# ON THE BOUNDEDNESS AND COMPACTNESS OF GENERALIZED TRUNCATED POTENTIALS

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In the present note we give necessary and sufficient conditions for positive $\sigma$-finite Borel measure $\nu$ on $\mathbb{R}_+^{n+1} \equiv \mathbb{R}^n \times [0,\infty)$, which guarantee the boundedness ( compactness) of the operator

$$K_\gamma f(x,t) = \int_{|y|\le 2|x|} \frac{f(y)}{(|x-y|+t)^{n-\gamma}} dy \quad (x \in \mathbb{R}^n, t \in [0,\infty))$$

from $L^p(\mathbb{R}^n)$ into $L^q_\nu(\mathbb{R}_+^{n+1})$, where $0 < p, q < \infty$, $p > 1$ and $\gamma > n/p$ (for $q < p$ it will be assumed that $\nu$ is absolutely continuous, i.e. $d\nu(x,t) = v(x,t)dxdt$, where $v$ is a Lebesgue-measurable almost everywhere positive function on $\mathbb{R}_+^{n+1}$).

An analogous problem for the generalized Riemann- Liouville operator

$$T_\alpha f(x,t) = \int_0^x \frac{f(\tau)}{(x-\tau+t)^{1-\alpha}} d\tau \quad (x,t \in [0,\infty))$$

was solved in [6] and [7] for $\alpha > 1/p$ (for the classical Riemann-Liouville operator $R_\alpha f(x) = T_\alpha(x,0)$ see [5]). A full characterization of couples of weights ($v$, $w$ ) which provide the two-weight weak $(p,q)$ $(1 < p < q < \infty)$ type inequality for generalized Riesz potential

$$I_\alpha f(x,t) = \int_{\mathbb{R}^n} \frac{f(y)}{(|x-y|+t)^{n-\gamma}} dy \quad (x \in \mathbb{R}^n, t \in [0,\infty))$$

where $0 < \gamma < 1$, is given in [3] (see also [4], Chapter 3) and the boundedness problem for $I_\alpha$ from $L^p(\mathbb{R}^n)$ into $L^q_\nu(\mathbb{R}_+^{n+1})$ was solved in [2].

The necessary and sufficient conditions for the boundedness of truncated Riesz potential $J_\gamma f(x) = K_\gamma f(x,0)$ from $L^p(\mathbb{R}^n)$ into $L^p_\nu(\mathbb{R}_+^{n+1})$, where $1 < p < \infty$ and $\gamma > 1/p$, was found in [8], while the solution of two-weight problem for the classical Riesz potential was presented in [3] and [4], Chapter 3. For the compactness of classical Riesz potential see [1].

Let $\nu$ be a positive $\sigma$-finite Borel measure on $\mathbb{R}^{n+1}$. In the sequel $L^q_\nu(\mathbb{R}_+^{n+1})$ $(0 < q < \infty)$ denotes a set of all $\nu$-measurable functions $g : \mathbb{R}_+^{n+1} \to \mathbb{R}^1$ for which

$$\|g\|_{L^q_\nu(\mathbb{R}_+^{n+1})} = \Big( \int_{\mathbb{R}_+^{n+1}} |g(x,t)|^q d\nu(x,t) \Big)^{1/q} < \infty.$$

If $\nu$ is absolutely continuous ( i.e. $d\nu = v(x,t)dxdt$, where $v$ is Lebesgue-measurable, a.e. positive function on $\mathbb{R}_+^{n+1}$ ), then instead of $L^q_\nu(\mathbb{R}_+^{n+1})$ we will use the notation $L^q_v(\mathbb{R}_+^{n+1})$.

Let $\mathbb{U}_r \equiv \{x \in \mathbb{R} : |x| \geq r\} \times [0, \infty)$, where $r > 0$. It is obvious that $\mathbb{U}_{2^k} \setminus \mathbb{U}_{2^{k+1}} = \{x \in \mathbb{R}^n : 2^k \leq |x| < 2^{k+1}\} \times [0, \infty)$ for every $k \in \mathbb{Z}$.

The following theorems are valid:

**Theorem 1.** *Let $1 < p \leq q < \infty$, $\gamma > \frac{n}{p}$. Then the following statements are equivalent:*

(i) $K_\gamma$ *is bounded from* $L^p(\mathbb{R}^n)$ *into* $L^q_\nu(\mathbb{R}^{n+1}_+)$;

(ii)

$$B \equiv \sup_{r>0} \left( \int_{\mathbb{U}_r} (|x| + t)^{(\gamma-n)q} d\nu(x,t) \right)^{\frac{1}{q}} r^{\frac{n}{p'}} < \infty;$$

(iii)

$$B_1 \equiv \sup_{k\in\mathbb{Z}} \left( \int_{\mathbb{U}_{2^k} \setminus \mathbb{U}_{2^{k+1}}} (|x| + t)^{(\gamma-n)q} |x|^{nq/p'} d\nu(x,t) \right)^{\frac{1}{q}} < \infty.$$

*Moreover,* $\|K_\gamma\| \approx B \approx B_1$.

