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PREDICTIVE DISTRIBUTION FOR POLYTOMOUS LOGISTIC MODELS WITH NONLINEAR LINK FUNCTIONS

M.A. Tabatabai and K.P. Singh

Summary. This article is concerned with predictive distribution for polytomous logistic models with nonlinear link functions. Within a Bayesian framework, an approximation of the predictive distribution for determining probabilities is presented. We describe this method using nonlinear link functions. The generalized linear link function is simply a special case of nonlinear models. All of the generalized logistic models require nonlinear estimation procedures to evaluate maximum likelihood estimates.

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1. INTRODUCTION.

Statistical methods based on logistic models are routinely used in experiments with binary outcome data. It can also be used to model the association between a polytomous response variable and some explanatory variable of interest. This article's emphasis is on obtaining a predictive distribution for future observation based on previous data. Prediction analysis plays the key role in the real decision-making process implicit in most medical survival studies. Discussions of prediction analysis can be found in Geisser (1971, 1980, 1982) and Aitchison and Dunsmore (1975). Data with polytomous response variables are frequently encountered in social and biomedical sciences. The response may be recorded on an ordinal or nominal scale and the link function can be linear or nonlinear functions. Here we use a Bayesian formulation to make predictions for future individuals. Given data

$$D = \{(x_i, y_i),\ i = 1, \dots, n\}$$

and x_{n+1} for future individual, we can predict the distribution of unobserved response y_{n+1}. The future response is polytomous response variable. This usage has previously been applied by Percy (1993) to find predictive distribution for linear models, including its application to survival data. Heitjan (1991) applied nonlinear modeling to serial immunologic data from a multiple sclerosis clinical trial. Rosner (1989) presented a polychotomous logistic model to analyze ophthalmologic data. We focus on polytomous logistic models with nonlinear links.

2. THE POLYTOMOUS LOGISTIC MODEL.

Let Y represent the response variable and assume $Y \in \{0, 1, \dots, k\}$, and let a sample of size n be taken. When the response is categorical, the following logit model may be used to relate the response

$$ln(P_{ij}/P_{io}) = g_j(x_i; \beta_j),\ j = 1, \dots, k, \tag{2.1}$$

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where $g_j(x_i;\beta_j)$ is a function of a parameter vector β_j and covariate vector x_i of dimension P, and

$$P_{ij} = P(y_i = j|x_i),\ i = 1, \ldots, n,\ j = 0, 1, \ldots, k \tag{2.2}$$

the conditional probabilities P_{ij} of each outcome category given the covariate vector are

$$P_{ij} = \exp[g_j(x_i;\beta_j)]/\left\{\sum_{l=0}^{k}\exp[g_l(x_i;\beta_l)]\right\},\ j = 0, 1, \ldots, k,\ i = 1, \ldots, n \tag{2.3}$$

where $g_0(x_i;\beta_0) \equiv 0, \sum_{j=0}^{k} P_{ij} = 1$, and $\beta_j = (\beta_{j1}, \beta_{j2}, \ldots, \beta_{jP_j})'$. To construct the likelihood function, we formulate $k+1$ binary variables coded as zero or 1 to indicate group membership of an observation. The variables are coded as follows:

If $y_i = j$, then the coded vector $y_{ij} = (y_{i0}, y_{i1}, \ldots y_{ih}, \ldots y_{ik})$ has $y_{ih} = 1$ if $h = j$ and 0 if $h \neq j$.

3. THE PROPOSED TECHNIQUE.

Let $y_1, y_2, \ldots, y_n$ be n independent random variable with joint probability mass function

$$f(y_1, y_2, \ldots, y_n|\,P_{1j}, P_{2j}, \ldots P_{nj},\ j = 0, 1, \ldots, k) = \prod_{i=1}^{n}\{(P_{i0})^{y_{io}}(P_{i1})^{y_{i1}}\ldots(P_{ik})^{y_{ik}}\} \tag{3.1}$$

where for each i, $\sum_{j=0}^{k} y_{ij} = 1$.

If we use (2.3), we obtain the joint distribution

$$f(y_1, y_2, \ldots, y_n|x_i,\ i = 1, \ldots, n;\ \beta_j,\ j = 1, \ldots, k) = \prod_{i=1}^{n}\left\{\exp[\sum_{j=0}^{k} y_{ij}g_j(x_i,\beta_j)]/\sum_{j=0}^{k}\exp[g_j(x_i;\beta_j)]\right\} \tag{3.2}$$

One major application of linear as well as nonlinear polytomous regression is to make forecasts. With these models, one can incorporate recent data

$$\{(y_i, x_i), i = 1, \ldots, n\} \text{ and } x_{n+1} \text{ into}$$

forecast for unobserved response y_{n+1}. To solve this problem and more importantly to estimate the predictive distribution we use Bayes theorem.

According to Bayes theorem, the predictive distribution

$$f(y_{n+1}|x_1, x_2, \ldots, x_{n+1}; y_1, \ldots, y_n) = \int_\beta f(y_{n+1}|x_{n+1};\beta)\ f(\beta|x_1, x_2, \ldots, x_n)d\beta \tag{3.3}$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ and $\dim(\beta_j) = P_j,\ j = 1, \ldots, k$ and $\sum_{j=1}^{k} P_j < n$.

To evaluate this integral we use an approximation similar to Percy's (1993) for binary linear logistic regression. Before proceeding, we must decide on a suitable prior distribution for the parameter matrix $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$. We assume $\beta_1, \beta_2, \ldots, \beta_k$ are k independent random vectors with each having a prior distribution as multivariate normal. This allows considerable flexibility for practical applications. The parameter values for these priors are easily equated to the first two moments of β based on hyperprior knowledge.

Let

$$f(\beta_j) = (\det A_j)^{\frac{1}{2}}/(2\pi)^{\frac{P_j}{2}} \exp\left\{-\frac{1}{2}(\beta_j - b_j)' A_j(\beta_j - b_j)\right\} \ ,j = 1, 2, \ldots, k \tag{3.4}$$

where A_j is symmetric positive definite matrix. Then the likelihood function is given by

$$L(\beta; x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n}\left\{\exp[\sum_{j=0}^{k} y_{ij} g_j(x_i; \beta_j)/\sum_{j=0}^{k}\exp[g_j(x_i; \beta_j)]\right\} \tag{3.5}$$

Now, let $D(\beta)$ denote the posterior density. Then

$$\begin{aligned} D(\beta) &= f(\beta | x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) \\ &\alpha L(\beta; x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n). \prod_{j=1}^{k} f(\beta_j) \\ &\alpha \prod_{i=1}^{n}\left\{\exp[\sum_{j=0}^{k} y_{ij} g_j(x_i; \beta_j)/\sum_{j=0}^{k}\exp[g_j(x_i; \beta_j)]\right\}. \\ &\quad \exp\left\{-\frac{1}{2}\sum_{j=1}^{k}(\beta_j - b_j)' A_j(\beta_j - b_j)\right\} \end{aligned} \tag{3.6}$$

By taking natural logarithm we obtain

$$\begin{aligned} ln\ D(\beta) &= \sum_{j=0}^{k}\left\{y_{ij} g_j(x_i; \beta_j) - ln(\sum_{j=0}^{k}\exp[g_j(x_i; \beta_j)])\right\} \\ &\quad - \frac{1}{2}\sum_{j=1}^{k}(\beta_j - b_j)' A_j(\beta_j - b_j) \end{aligned} \tag{3.7}$$

Now, consider the vector $w(\beta)$ defined as

$$\begin{aligned} w(\beta) &= \frac{\partial}{\partial\beta} ln\ D(\beta) = \begin{bmatrix} w_1(\beta) \\ \vdots \\ w_k(\beta) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial}{\partial\beta_1} ln\ D(\beta) \\ \vdots \\ \frac{\partial}{\partial\beta_k} ln\ D(\beta) \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} w_j(\beta) = \sum_{i=1}^{n}[y_{ij} - \exp(g_j(x_i; \beta_j)/\sum_{l=0}^{k}\exp[g_l(x_i; \beta_l)] \cdot \frac{\partial g_j}{\partial\beta_j}(x_i; \beta_j) \\ -A_j(\beta_j - b_j) \end{aligned}$$

and

$$\frac{\partial g_j}{\partial \beta_j}(x_i;\beta_j) = \begin{bmatrix} \frac{\partial g_j}{\partial \beta_{j1}}(x_i;\beta_j) \\ \vdots \\ \frac{\partial g_j}{\partial \beta_{jP_j}}(x_i,\beta_j) \end{bmatrix} \tag{3.8}$$

