ON A ONE INTEGRO-DIFFERENTIAL EQUATIONS

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The question of investigation of a bending of cusped plates (see, e.g., [1-3]) can be reduced to some classes of integro-differential equation.

Let us consider the following integro-differential equation

$$\varphi(x,t)-\sum_{i=1}^{n}\int_0^{\ell}P_i(x)Q_i(\xi)\varphi_{,tt}(\xi,t)d\xi=f(x,t), \tag{1}$$

under the initial conditions

$$\varphi(x,0)=\psi_1(x),\quad \varphi_{,t}(x,0)=\psi_2(x), \tag{2}$$

where $P_i(x)$, $Q_i(x)$, ψ_j $(i=1,2,...,n,\ \ j=1,2)$, and $f(x,t)$ are given functions,

$$\begin{aligned}&P_i(x),\ Q_i(x),\ \psi_j\in C[0,\ell]\ \ (i=1,2,...,n,\ \ j=1,2),\ \ f(x,t)\in C(\overline{\Omega})\\ &\varphi(x,t)\in C(\overline{\Omega})\cap C^1(t\geq 0)\cap C^2(t>0),\ \ \Omega=[0,\ell]\cap(t>0).\end{aligned} \tag{3}$$

We can express $\varphi(x,t)$ as follows

$$\varphi(x,t)=f(x,t)+\sum_{i=1}^{n}P_i(x)a_{i,tt}(t), \tag{4}$$

here

$$a_i(t)=\int_0^{\ell}Q_i(\xi)\varphi(\xi,t)d\xi,\quad i=\overline{1,n}.$$

After multiplying (4) by $Q_j(x)$ and integrating it in the limits 0 and ℓ, for $a_i(t)$ we obtain the following system of ordinary differential equations

$$a_j(t)=f_j(t)+\sum_{i=1}^{n}a_{i,tt}(t)\alpha_{ij}, \tag{5}$$

$$\alpha_{ij}:=\int_0^{\ell}P_i(\xi)Q_j(\xi)d\xi,\quad f_j(t):=\int_0^{\ell}f(\xi,t)Q_j(\xi)d\xi,\quad i,j=\overline{1,n}. \tag{6}$$

Taking into (2), initial conditions for $a_i(t)$ will have the form

$$a_i(0)=\int_0^{\ell}\psi_1(\xi)Q_i(\xi)d\xi:=c_i,\quad a_{i,t}(0)=\int_0^{\ell}\psi_2(\xi)Q_i(\xi)d\xi:=c_i',\qquad i,j=\overline{1,n}. \tag{7}$$

Let us denote by Δ the following determinant

$$\Delta := \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix}.$$

If $\Delta \neq 0$ we can reduce system (5) to the normal form as follows (see, e.g., [4])

$$\begin{gathered} a_{i,tt}(t) - \sum_{i=1}^{n} a_i(t)\beta_{ij} = g_j(t), \\ \beta_{ij} := \frac{\Delta_{ij}}{\Delta}, \ \ g_j := \frac{\sum\limits_{i=1}^{n} f_i(t)\Delta_{ij}}{\Delta}, \ \ i,j = \overline{1,n}, \end{gathered} \tag{8}$$

$$\Delta_{ij} := \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1,j-1} & \alpha_{1,j-1} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2,j-1} & \alpha_{2,j-1} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{i-1,1} & \alpha_{i-1,2} & \dots & \alpha_{i-1,j-1} & \alpha_{i-1,j-1} & \dots & \alpha_{i-1,n} \\ \alpha_{i+1,1} & \alpha_{i+1,2} & \dots & \alpha_{i+1,j-1} & \alpha_{i+1,j-1} & \dots & \alpha_{i+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{n,j-1} & \alpha_{n,j-1} & \dots & \alpha_{nn} \end{vmatrix}.$$

But it is well-known (see [4]) that (8) under initial conditions (7) is uniquely solvable, since $g_i(t) \in C(t \geq 0)$ because of (3).

Thus, we have to prove the following

Proposition 1. *In case $\Delta \neq 0$, if conditions (3) are fulfilled, problem (1), (2) has a unique solution, which has the form (4).*

Let, now $\Delta = 0$. We can rewrite (5) as follows

$$a(t) - Aa_{,tt}(t) = F(t), \tag{9}$$

where $A := (\alpha_{ij})$, $a(t) := (a_i(t))$, $F(t) := (f_i(t))$. We denote by $c := (c_i)$, $c' := (c'_i)$ $(i = \overline{1,n})$.

