SOLUTIONS OF DIFFERENTIAL GAMES UNDER UNCERTAINTY

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Since J. Von Neumann's pioneering studies (see Von Neumann, J., and Morgeshtern, O., Theory of Games and Economic Behavior, Nauka, Moscow, 1970), mathematical models of game problems have been considered under "ideal" settings. The reported results hold for a case in which a given situation, chosen and used by players, generates a specific point in a criterion space. In other words, the influence of possible perturbations, measurement errors, incomplete information and other sources of uncertainties are ignored. At the same time construction of fairly adequate models of actual processes and objects requires recognizing the influence of uncertain factors. As a rule, in game theory the influence of such kind of uncertainties has not been recognized.

Some typical reasons for uncertainty in conflict (game) problems are as follows:

- It is not possible to identify precisely parameters of the controlled object, hence we know only the ranges of their values and at the same time we have no statistical characteristics of uncertainty.
- Measurement errors as well as computational and run-time errors may take place when dealing with dynamic systems with closed-loop feedback.
- The preferences of players are not completely determined and objective functions contain some uncertain parameters.
- Multi-level structure in hierarchical games with lower level players, who perform vector-valued optimization of their criteria (using Stackelberg solutions), yields multivaluence of highlevel player's vector criterion. Similar situations occur if the upper level player does not possess perfect information on the lower level's utility function.
- Uncertainty may be induced by some specifics of mathematical techniques.

When uncertainty is recognized, any situation is associated with a set of points ("a cloud") in the criterion space, which are possible values of the players' payoff functions. That is why the players under the assumption that any point of this cloud can take place should choose an optimal (in some respect) solution. From a game theoretic standpoint this effect of uncertainty can be treated as the strategy of an additional player, who has no payoff function of his own and tries to do maximal harm to all other players by making their payoff functions as small as possible.

Therefore the problem to develop strong theoretical foundations of games (both statical and differential ones) under uncertainty is an actual one.

The presentation of the current investigation involves some general points on differential games under uncertainty conditions. First formalization of differential games under uncertainties is carried out, some solutions concepts for such games are introduced and their general properties are investigated, etc.

We discuss differential, positional, linear-square games of two persons under uncertainty. Two different types of games can be pointed:

- *coalitionless games,* when the players act independently one from the other
- *cooperative games,* when the players can agree on joint choice of their own strategies after some negotiations.

Some examples are widely discussed. In the case of competition or pursuit of one player by the other the game has a coalitionless nature. In this situation both players deny to negotiate and act independently. When we have a reapproachement problem of two objects the players are able to negotiate about the joint choice of strategies and then the game becomes cooperative.

In the coalitionless case players can follow an arbitrary optimal concept from the theory of coalitionless differential games (Nash equilibrium, Berge equilibrium, threats and counter-threats equilibrium, absolute and active equilibrium).

In the cooperative case players can use an arbitrary solution from the theory of cooperative differential games (divisions, C-kernel, Shepli vector, etc.).

A nonunique notion of solution is a peculiarity of game theory and it can be described by both external conditions, such that the game can "be sunk into them", and different social, economical, etc. conditions under which the game takes place.

Further on we introduce one of the solution concepts for the game based on the notions of a vector guarantee and a minimal vector guarantee. The positive and negative peculiarities of such solutions are discussed widely. In particular, the vector guarantee, as a rule, can not be achieved for any uncertainty. Therefore we have to consider only vector guarantees that are realized under the same uncertainty. The other lack of the minimal vector guarantee is that the values of its components are the minimal ones. The players try their best to increase their guarantees, as the guarantees are exactly the values of the player's payoff functions, which they provide to themselves by using the situations.

Further the formalization of vector guarantees is carried out. Different types of guarantees are considered based on different types uncertainties: Slater minimal, Pareto minimal, Borwein minimal, Geoffrion minimal, A-minimal. Connection among all of these types of guarantees is widely discussed.

If we consider the fixed game under uncertainty we can make some conclusions:

- Sufficient conditions for the existence of an A-minimal uncertainty for the game still stay sufficient for the four other solutions of this problem, sufficient conditions for the existence of a Geoffrion minimal uncertainty for the game still stay sufficient for Borwein, Pareto and Slater solutions of this problem, etc.
- Necessary conditions for some uncertainty to be a Slater minimal one in the game still are necessary uncertainties for the four other solutions of this problem, necessary conditions for an uncertainty to be a Pareto minimal one in the game still are necessary uncertainties for it to be a Borwein, Geoffrion and A-minimal one, etc.
- If Slater minimal uncertainties are absent in the game then the uncertainties of the four other types for this problem are absent too.
- If an A-minimal uncertainty exists in the game then the four other types of solutions for this problem exist too.
- The properties, which possess Slater minimal uncertainties of the game, are still valid for the four other solutions of the game.

Geometry interpretation of vector guarantees is also widely discussed.

Different sufficient conditions for vector guarantees existence are derived as well. We formulate sufficient conditions such that there exists an A-minimal uncertainty. It is shown that the A-minimal uncertainty will be a Geoffrion, Borwein, Pareto and Slater minimal one for the problem too.

The conditions are presented such that the uncertainty is a Geoffrion minimal one for the game. Special cases are discussed. It is shown that there exists an A-minimal uncertainty in the game under specific assumptions. The A-minimal uncertainty is constructed in an explicit way. The same is done with respect to Geoffrion minimal uncertainties as well.