$j = 1, \ldots, k$.
Define

$$l(\beta) = \frac{\partial}{\partial \beta} w(\beta) = \begin{bmatrix} \frac{\partial}{\partial \beta_1} w_1(\beta) & \ldots & \frac{\partial}{\partial \beta_k} w_1(\beta) \\ \vdots & & \\ \frac{\partial}{\partial \beta_1} w_k(\beta) & \ldots & \frac{\partial}{\partial \beta_k} w_k(\beta) \end{bmatrix} \tag{3.9}$$

where $l(\beta)$ is the information matrix of order $(\sum_{j=1}^{k} P_j)$ by $(\sum_{j=1}^{k} P_j)$ which is adjusted for prior information and if $j = j'$, $j = 1, \ldots, k$, then

$$\begin{aligned} \frac{\partial}{\partial \beta_j} w_j(\beta) = \sum_{i=1}^{n} \{ & \frac{\exp[2g_j(x_i;\beta_j)] - \exp[g_j(x_i;\beta_j)][\sum_{l=0}^{k} \exp[g_l(x_i;\beta_l)]}{[\sum_{l=0}^{k} \exp[g_l(x_i;\beta_l)]]^2} \cdot \\ & \left[\frac{\partial}{\partial \beta_j} g_j(x_i;\beta_j)\right] \cdot \left[\frac{\partial}{\partial \beta_j} g_j(x_i;\beta_j)\right]' \\ & + \left[y_{ij} - \frac{\exp[g_j(x_i;\beta_j)}{\sum_{l=0}^{k} \exp[g_l(x_i;\beta_l)]} \right] \cdot \frac{\partial^2 g_j(x_i;\beta_j)}{\partial \beta_j \partial \beta_{j'}'} \} - A_j \end{aligned} \tag{3.10}$$

But if $j \neq j', j = 1, \ldots, k, j' = 1, \ldots, k$, then we have

$$\frac{\partial}{\partial \beta_{j'}} w_j(\beta) = \sum_{i=1}^{n} \frac{\exp[g_j(x_i;\beta_j) + g_{j,}(x_i;\beta_{j'})]}{\left[\sum_{l=0}^{k} \exp[g_l(x_i;\beta_l)]\right]^2} \cdot \left(\frac{\partial}{\partial \beta_{j'}} g_{j'}(x_i;\beta_{j'})\right) \cdot \left(\frac{\partial}{\partial \beta_j} g_j(x_i;\beta_j)\right)' \tag{3.11}$$

The first order approximation is defined by

$$f(y_{n+1}|x_1, x_2, \ldots, x_{n+1}; y_1, y_2, \ldots, y_n) = f(y_{n+1}|x_{n+1}, \beta^*)$$

where β^* denotes the posterior mode, which can be found by solving the vector equation

$$w(\beta) = 0 \tag{3.12}$$

For this, an initial guess β^0 is made for β^* and the iterative algorithm is given by

$$\beta^{(l)} = \beta^{(l-1)} + [l(\beta^{(l-1)})]^{-1} w(\beta^{(l-1)}) \tag{3.13}$$

for $l = 1, 2, \ldots$ until convergence is achieved according to satisfaction of the criterion $\| \beta^{(l)} - \beta^{(l-1)} \| \leq \in$ for a chosen norm $\| \cdot \|$ and small positive value $\in$.

Having achieved convergence, β^* is taken to be $\beta^{(l)}$ having estimated $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$, our apparent predictive distribution for polytomous nonlinear logistic model is

$$
\begin{aligned}
&f(y_{n+1}|x_1, x_2, \ldots, x_{n+1};\, y_1, y_2, \ldots, y_n)\\
&\quad = \exp\{\sum_{j=0}^{k}[y_{n+1,j}\, g_j(x_{n+1}; \beta_j)]\}/\sum_{j=0}^{k}\exp\{g_j(x_{n+1}; \beta_j)\}
\end{aligned}
$$

It should be mentioned that the above method takes into account full prior knowledge which is available in most studies. If no prior information is available, set $A_j = 0$, and $b_j = 0$ for $j = 1, \ldots, k$ in equation (2.6).

4. AN EXAMPLE.

As an example, consider the data appearing in Percy (1993) consisting of a sample of $n = 100$ patients who were admitted to intensive care units in the UK with acute respiratory and renal failure. The respose y is died (0) or survived (1) and covariates are age, shunt (a measure of lung damage), cieatinine (mg/100ml), albumin (g/100ml) and sepsis (0 = absent or 1 = present).

Percy used linear logistic model with linear link function ($g(x, \beta) = \beta_0 + \beta_1 x_1 + \ldots + \beta_5 x_5$) with prior information $A = \text{diag}\ (1, 100, .01, 10000, 100, 1)$ and $\beta = (-1, -0.1, -10, 0.01, 0.1, 1)'$. For this data he obtained $\beta^* = (-1.2574, -0.052341, -7.8132, 0.0023812, 0.084193, 1.4057)'$ if the vector of covariates for the next patient admitted to an intensive care unit with acute respiratory and renal failure is

$$x_{101} = (1,\ 49,\ 0.21,\ 706,\ 38,\ 1)'$$

then Percy approximated the predictive distribution of y_{101} by

$$f(y_{101}|x-1, x-2, \ldots, x_{101};\, y_1, y_2, \ldots, y_{100}) = \begin{cases} 0.305 & \text{if } y_{101} = 0 \\ 0.695 & \text{if } y_{101} = 1. \end{cases}$$

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A COMMON FIXED POINT THEOREM IN COMPLETE METRIC SPACES

R.A.RASHWAN AND A.M.SADEEK

ABSTRACT. In this note we establish a common fixed point theorem in complete metric spaces which generalizes some results in [2].

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Key Words and Phrases. Common fixed points, complete metric spaces-generalized contractive conditions.

1. INTRODUCTION

Delbosco [1] and Skof [3]have established fixed point theorems for self - maps of complete metric spaces by altering the distances between the points with the use of a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

(i) $\phi : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing in R^+;

(ii) $\phi(t) = 0$ iff $t = 0$;

(iii) $\phi(t) \geq Mt^u$ for every $t > 0, u > 0$ are constants.

In 1977, F. Skof [3] gave the following theorem

THEOREM A. Let T be a self-map of a complete metric space (X, d) and $\phi \in \Phi$ such that for every $x, y \in X$,

$$(I) \quad \phi(d(Tx, Ty)) \leq a\phi(d(x, y)) + b\phi(d(x, Tx)) + c\phi(d(y, Ty)),$$

where a, b and c are three nonnegative constants satisfying $a + b + c < 1$. Then T has a unique fixed point.

In [1], the author has considered functions $\phi \in \Phi$ such that $\phi(t) = t^n, n \in N$ for every $t \geq 0$. Clearly, ϕ is not a metric since ϕ has only a argument.

In 1984, M.S.Khan et al. [2] generalized Theorem (A) by using a much extensive condition than (I) and removed the hypothesis (iii) which seems superfluous. They proved the following theorem:

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THEOREM B. Let T be a self-map of a complete metric space (X,d) and ϕ satisfying (i) and (ii). Furthermore, let a,b,c be three decreasing functions from R^+ 0 into $[0,1)$ such that

$$a(t)+2b(t)+c(t)<1$$

for every $t>0$.

Suppose that T satisfies the following condition:

$$\begin{aligned}(II)\quad \phi(d(Tx,Ty)) &\le a(d(x,y))\phi(d(x,y))+b(d(x,y))[\phi(d(x,Tx))+\phi(d(y,Ty))]\\ &+c(d(x,y))\min\{\phi(d(x,Ty)),\phi(d(y,Tx))\},\end{aligned}$$

where $x,y\in X$ and $x\ne y$. Then T has a unique fixed point.