On the other hand, it is obvious that the following equation

$$Ax = \lambda x, \quad x := (x_i), \quad i = \overline{1,n},$$

has a finite number of eigenvalues λ_1, λ_2,...λ_m $(m \leq n)$, we denote by m_k orders of λ_k. Let $|\lambda_1| < |\lambda_2| < ... < |\lambda_m|$ (it can be assumed without loss of generality). It is easy to show that $\lambda_1 = 0$.

For A there exists such a non-degenerate $n \times n$ matrix T, that $n \times n$ matrix $T^{-1}AT$ is a Jordan matrix (see [5]), i.e.,

$$\left(T^{-1}AT\right) = J = \begin{pmatrix} J_1 & O_{12} & \dots & O_{1m} \\ O_{22} & J_2 & \dots & O_{2m} \\ \dots & \dots & \dots & \dots \\ O_{m1} & O_{m2} & \dots & J_m \end{pmatrix}, \tag{10}$$

where O_{ij} are $m_i \times m_j$ null matrices and

$$J_k = \begin{pmatrix} \lambda_k & 1 & 0 & ... & O & 0 \\ 0 & \lambda_k & 1 & ... & O & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda_k & 1 \\ 0 & 0 & 0 & ... & 0 & \lambda_k \end{pmatrix}, \quad k = \overline{1,m},$$

are $(m_k \times m_k)$ matrices.

We introduce a new unknown vector $b(t) := (b_i(t)) \in C^1(t \geq 0) \cap C^2(t > 0)$, $i = \overline{1,n}$, in this way

$$a(t) = Tb(t).$$

Then taking into account (9), for $b(t)$ we have

$$Tb(t) - ATb,_{tt}(t) = F(t).$$

After multiplying this equation from the left by T^{-1}, in view of (10), we obtain

$$b(t) - Jb,_{tt} = T^{-1}F(t) =: G(t).$$

$$b(0) = T^{-1}a(0), \quad b,_t(0) = T^{-1}a,_t(0) \tag{11}$$

For $b_i(t)$ $(i = \overline{m_1+1, m})$ we have

$$b(t) - \sum_{j=m_1+1}^{m} J_{ij} b_{j,tt} = G_i(t), \tag{12}$$

here

$$J_{ij} := \begin{pmatrix} J_2 & 0 & ... & 0 \\ 0 & J_3 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & J_m \end{pmatrix}.$$

It is obvious that $det(J_{ij}) \neq 0$ and system (12) can be reduced to the normal form which under initial conditions (11) has a unique solution.

Let us now consider the first m_1 equations,

$$b_i(t) - J_1 b_{i,tt} = G_i(t), \quad i = 1, ..., m_1. \tag{13}$$

$$det(J_1) = \begin{vmatrix} \lambda_1 & 1 & 0 & ... & 0 & 0 \\ 0 & \lambda_1 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda_1 & 1 \\ 0 & 0 & 0 & ... & 0 & \lambda_1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 \end{vmatrix} = 0. \tag{14}$$

From (13) and (14) we get

$$b_{m_1}(t) = G_{m_1}(t). \tag{15}$$

In view of initial conditions (11), G_{m_1} must satisfy the following condition

$$G_{m_1}(0) = \left(T^{-1} \int_0^\ell Q(\xi)\psi_1(\xi) d\xi \right)_{m_1},$$

$$G'_{m_1}(0) = \left(T^{-1} \int_0^\ell Q(\xi)\psi_2(\xi) d\xi \right)_{m_1}.$$

Taking into account $G_{m_1} = (T^{-1}F(t))_{m_1}$, and (6) we have

$$\left(T^{-1} \int_0^\ell Q(\xi) \left(f_t^{(i-1)}(\xi,0) - \psi_i(\xi) \right) d\xi \right)_{m_1} = 0, \quad i = 1,2.$$

Further, by virtue of (13), (14), (15), we have

$$b_{m_1-j}(t) = G_{m_1-j}(t) + \left(G_{m_1-j+1}(t) + G''_{m_1-j+2}(t) + \ldots + G_{m_1}^{(2j-2)}(t) \right)'',$$

$$j = 1, \ldots, m_1 - 1.$$

Initial conditions are fulfilled if

$$\left(T^{-1} \int_0^\ell Q(\xi) \left(f_t^{(i-1)}(\xi,0) - \psi_i(\xi) \right) d\xi \right)_{m_1} =$$
$$\sum_{n=m_1-j+1}^{m_1} \left(T^{-1} \int_0^\ell Q(\xi) f^{4(j-1)}(\xi,0) d\xi \right)_n, i = 1,2, \quad j = 1, \ldots, m_1 - 1. \tag{16}$$

Proposition 2. *In case $\Delta = 0$, problem (1), (2) is solvable if and only if conditions (16) are fulfilled.*

R E F E R E N C E S

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