectral framework, with some hints to its application in finite elements. A further argument seems necessary to handle axisymmetric geometries: three-dimensional equations in such domains can be reduced to an infinite family of two-dimensional problems thanks to the use of cylindrical coordinates and Fourier expansion with respect to the angular variable. In the case of axisymmetric data, only one problem in this family must be handled. So the computational cost is widely reduced in comparison with a three-dimensional computation. The only drawback is that, due to the change of measure, weighted Sobolev spaces appear in the variational formulation of the reduced problems. So, corresponding spectral tools must be analyzed (orthogonal polynomials or quadrature formulas for the weighted measure). We only present the main results also for a model problem.