In this note we shall establish a common fixed point theorem for a pair of mappings S and T which satisfying the contractive defintion corresponding to (II), i.e., for all $x,y\in X$ with $x\ne y$

$$\begin{aligned}(III)\quad \phi(d(Sx,Ty)) &\le a(d(x,y))\phi(d(x,y))+b(d(x,y))[\phi(d(x,Sx))+\phi(d(y,Ty))]\\ &+c(d(x,y))\min\{\phi(d(x,Ty)),\phi(d(y,Sx))\},\end{aligned}$$

A COMMON FIXED POINT THEOREM

We establish the following:

THEOREM 2.1. Let S and T be self-mappings on a complete metric space (X,d) and ϕ satisfying the conditions (i) and (ii). Furthermore, let a,b,c be three decreasing functions from R^+ into $[0,1)$ such that

$$a(t)+2b(t)+c(t)<1$$

for every $t>0$.

Suppose that S and T satisfy (III). Then S and T have a unique fixed point in X.

Proof. Let x_0 be some point in X, define

$$\begin{aligned}x_{2n}&=Sx_{2n+1},\quad n=0,1,2,...\\ x_{2n+1}&=Tx_{2n},\quad n=0,1,2,...\end{aligned}$$

Let $\tau_n=d(x_n,x_{n+1})$. We assume that $\tau_n>0$ for every n. Now, for an even integer n, we have

$$\begin{aligned}\phi(\tau_n)&=\phi(d(x_n,x_{n+1}))=\phi(d(Sx_{n-1},Tx_n))\\ &\le A_1+A_2+A_3\end{aligned}$$

where

$$A_1=a(d(x_{n-1},x_n))\phi(d(x_{n-1},x_n)),$$

$$A_2=b(d(x_{n-1},x_n))[\phi(d(x_{n-1},Sx_{n-1}))+\phi(d(x_n,Tx_n))],$$

$$A_3=c(d(x_{n-1},x_n))\min\{\phi(d(x_{n-1},Tx_n)),\phi(d(x_n,Sx_{n-1}))\}.$$

So

$$\phi(\tau_n) = \leq a(\tau_{n-1})\phi(\tau_{n-1}) + b(\tau_{n-1})[\phi(\tau_{n-1}) + \phi(\tau_n)].$$

Hence we obtain:

$$\phi(\tau_n) \leq \frac{a(\tau_{n-1}) + b(\tau_{n-1})}{1 - b(\tau_{n-1})}\phi(\tau_{n-1}) < \phi(\tau_{n-1}). \tag{1}$$

Since ϕ is increasing, $\{\tau_n\}$ is a decreasing sequence. Put $\lim_n \tau_n = \tau$ with $\tau > 0$. By (1), then $\tau_n \geq \tau$ implies that

$$\phi(\tau_n) \leq \frac{a(\tau) + b(\tau)}{1 - b(\tau)}\phi(\tau_{n-1}).$$

By letting $n \to \infty$, Since ϕ is continuous, we have:

$$\phi(\tau) \leq \frac{a(\tau) + b(\tau)}{1 - b(\tau)}\phi(\tau) < \phi(\tau),$$

which is a contradiction, hence $\tau = 0$.

Similarly for an odd integer n one can show that $\tau = 0$. Now, we wish to show that $\{x_n\}$ is a Cauchy sequence. Since $\tau = 0$ it suffices to prove that $\{x_{2n}\}$ is a Cauchy sequence we proceed by contradiction. Then there is an $\epsilon > 0$ such that for each even integer $2k$, $k = 0, 1, 2, ...$, there exists two sequences $\{2p(k)\}$, $\{2q(k)\}$ with $2k \leq 2q(k) < 2p(k)$ for which

$$d(x_{2p(k)}, x_{2q(k)}) > \epsilon. \tag{2}$$

For each even integer $2k$, let $2p(k)$ be the least even integer exceeding $2q(k)$ and satisfy (2). Then

$$d(x_{2q(k)}, x_{2p(k)-2}) \leq \epsilon \quad and \quad d(x_{2q(k)}, x_{2p(k)}) > \epsilon.$$

Put $d_{2k} = d(x_{2q(k)}, x_{2p(k)})$, then we have

$$\begin{aligned}
&\epsilon < d_{2k} = d(x_{2q(k)}, x_{2p(k)}) \\
&\leq d(x_{2q(k)}, x_{2p(k)}) + d(x_{2p(k)-2}, x_{2p(k)-1}) + d(x_{2p(k)-1}, x_{2p(k)}) \\
&\leq \epsilon + \tau_{2p(k)-2} + \tau_{2p(k)-1}.
\end{aligned} \tag{3}$$

Since $\{\tau_n\}$ converges to 0, $d_{2k} \to \epsilon$. Furthermore, by triangular inequality, it follows that

$$\begin{aligned}
-\tau_{2p(k)} - \tau_{2q(k)} + d_{2k} &\leq d(x_{2q(k)+1}, x_{2p(k)+1}) \\
&\leq \tau_{2p(k)} + \tau_{2q(k)} + d_{2k},
\end{aligned}$$

and therefore the sequence $\{d(x_{2q(k)+1}, x_{2p(k)+1})\}$ converges to ϵ.

Moreover, by triangle inequality we have

$$\begin{aligned}
&d(x_{2p(k)+1}, x_{2q(k)}) \leq d(x_{2p(k)+1}, x_{2p(k)}) + d(x_{2p(k)}, x_{2q(k)}) \\
&\leq \tau_{2p(k)} + d_{2k} \to \epsilon, \quad as \quad k \to \infty
\end{aligned}$$

and

$$d(x_{2p(k)+2}, x_{2q(k)+1}) \leq d(x_{2p(k)+2}, x_{2p(k)+1}) + d(x_{2p(k)+1}, x_{2q(k)+1}) \leq A_4$$

So

$$A_4 = d(x_{2p(k)+2}, x_{2q(k)+1}) \leq \tau_{2p(k)+1} + d(x_{2p(k)+1, x_{2q(k)+1}})$$

Hence

$$A_4 \to \epsilon \quad as \quad k \to \infty$$

From (III), we also deduce:

$$\begin{aligned}\phi(d(x_{2p(k)+2}, x_{2q(k)+1})) &= \phi(d(Sx_{2p(k)+1}, Tx_{2q(k)})) \\ &\leq L_1 + L_2 + L_3 \leq L_4 + L_5 + L_6\end{aligned}$$

where

$$L_1 = a(d(x_{2p(k)+1}, x_{2q(k)}))\phi(d(x_{2p(k)+1}, x_{2q(k)})),$$

$$L_2 = b(d(x_{2p(k)+1}, x_{2q(k)}))[\phi(d(x_{2p(k)+1}, Sx_{2p(k)+1})) + \phi(d(x_{2q(k)}, Tx_{2q(k)}))],$$

$$L_3 = c(d(x_{2p(k)+1}, x_{2q(k)}))\min\{\phi(d(x_{2p(k)+1}, Tx_{2q(k)})), \phi(d(x_{2q(k)}, Sx_{2p(k)+1}))\},$$

$$L_4 = a(\tau_{2p(k)} + d_{2k})\phi(\tau_{2p(k)} + d_{2k}),$$

$$L_5 = b(\tau_{2p(k)} + d_{2k})[\phi(\tau_{2p(k)+1}) + \phi(\tau_{2q(k)})],$$

$$\begin{aligned}L_6 = c(\tau_{2p(k)} + d_{2k})\min\{&\phi(d(x_{2p(k)+1}, x_{2q(k)}) + \tau_{2q(k)}), \\ &\phi(\tau_{2q(k)} + d(x_{2q(k)+1}, x_{2p(k)+2}))\}.\end{aligned}$$

As $k \to \infty$, we have

$$\begin{aligned}\phi(\epsilon) &\leq a(\epsilon)\phi(\epsilon) + c(\epsilon)\phi(\epsilon) \\ &= \{a(\epsilon) + b(\epsilon)\}\phi(\epsilon) < \phi(\epsilon).\end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in a complete metric space X, then there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now, we shall show that $Sz = z = Tz$.