**Theorem 2.***Let $1 < p \leq q < \infty$, $\gamma > \frac{n}{p}$. Suppose that $L^q_\nu(\mathbb{R}^{n+1}_+)$ is separable. Then the following conditions are equivalent:*

(i) $K_\gamma$ *is compact from* $L^p(\mathbb{R}^n)$ *into* $L^q_\nu(\mathbb{R}^{n+1}_+)$ ;

(ii) $B < \infty$ and $\lim_{r\to 0} B(r) = \lim_{r\to+\infty} B(r) = 0$, *where*

$$B(r) = \left( \int_{\mathbb{U}_r} (|x| + t)^{(\gamma-n)q} d\nu(x,t) \right)^{\frac{1}{q}} r^{\frac{n}{p'}};$$

(iii) $B_1 < \infty$ and $\lim_{k\to-\infty} B_1(k) = \lim_{k\to+\infty} B_1(k) = 0$, *where*

$$B_1(k) = \left( \int_{\mathbb{U}_{2^k} \setminus \mathbb{U}_{2^{k+1}}} (|x| + t)^{(\gamma-n)q} |x|^{qn/p'} d\nu(x,t) \right)^{\frac{1}{q}}.$$

Now we consider the case $q < p$.

**Theorem 3.** *Let $0 < q < p < \infty$, $p > 1$ and $\gamma > n/p$. Assume that $\nu$ is absolutely continuous. Then the following statements are equivalent:*

(i) $K_\gamma$ *is bounded from* $L^p(\mathbb{R}^n)$ *into* $L^q_v(\mathbb{R}^{n+1}_+)$;

(ii) $K_\gamma$ *is compact from* $L^p(\mathbb{R}^n)$ *into* $L^q_v(\mathbb{R}^{n+1}_+)$;

(iii)

$$D = \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{U}_x} (|y| + t)^{(\gamma-n)q} v(y,t) dy dt \right)^{\frac{p}{p-q}} |x|^{\frac{(q-1)pn}{p-q}} dx \right)^{\frac{p-q}{pq}} < \infty.$$

*Moreover,* $\|K_\gamma\| \approx D$.

Now we give the upper and lower estimates of the distance of the operator $K_\gamma$ from a space of compact operators.

Let $X$ and $Y$ be Banach spaces. Denote by $\mathbb{K}(X, Y)$ a space of all compact operators from $X$ into $Y$. We will assume that $\nu$ is absolutely continuous.

**Theorem 4.** *Let $1 < p \le q < \infty$, $\gamma > n/p$ and let $X = L^p(\mathbb{R}^n)$, $Y = L^q_v(\mathbb{R}^{n+1}_+)$. Assume that $B < \infty$ for $d\nu(x,y) = v(x,t)dxdt$, where $v$ is Lebesgue-measurable a.e. positive function on $\mathbb{R}^{n+1}_+$. Then there exist positive constants $b_1$ and $b_2$ depending only on $p$, $q$ and $\gamma$ such that*

$$b_1 J \le dist(K_\gamma, \mathbb{K}(X,Y)) \le b_2 J,$$

*where* $J = \lim\limits_{a\to 0} J^{(a)} + \lim\limits_{d\to+\infty} J^{(d)}$,

$$J^{(a)} \equiv \sup_{0<r<a} \left( \int_{\mathbb{U}_r \setminus \mathbb{U}_a} v(x,t)(|x|+t)^{(\gamma-n)q} dxdt \right)^{1/q} r^{n/p'},$$

$$J^{(d)} \equiv \sup_{r>d} \left( \int_{\mathbb{U}_r} v(x,t)(|x|+t)^{(\gamma-n)q} dxdt \right)^{1/q} (r^n - d^n)^{1/p'}.$$

**Remark.** Let $(X,d,\mu)$ be the space of homogeneous type, i.e. the space with quasimetric $d$ and complete measure $\mu$ satisfying the doubling condition

$$\mu B(x,2r) \le b\mu B(x,r),$$

for all $x \in X$ and $r > 0$, where $B(x,r) = \{y \in X : d(x,y) < r\}$. Then analogous results for the operator

$$K_\gamma f(x,t) = \int_{d(x_0,y)\le 2d(x_0,x)} \frac{f(y)}{\mu B(x, d(x,y)+t)^{1-\gamma}} d\mu(y)$$

($\gamma > 1/p$, $x_0 \in X$ and $(x,t) \in X \times [0,\infty)$)) can be obtained.

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