Hence using (III), we obtain:

$$\begin{aligned}&\phi(d(x_{2p(k)+2}, Tz)) = \phi(d(Sx_{2p(k)+1}, Tz)) \\ &\leq a(d(x_{2p(k)+1}, z))\phi(d(x_{2p(k)+1}, z)) \\ &+ b(d(x_{2p(k)+1}, z))[\phi(d(x_{2p(k)+1}, x_{2p(k)+2})) + \phi(d(z, Tz))] \\ &+ c(d(x_{2p(k)+1}, z))\min\{\phi(d(x_{2p(k)+1}, Tz)), \phi(d(z, x_{2p(k)+2}))\}.\end{aligned} \tag{4}$$

Letting $k \to \infty$, and use $b < \frac{1}{2}$ in (4), we have

$$\phi(d(z, Tz)) < \frac{1}{2}\phi(d(z, Tz)).$$

That is, $\phi(d(z,Tz)) = 0$ and therefore $d(z,Tz) = 0$, i.e., $z = Tz$. Similarly, it can be shown that $Sz = z$. Hence S and T have a common fixed point $z \in X$. We claim that z is the unique common fixed point of S and T. For this, we suppose that $w(w \neq z)$ is another fixed point of S and T. Then

$$\begin{aligned}
&\phi(d(z,w)) = \phi(d(Sz,Tw)) \\
&\leq a(d(z,w))\phi(d(z,w)) + b(d(z,w))[\phi(d(z,Sz)) + \phi(d(w,Tw))] \\
&+ c(d(z,w)) \min\{\phi(d(z,Tw)), \phi(d(w,Sz))\} \\
&\leq [a(d(z,w)) + c(d(z,w))]\phi(d(z,Sz)) \\
&< \phi(d(z,Tw))
\end{aligned}$$

a contradiction. This completes the proof of the theorem.

Finally, we give an example to discuss the validity of the hypothesis of Theorem 2.1.

EXAMPLE 2.1

Let $X = \{1,2,3\}$ be with the Euclidean metric induced by d which is defined by

$$d(1,2) = d(2,3) = 2, d(1,3) = 1.$$

Suppose S and T are maps on X defined by

$$S1 = S2 = S3 = 1, \quad T1 = 1, \quad T2 = 3, \quad T3 = 1.$$

Then it is clear that X is complete with respect to d and S,T satisfy all the conditions of Theorem 2.1. Now setting

$$\begin{aligned}
&x = 1, \quad y = 2, \quad a(t) = \frac{2}{5}, \quad b(t) = \frac{1}{5}, \quad c(t) = \frac{1}{6}, \\
&\phi(t) = t^2, \quad t \in R^+.
\end{aligned}$$

We observe that S and T satisfy condition (III), and 1 is the unique common fixed point of S and T.

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COMMON FIXED POINTS BY ISHIKAWA ITERATES IN METRIC LINEAR SPACES

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Abstract. In this paper we obtain a common fixed point theorem for Ishikawa iterates of two self-mappings on a metric linear space under various contractive conditions.

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Key Words and Phrases. Metric linear space, F - norm, Contractive mappings, Ishikawa iterates, Common fixed point.

1. INTRODUCTION. In [3], [4], it has shown that for a self-mapping T on a normed space or a Banach space X satisfying variuos contactive conditions, if the sequence of Ishikawa iterates associated with T converges, it coverges to a fixed point of T. These results have recently extended by L.A.Khan in [2] to the case of metric linear spaces .

In this paper, we consider two self-mappigs S and T on a metric linear space X and show that if the sequence of Ishikawa iterates associated with S and T converges, it coverges to a common fixed point of S and T.

In the sequel we assume that the topology on X is generated by an F -norm q which has the following properties (see [5], pp.28-29).

(a) $q(x) \geq 0$ and $q(x) = 0$ iff $x = 0$

(b) $q(x + y) \leq q(x) + q(y)$

(c) $q(ax) \leq q(x)$ for all (real or complex) scalars a with $|a| \leq 1$

(d)If $a_n \to a$ and $x_n \to x$, then $q(a_n x_n - ax) \to 0$.

Note that q is continuous on X, the equation $d(x, y) = q(x - y)$ defines a translation-invariant metric on X, and $q(ax) \leq q(bx)$ for all scalars a, b with $|a| \leq |b|$.

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Let C be a nonempty covex subset of X and S, T self-maps on C. An Ishikawa scheme [1], for S, T is defined by

$$x_0 \in C$$

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n S x_n \quad , n \geq 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \quad , n \geq 0, \end{aligned} \tag{1}$$

where the real sequences $\{\alpha_n\}, \{\beta_n\}$ satisfy

(i) $0 \leq \alpha_n \leq 1, \quad 0 \leq \beta_n \leq 1$, for $n \geq 0$

(ii) $\underline{\lim}_{n\to\infty} \alpha_n = \alpha > 0$

From (1) and condition (i) we obtain some inequalities which will be used later

$$\begin{aligned} q(y_n - x_n) &= q(\beta_n(Sx_n - x_n)) \\ &\leq q(Sx_n - Ty_n) + q(Ty_n - x_n), \end{aligned} \tag{2}$$

$$\begin{aligned} q(y_n - Ty_n) &= q(\beta_n(Sx_n - Ty_n)) + (1-\beta_n)(x_n - Ty_n) \\ &\leq q(Sx_n - Ty_n) + q(x_n - Ty_n), \end{aligned} \tag{3}$$

$$q(x_n - Sx_n) \leq q(x_n - Ty_n) + q(Ty_n - Sx_n) \tag{4}$$

$$\begin{aligned} q(y_n - Sx_n) &= q(1-\beta_n)(x_n - Sx_n) \\ &\leq q(x_n - Ty_n) + q(Ty_n - Sx_n) \end{aligned} \tag{5}$$

2. MAIN RESULT

We present our result in the form of the following theorem:

Theorem 2.1. Let C be a closed convex subset of X and let S, T be two self-mappings of C satisfying at least one of the folloing conditions

$$(I) q(Sx - Ty) \leq k \max\{q(x-y), q(x-Sx), q(y-Ty), q(x-Ty) + q(y-Sx)\}$$

$$(II) q(Sx - Ty) + q(x - Sx) + q(y - Ty) \leq c[q(x-Ty) + q(y-Sx)], 0 \leq c < 2$$

$$(III) q(Sx - Ty) \leq k \max\{q(x-y), q(x-Sx), q(y-Ty), q(x-Ty), q(y-Sx)\},$$

for all $x, y \in C$, where $0 \leq k < 1$ and $0 \leq c < 2$. Suppose that for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ of Ishikawa iterates converges to a point u. Then u is a common fixed point of S and T. Moreover if (III) holds, then the common fixed point u is unique.

Proof. Suppose first that $Su = u$ for a point u in C. Then putting $x = y = u$ into any of the inequalities (I)-(III) we easily see that $Tu = u$. Similarly $Tu = u$ implies $Su = u$. Now let $\{x_n\}$ be a sequence of Ishikawa iterates with S and T such that

$$\lim_{n\to\infty} x_n = u.$$

From (1),we see that

$$Ty_n - x_n = \frac{1}{\alpha_n}\alpha_n(x_{n+1} - x_n).$$

Since

$$\underline{\lim}_{n\to\infty}\ \alpha_n = \alpha > 0,$$

there exists an integer $N \geq 1$ such that

$$\frac{\alpha}{2} \leq \alpha_n$$

for all $n \geq N$. Hence for $n \geq N$

$$q(Ty_n - x_n) = q(\frac{\alpha}{2}(x_{n+1} - x_n)).$$

Since

$$\lim_{n\to\infty} x_n = u,$$

we have

$$q(Ty_n - x_n) \to 0 \quad as \quad n \to \infty$$

and hence

$$q(Ty_n - u) \to 0 \quad as \quad n \to \infty.$$

Now for $n \geq N$,

$$q(u - Tu) \leq q(u - Ty_n) + q(Sx_n - Ty_n) + q(Sx_n - Tu). \tag{6}$$

Now we need to show that

$$q(Sx_n - Ty_n) \to 0 \quad and \quad q(Sx_n - Tu) \to o \quad as \quad n \to \infty.$$

If x_n, y_n satisfy (I),then

$$\begin{aligned} q(Sx_n - Ty_n) \leq k\max\{&q(Sx_n - Ty_n) + q(x_n - Ty_n), \\ &q(x_n - Ty_n) + q(Sx_n - Ty_n), \\ &q(Sx_n - Ty_n) + q(x_n - Ty_n), \\ &2q(Sx_n - Ty_n) + q(x_n - Ty_n)\} \\ &= k[2q(x_n - Ty_n) + q(Sx_n - Ty_n)]. \end{aligned}$$

If x_n, y_n satisfy (II),then

$$(3 - c)q(Sx_n - Ty_n) \leq 2(c - 1)q(x_n - Ty_n).$$

If x_n, y_n satisfy (III),then

$$\begin{aligned}q(Sx_n - Ty_n) \leq k \max\{&q(x_n - Ty_n) + q(Sx_n - Ty_n),\\ &q(x_n - Ty_n) + q(Sx_n - Ty_n),\\ &q(Sx_n - Ty_n) + q(x_n - Ty_n),\\ &q(Sx_n - Ty_n) + q(x_n - Ty_n),\\ &q(Sx_n - Ty_n) + q(x_n - Ty_n)\}\\ &= k[q(x_n - Ty_n) + q(Sx_n - Ty_n)].\end{aligned}$$

Hence in any case,

$$q(Sx_n - Ty_n) \leq \max\{\frac{2k}{1-k}, \frac{2(1-c)}{3-c}, \frac{k}{1-k}\}q(x_n - Ty_n).$$

Letting $n \to \infty$, we obtain $q(Sx_n - Ty_n) \to o$. Then further $q(x_n - Sx_n)$ and $q(u - Sx_n)$ tend to zero as $n \to \infty$. Next if x_n, u satisfy (I), then

$$q(Sx_n - Tu) \leq k[q(x_n - u) + q(x_n - Sx_n) + q(Sx_n - Tu) + q(u - Sx_n].$$

If x_n, u satisfy (II),then

$$(2-c)q(Sx_n - Tu) \leq (c-1)[q(x_n - Sx_n) + q(u - Sx_n)].$$

If x_n, u satisfy (III), then obviously satisfy (I) as well. Hence, we have

$$q(Sx_n - Tu) \leq \max\{\frac{k}{1-k}, \frac{(1-c)}{2-c}\}[q(x_n - u) + q(x_n - Sx_n) + q(u - Sx_n)].$$

Letting $n \to \infty$, we have $q(Sx_n - Tu) \to 0$. Thus it follows from (6) that $Tu = u$. In view our remark at the beginning of the proof u is a fixed point of S as well. In order to show the uniqueness of u in the case (III), let $v(v \neq u)$ be another common fixed point of S and T, then using (III) we have

$$q(u - v) = q(Su - Tv) \leq \max\{q(u - v), 0, 0, q(u - v)\},$$

whence $u = v$.

Finally we give some examples of a metric linear space and two mappings which satisfy the contactive conditions of Theorem 2.1.

Example 2.1. Let $X = R$, the set of real numbers and q be the F -norm defined by

$$q(x) = \frac{|x|}{1+|x|}$$

Let $C = [0, 1]$ and $S, T : C \to C$ be defined by:

$$Sx = \begin{cases} 0, & 0 \leq x < 1 \\ \frac{1}{4}, & x = 1. \end{cases}, \qquad Tx = \begin{cases} 0, & 0 \leq x < 1 \\ \frac{1}{2}, & x = 1. \end{cases}.$$

Then S, T satisfy condition (III) of Theorem 2.1 with $k = \frac{2}{3}$ as follows
(i) If $x = y = 1$, then

$$q(Sx - Ty) = \frac{1}{5},$$
$$\frac{2}{3}q(y - Sx) = \frac{2}{3}q(\frac{3}{4}) = \frac{2}{7} > \frac{1}{5},$$
$$\frac{2}{3}q(x - Ty) = \frac{2}{3}q(\frac{1}{2}) = \frac{2}{9} > \frac{1}{5},$$

(ii)If $0 \leq x, y \leq 1$, then (III) is trivially satisfied.
(iii)If $x = 1$, and $0 \leq y < 1$, then

$$q(Sx - Ty) = q(\frac{1}{4}) = \frac{1}{5}$$
$$\frac{2}{3}q(x - Ty) = \frac{2}{3}q(1) = \frac{1}{3} > \frac{1}{5}.$$
$$\frac{2}{3}q(x - Sx) = \frac{2}{3}q(\frac{3}{4}) = \frac{2}{7} > \frac{1}{5}.$$

(iv) If $0 \leq x \leq 1, y = 1$, then (III) (Similarly as in (iii)).
Note that 0 is the unique common fixed point of S and T.

Example 2.2. Let X, q, C be as in example 2.1 and let $S, T : C \to C$ be defined by:

$$Sx = \begin{cases} 1 - x, & 0 \leq x < 1 \\ 0, & x = 1. \end{cases}, \qquad Tx = \begin{cases} \frac{1}{2}, & 0 \leq x < 1 \\ 0, & x = 1. \end{cases}.$$

Then S, T satisfy condition (II) of Theorem 2.2 with $c = \frac{3}{2}$ as follows If $x = 1$ and $0 \leq y < 1$, then

$$\begin{aligned} L &= q(Sx - Ty) + q(x - Sx) + q(y - Ty) \\ &= q(\frac{1}{2}) + q(1) + q(y - \frac{1}{2}) = q(1) + q(y) = \frac{1}{2} + q(y) \\ R &= q(x - Ty) + q(y - Sx) = q(\frac{1}{2}) + q(y) = \frac{1}{3} + q(y), \end{aligned}$$

So that

$$\frac{L}{R} = \frac{\frac{1}{2} + q(y)}{\frac{1}{3} + q(y)} \leq \frac{3}{2}.$$

In this case $x = \frac{1}{2}$ is a common fixed point of S and T.

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ON THE CONVERGENCE OF PERTURBED NEWTON-LIKE METHODS IN BANACH SPACE AND APPLICATIONS

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Abstract. In this study we use disturbed Newton-like methods to approximate a locally unique solution of a nonlinear equation containing a nondifferentiable term. We choose the linear operators to be inverted at every step of the computation of the Newton iterates to be associated not only with the differentiable part of the equation as it has happened so far but also with the nondifferentiable part. This way the upper error bounds on the distances are smaller than before.

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Key Words and Phrases. Disturbed Newton-like methods, Banach space, Mysovskii theorem, divided difference of order one, Fréchet-derivative.

I. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) + Q(x) = 0, \tag{1}$$

where the operators F, Q are defined on an open, convex subset D of a Banach space E_1 with values in a Banach space E_2, and are continuous on D.

We introduce disturbed Newton-like methods of the form

$$x_{n+1} = x_n - A_n^{-1}(F(x_n) + Q(x_n)) - z_n \quad (n \geq 0), \tag{2}$$

where

$$A_n = A(x_{n-1}, x_n) = L_1(x_{n-1}, x_n) + L_2(x_{n-1}, x_n) + L_3(x_{n-1}, x_n) \quad (n \geq 0)$$

for $x_{-1}, x_0 \in D$ and $A, L_1, L_2, L_3 : D \times D \to L(E_1, E_2)$.

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The points $z_n \in E_1$ $(n \geq 0)$ are determined in such a way that the iteration $\{x_n\}$ $(n \geq 0)$ converges to a solution x^* of equation (1). The importance of studying disturbed Newton-like methods comes from the fact that many commonly used variants of Newton's method can be considered procedures of this type. Indeed, approximation (2) characterizes any iterative process in which the corrections are taken as approximate solutions of the Newton equations. Note that if for example an equation on the real line is solved, $F(x_n) + Q(x_n) \geq 0$, and A_n overestimates the derivative, then $x_n - A_n^{-1}(F(x_n) + Q(x_n))$ $(n \geq 0)$ is always larger than the corresponding Newton iterate. In such cases, a positive z_n $(n \geq 0)$ correction term is appropriate. We assume that $L_1(x, y)$ is used to approximate the difference $F(z) - F(y)$ for $x, y, z \in D$; $L_2(x, y)$ is used to approximate the difference $Q(z) - Q(y)$ for all $x, y, z \in D$ and $L_3(x_{n-1}, x_n)$ is a linear operator depending on F, Q, x_n $(n \geq -1)$ and z_n $(n \geq 0)$. Special choices of operators A_n $(n \geq 0)$ and points z_n $(n \geq 0)$ lead to iterations that have already been examined by many authors. Indeed: (a) for $Q(x) = 0$ $(x \in D)$, $L_1 = F'(\cdot)$, $L_2 = L_3 = 0$ (linear operator), and $z_n = 0$ $(n \geq 0)$ iteration (2) reduces to Newton's method [3], [5], [10]; (b) for $Q(x) = 0$ $(x \in D)$, $L_1(x, y) = [x, y; F]$ (divided difference of order one of F on D at the points $x, y \in D$), $L_2 = L_3 = 0$, and $z_n = 0$ $(n \geq 0)$ iteration (2) reduces to the secant method [3], [5], [6]; (c) for $Q(x) = 0$ $(x \in D)$, $L_1(x, y) = [g_1(x), g_2(y); F]$, $E_1 = E_2 = E$, $g_1, g_2 : D \to E$, $L_2 = L_3 = 0$ and z_n $(n \geq 0)$ we get the Steffensen–Aitken method [3], [8], [9]; (d) for $L_2 = L_3 = 0$, $L_1 = F'(\cdot)$, and $z_n = 0$ $(n \geq 0)$ we obtain the Zincenko iteration [3], [9], [10]; (e) for $E_1 = E_2 = E$, $L_1(x_{n-1}, x_n) = F'(x_n)$ $(n \geq 0)$, $L_2(x_{n-1}, x_n) = [x_{n-1}, x_n; Q]$ $(n \geq 0)$, $L_3 = 0$ and $z_n = 0$ $(n \geq 0)$ we obtain the iteration introduced in [4]; (f) for $L_2 = L_3 = 0$, $L_1'(x, y) = A(x)$, (A approximating $F'(\cdot)$) and $z_n = 0$ $(n \geq 0)$ we get iterations considered in [1], [2], [3], [9].

We show under more general and weaker conditions that our iteration (2) is faster than the ones mentioned above.

II. CONVERGENCE ANALYSIS

Let $x_{-1}, x_0 \in D$, $z_n \in E_1$ $(n \geq 0)$, and $a_1, a_2, a_3 \geq 0$, $r \in [0, R]$, $R \geq 0$, $U(x_0, R) = \{x \in E_1 \mid \|x - x_0\| \leq R\}$ with $U(x_0, R) \subseteq D$, $C_1, C_2, C_3 : [0, +\infty) \to [0, +\infty)$ be continuous nondecreasing functions that vanish at the origin, and $A, L_1, L_2, L_3 : D \times D \to L(E_1, E_2)$ such that

$$(3) \quad \begin{aligned} &\|A(y, z)^{-1}[F(z) - F(y) - L_1(x, y)(z - y)]\| \\ &\quad \leq [C_1(\|x - x_0\|, \|y - x_0\|, \|z - x_0\|, \|x - y\|, \|z - y\|) + a_1]\|z - y\|, \end{aligned}$$

$$(4) \quad \begin{aligned} &\|A(y, z)^{-1}[Q(z) - Q(y) - L_2(x, y)(z - y)]\| \\ &\quad \leq [C_2(\|x - x_0\|, \|y - x_0\|, \|z - x_0\|, \|x - y\|, \|z - y\|) + a_2]\|z - y\|, \end{aligned}$$

and

$$(5) \quad \begin{aligned} &\|A(x_{n+1}, x_n)^{-1}[A_{n+1}(z_{n+1}) - A_n(z_n) - L_3(x_{n-1}, x_n)(x_{n+1} - x_n)]\| \\ &\quad \leq [C_3(\|x_{n-1} - x_0\|, \|x_n - x_0\|, \|x_{n+1} - x_0\|, \\ &\qquad \|x_{n-1} - x_n\|, \|x_{n+1} - x_n\|) + a_3]\|x_{n+1} - x_n\| \quad (n \geq 0) \end{aligned}$$

for all $x, y, z \in U(x_0, r)$ where x_n $(n \geq -1)$ are given by (2).

Let a_4, a_5 be constants such that

$$\|x_0 - x_{-1}\| \leq a_4 \quad \text{and} \quad \|A_0^{-1}(F(x_0) + Q(x_0)) - z_0\| \leq a_5. \tag{6}$$

It is convenient to introduce the iterations

$$\begin{aligned}\bar{h}_{n+1} = C(&\|x_{n-1} - x_0\|, \|x_n - x_0\|, \|x_{n+1} - x_0\|,\\ &\|x_{n-1} - x_n\|, \|x_{n+1} - x_n\|)\|x_{n+1} - x_n\| \ (n \geq 0)\end{aligned}$$

where

$$C(s_1, s_2, s_3, s_4, s_5) = (C_1 + C_2 + C_3)(s_1, s_2, s_3, s_4, s_5) + a_1 + a_2 + a_3$$

for all $s_1, s_2, s_3, s_4, s_5 \in [0, +\infty)$,

$$t_{n+1} = t_{n+1} + h_{n+1}, \ t_{-1} = 0 \quad (n \geq 0), \tag{7}$$

where

$$h_{n+1} = C(|t_{n-1} - t_0|, t_n - t_0, t_{n+1} - t_0, t_n - t_{n-1}, t_{n+1} - t_n)(t_{n+1} - t_n) \quad (n \geq 0),$$

and the functions $T, C_4 : [0, +\infty) \to [0, +\infty)$ by

$$T(r) = t_1 + C(r, r, r, r, r)(r - t_1) + C(t_0, 0, t_1 - t_0, t_0, t_1 - t_0)(t_1 - t_0) \tag{8}$$

and

$$C_4(s_1, s_2, s_3, s_4, s_5) = (C_1 + C_2)(s_1, s_2, s_3, s_4, s_5) + a_1 + a_2. \tag{9}$$

We can now state and prove the following Newton–Mysovskii [6] semilocal convergence theorem for the disturbed Newton like method generated by (2).

Theorem. *Assume:*

(a) *operators F, Q are defined on some open convex subset D of a Banach space E_1 with values in E_2 and continuous on D;*
(b) *there exist r, R with $r \in [0, R]$ such that $A(x, y) \in L(E_1, E_2)$ for all $x, y \in U(x_0, r)$ is boundedly invertible, $U(x_0, R) \subseteq D$ and conditions* (3)–(5) *are satisfied;*
(c) *sequence $\{z_n\}$ $(n \geq 0) \in E_1$ is null;*
(d) *there exist numbers t_0 and t_1 such that*

$$a_5 \leq t_1 - t_0 \leq \|x_0 - x_{-1}\| \leq a_4 \leq t_0 - t_{-1}; \tag{10}$$

(e) *there exists a minimum nonnegative number r^* satisfying*

$$r^* \leq R \tag{11}$$

and

$$T(r^*) \leq r^* \tag{12}$$

where function T is given by (8);
(f) *moreover r^*, R satisfy*

$$C_4(r^*, r^*, R, 2r^*, r^* + R) < 1 \tag{13}$$

where function C_4 is given by (9).

Then

(i) *Scalar iteration* $\{t_n\}$ $(n \geq -1)$ *generated by* (7) *is monotonically increasing, bounded above by* r^* *and* $\lim_{n\to\infty} t_n = r^*$;

(ii) *disturbed Newton-like method* $\{x_n\}$ $(n \geq -1)$ *generated by* (2) *is well defined, remains in* $U(x_0, r^*)$ *for all* $n \geq -1$, *and converges to a unique solution* x^* *of equation* $F(x) + Q(x) = 0$ *in* $U(x_0, R)$.

Moreover the following error estimates are true for all $n \geq 0$

$$\|x_{n+1} - x_n\| \leq \bar{h}_{n+1} \leq h_{n+1} = t_{n+1} - t_n \tag{14}$$

and

$$\|x_n - x^*\| \leq r^* - t_n. \tag{15}$$

Proof. (i) It follows from (7), (10) and (12) that $t_{-1} \leq t_0 \leq t_1 \leq r^*$. Let us assume that $t_k \leq t_{k+1} \leq r^*$ for $k = -1, 0, 1, 2, \dots, n$. The C function is nonnegative and nondecreasing, hence $h_{k+1} \geq 0$. That is $t_{k+1} \leq t_{k+2}$ (by (7)). Moreover from (7), (12) and the induction hypothesis we obtain in turn

$$\begin{aligned} t_{k+2} &= t_{k+1} + h_{k+1} \leq t_{k+1} + C(r^*, r^*, r^*, r^*, r^*)(t_{k+1} - t_k) \\ &\leq \cdots \leq t_1 + C(r^*, r^*, r^*, r^*, r^*)(t_{k+1} - t_1) \\ &\qquad + C(t_0, 0, t_1 - t_0, t_0, t_1 - t_0)(t_1 - t_0) \\ &= T(r^*) \leq r^*. \end{aligned}$$

Hence, iteration $\{t_n\}$ $(n \geq -1)$ is also bounded above by r^*. Since r^* is the minimum nonnegative number on $[0, r^*]$ satisfying (12) it follows that $\lim_{n\to\infty} t_n = r^*$.

(ii) It follows from (10), (12) that $x_{-1}, x_1 \in U(x_0, r^*)$, and that (14) is satisfied for $n = -1, 0$. Assume that $x_k \in U(x_0, r^*)$ and that (14) is satisfied for $k = -1, 0, \dots, n$. Using (14), the triangle inequality and the induction hypothesis we obtain

$$\|x_{k+1} - x_0\| \leq \sum_{i=0}^{k} \|x_{k+1} - x_i\| \leq \sum_{i=0}^{k} (t_{i+1} - t_i) = t_{k+1} - t_0 \leq r^*.$$

That is $x_{k+1} \in U(x_0, r^*)$. We will show that (14) is true for $k = n + 1$. Using (2)–(5), (7), the induction hypothesis we can obtain the approximation

$$\begin{aligned} F(x_{k+1}) + Q(x_{k+1}) + A_{k+1}(z_{k+1}) &= F(x_{k+1}) + Q(x_{k+1}) \\ &\quad + A_{k+1}(z_{k+1}) - F(x_k) - Q(x_k) - A_k(z_k) - A_k(x_{k+1} - x_k) \\ &= [F(x_{k+1}) - F(x_k) - L_1(x_{k-1}, x_k)(x_{k+1} - x_k)] \\ &\quad + [Q(x_{k+1}) - Q(x_k) - L_2(x_{k-1}, x_k)(x_{k+1} - x_k)] \\ &\quad + [A_{k+1}(z_{k+1}) - A_k(z_k) - L_3(x_{k-1}, x_k)(x_{k+1} - x_k)], \end{aligned} \tag{16}$$

and

(17)
$$
\begin{aligned}
\|x_{k+2}-x_{k+1}\| &= \|A_{k+1}^{-1}[F(x_{k+1})+Q(x_{k+1})-A_{k+1}(z_{k+1})]\|\\
&\le [C_1(\|x_{k-1}-x_0\|,\|x_k-x_0\|,\|x_{k+1}-x_0\|,\\
&\qquad \|x_{k-1}-x_k\|,\|x_{k+1}-x_k\|)+a_1]\|x_{k+1}-x_k\|\\
&\quad + [C_2(\|x_{k-1}-x_0\|,\|x_k-x_0\|,\|x_{k+1}-x_0\|,\|x_{k-1}-x_k\|,\\
&\qquad \|x_{k+1}-x_k\|)+a_2]\|x_{k+1}-x_k\|\\
&\quad + [C_3(\|x_{k-1}-x_0\|,\|x_k-x_0\|,\|x_{k+1}-x_0\|,\|x_{k-1}-x_k\|,\\
&\qquad \|x_{k+1}-x_k\|)+a_3]\|x_{k+1}-x_k\|\\
&= C(\|x_{k-1}-x_0\|,\|x_k-x_0\|,\|x_{k+1}-x_0\|,\|x_{k-1}-x_k\|,\\
&\qquad \|x_{k+1}-x_k\|)\|x_{k+1}-x_k\| = \bar{h}_{k+1}\\
&\le C(|t_{k-1}-t_0|,t_k-t_0,t_{k+1}-t_0,t_k-t_{k-1},t_{k+1}-t_k)(t_{k+1}-t_k)\\
&= h_{k+1} = t_{k+2}-t_{k+1},
\end{aligned}
$$

which shows (14) for all $n \ge 0$.

It now follows from part (i) and (14) that the inexact Newton-like iteration $\{x_n\}$ $(n \ge -1)$ is Cauchy in a Banach space E_1 and as such it converges to some $x^* \in U(x_0, r^*)$ (since $U(x_0, r^*)$ is a closed set). By letting $n \to \infty$ in (16) and using hypothesis (c) we deduce that $F(x^*) + Q(x^*) = 0$. That is x^* is a solution of equation (1). Estimate (15) follows immediately from (14) by using standard majorization techniques [3], [5].

To show uniqueness, we reason as follows: under the hypotheses of the theorem iteration (2) for $z_n = 0$ $(n \ge 0)$ becomes

(18)
$$v_{n+1} = v_n - A_n^{-1}(F(v_n)+Q(v_n)),\ v_{-1} = x_{-1}, v_0 = x_0 \quad (n \ge 0)$$

and converges to some $x_1^* \in U(x_0, r^*)$. Let us assume that there exists $y^* \in U(x_0, R)$ such that $F(y^*) + Q(y^*) = 0$. Using (18), (3), (4) and the approximation

$$
\begin{aligned}
v_{n+1}-y^* &= v_n - A_n^{-1}(F(v_n)+Q(v_n)) - y^*\\
&= A_n^{-1}[F(v_n)-F(y^*)-L_1(v_{n-1},v_n)(v_n-y^*)]\\
&\quad + A_n^{-1}[Q(v_n)-Q(y^*)-L_2(v_{n-1},v_n)(v_n-y^*)]
\end{aligned}
$$

we obtain

(19)
$$
\begin{aligned}
\|v_{n+1}-y^*\| &\le [C_1(\|v_{n-1}-v_0\|,\|v_n-v_0\|,\|y^*-v_0\|,\|v_{n-1}-v_n\|,\\
&\qquad \|v_n-y^*\|)+a_1]\|v_n-y^*\|\\
&\quad + [C_2(\|v_{n-1}-v_0\|,\|v_n-v_0\|,\|y^*-v_0\|,\|v_{n-1}-v_n\|,\\
&\qquad \|v_n-y^*\|)+a_2]\|v_n-y^*\|\\
&\le C_4(\|v_{n-1}-v_0\|,\|v_n-v_0\|,\|y^*-v_0\|,\|v_{n-1}-v_n\|,\\
&\qquad \|v_n-y^*\|)\|v_n-y^*\|.
\end{aligned}
$$

By (13) there exist $r^*, q \in [0,1]$ such that

$$C(r^*, r^*, R, 2r^*, r^* + R) \le q < 1. \tag{20}$$

It follows from (19) and (20) that

$$\|v_{n+1} - y^*\| \le q^{n+1}\|v_0 - y^*\| \le q^{n+1}R \to 0 \quad \text{as } n \to \infty.$$

That is, $y^* = \lim_{n\to\infty} v_{n+1} = x_1^*$. Hence we deduce $x^* = x_1^* = y^*$.

The proof of the Theorem is now complete.

Remarks. (1) Hypotheses (3)–(5) can be replaced by weaker $(3)'$–$(5)'$ if $A(y,z)^{-1}$ is replaced by A_0^{-1} for all $y, z \in U(x_0, r)$. In this case we need to add a hypothesis of the form

$$\|A_0^{-1}(A(x,y) - A_0)\| \le C_5(\|x - x_{-1}\|, \|y - x_0\|) + a_6, \tag{21}$$

with $C_5 : [0,+\infty)^2 \to [0,+\infty)$ be a continuous nondecreasing function vanishing at the origin and $a_6 \ge 0$. Parameter r^* must satisfy the additional hypothesis

$$C_5(r^*, r^*) + a_6 < 1.$$

It then follows from the Banach Lemma on invertible operators [5] that

$$\|A_{n+1}^{-1}A_0\| \le \bar{b}_{n+1} \le b_{n+1} \quad (n \ge 0)$$

where

$$\bar{b}_{n+1} = \{1 - [a_6 + C_5(\|x_n - x_{-1}\|, \|x_{n+1} - x_0\|)]\}^{-1}$$

and

$$b_{n+1} = \{1 - [a_6 + C_5(t_n, t_{n+1} - t_0)]\}^{-1}.$$

The following modifications must also take place: iterations $\bar{h}_{n+1}$, h_{n+1} must be multiplied by $\bar{b}_{n+1}$, b_{n+1} $(n \ge 0)$ respectively, and functions C, C_4 must be multiplied by

$$b(r) = \{1 - [a_6 + C_5(r,r)]\}^{-1}.$$

It can then be easily seen that with the above modifications the conclusions of the Theorem under the above-mentioned weaker assumptions will also hold.

(2) Several choices for the "C" functions appearing in conditions (3)–(5) can be suggested. To compare with conditions suggested by Yamamoto and Chen in [9] under the special choice (f) stated at the introduction of this study, we choose

(22)

$$C_1(s_1,s_2,s_3,s_4,s_5) = \int_0^1 [w_1(s_1 + ts_4) - w_2(s_1) + w_3(s_2 + ts_5) - w_4(s_2) + a_1]s_5\,dt,$$

$$C_2(s_1,s_2,s_3,s_4,s_5) = [w_5(s_1 + s_4) - w_6(s_1) + w_7(s_2 + s_5) - w_8(s_2) + a_2]s_5 \tag{23}$$

and

$$(24)\quad C_3(s_1,s_2,s_3,s_4,s_5)=[w_9(s_1+s_4)-w_{10}(s_1)+w_{11}(s_2+s_5)-w_{12}(s_2)+a_3]s_5$$

where $w_m:[0,+\infty)\to[0,+\infty)$, $m=1,2,\dots,12$ are continuous functions vanishing at the origin and satisfying

$$0\le w_i(d_1+d_2)-w_{i+1}(d_1)\le w_i(d_3+d_4)-w_{i+1}(d_3),\, i=1,3,5,7,9,11$$

for all $r_1\le r_3$ and $r_2\le r_4$. Yamamoto–Chen conditions (4), (5) and (6) in [9] are given respectively by (following their notation)

$$\|A(x_0)^{-1}(A(x)-A(x_0))\|\le w_0(\|x-x_0\|)+b$$

$$\|A(x_0)^{-1}(F'(x+t(y-x))-A(x))\|\le w(\|x-x_0\|+(t\|y-x\|)-w_0(\|x-x_0\|)+c$$

and

$$\|A(x_0)^{-1}(Q(x)-Q(y))\|\le e(r)\|x-y\|.$$

These conditions correspond to our conditions (21), (3) and (4) respectively. It can easily be seen from (22)–(24) that for special choices of the w_i functions our conditions (21) and (4) reduce to Yamamoto–Chen (4) and (6), whereas our condition (4) is weaker than Yamamoto–Chen (5). Moreover since the left-hand side of Yamamoto–Chen condition (5) (times $(y-x)$ inside the norm) is larger than the left-hand side of our condition (3), our upper bounds on the distances $\|x_{n+1}-x_n\|$ and $\|x_n-x^*\|$ $(n\ge 0)$ are smaller than Yamamoto–Chen upper bounds.

(3) Favorable comparisons can also be made with the rest of the cases (a)–(e) mentioned at the introductions of this study. The details are left to the motivated reader. Numerical examples for the usefulness of iteration (2) can also be found in [1]–[4].

(4) Define the residuals $r_n=-A_n(z_n)$ and set $\Delta_n=x_{n+1}-x_n$ $(n\ge 0)$. Since the sequence $\{z_n\}$ $(n\ge 0)$ is null and the linear operators $\{A_n\}$ $(n\ge 0)$ are bounded it follows that the sequence $\{r_n\}$ $(n\ge 0)$ is null also. Consequently the conclusions of the Theorem hold for the system

$$x_{n+1}-x_n=\Delta_n,$$

and

$$A_n\Delta_n=-(F(x_n)+Q(x_n))+r_n\quad (n\ge 0)\ \ [1],[3].$$

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SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF ITERATIONS TO POINTS OF ATTRACTION IN BANACH SPACE

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Abstract. In this study we provide weak sufficient conditions for the convergence of iterations to points of attraction in Banach space. Our results are used to show existence and uniqueness of solutions of nonlinear equations. We complete our study with an example that shows that our convergence results apply where earlier ones do not.

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I. INTRODUCTION

In this study, we are concerned with approximating a locally unique solution x^* of the nonlinear equation

$$F(x, h) = 0 \quad \text{for all} \quad h \in E_0, \tag{1}$$

where $F\colon D \times E_0 \subseteq E^{m+1} \to E$ is a continuous operator, E is a Banach space, D, E_0 are subsets of E and E^m respectively and $m \in N \cup \{0\}$.

We introduce the iteration

$$x_{n+1} = G(x_n, h_n) \quad (n \geq 0), \tag{2}$$

to approximate the solution x^* of equation (1), where

$$G(x, h) = x - F(x, h) \quad \text{for all} \quad (x, h) \in D \times E_0. \tag{3}$$

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For $m = 0$ equation (1) does not depend on h, setting $F(x,h) = P'(x)^{-1}P(x)$ for all $x \in D$, and using (2) and (3), we obtain Newton's method

$$x_{n+1} = x_n - P'(x_n)^{-1}P(x_n) \quad (n \geq 0), \tag{4}$$

where $P: D \to E$, $P'(x_n) \in L(E,E)$ denotes the Fréchet-derivative of P evaluated at $x = x_n$ $(n \geq 0)$ [1], [3], [4].

For $m = 1$ set $F(x,h) = [h,x;Q]^{-1}Q(x)$ for all $(x,h) \in D \times E_0$, where $Q: D \to E$, $[h,x;Q]$ is a divided difference of order one on E satisfying

$$[h,x;Q](x-h) = Q(x) - Q(h) \quad \text{for all} \quad (x,h) \in D \times E_0. \tag{5}$$

Then for $h_n = x_{n-1}$ $(n \geq 0)$ we obtain the secant method

$$x_{n+1} = x_n - [x_{n-1}, x_n; Q]^{-1}Q(x_n) \quad (n \geq 0), \tag{6}$$

[1], [3], [4]. Many other choices for the operator F are also possible. It follows immediately from (1) and (3) that any solution x^* of equation (1) is a fixed point of equation (3) and vice versa. If for some $x_0 \in D$ and any $h_n \in E^m$ iteration $\{x_n\}$ $(n \geq 0)$ is well defined and converges to some point $y^* \in D$ then y^* is called a point of attraction for this iteration. In this case by taking the limit as $n \to \infty$ in (2) we deduce

$$y^* = G(y^*, h) \quad \text{for all} \quad h \in E_0. \tag{7}$$

That is y^* is a fixed point of the operator $G(\cdot, h)$ for all $h \in E_0$.

The study of equations of the form (1) is very important since many real life problems can be reduced to solving such equations [1], [3], [4], via iterations of the form (2).

Sufficient conditions for the convergence of (2) for $m = 0$ (or not) have been given by various authors (see, e.g., [1]–[4] and the references there). Even under existing conditions one can construct examples where, say Newton's method (4) converges despite the fact that the conditions of existing theorems are violated. See Remark 2 for such an example. This important observation creates the need of the development of weaker sufficient conditions for the convergence of (2) than the ones already in the literature. This is what we have done in this study.

Finally we conclude our study with an example to show that our convergence results apply where previous ones fail.

II. CONVERGENCE ANALYSIS

Definition. Let $(E, \| \ \|)$ be a metric space. An operator $G: E^{m+1} \to E$ is called a generalized contractive operator if there exist $z \in E$ and a number $c \in [0,1)$ such that

$$\|G(x,h) - G(z,h)\| \leq c\|x - z\| \quad \text{for all} \quad x \in E, \ h \in E_0. \tag{8}$$

If z is such that $z = G(z,h)$ for all $h \in E$ and G satisfies (8) for $c = 1$, then G is called a generalized quasinonexpansive operator. We note that if $P: E \to E$ is such that $P(x) = G(x,h)$ for all $x \in E$, $h \in E_0$ then P is called a quasicontractive operator if (8) is satisfied [1], [3], [4].

Theorem 1. *Let $(E, \|\ \|)$ be a Banach space and $G: E^{m+1} \to E$ be a generalized quasicontractive operator with x^* such that $G(x^*, h) = x^*$ for all $h \in E_0$.*

Then x^* is the unique point in E such that $G(x^*, h) = x^*$ for all $h \in E_0$ and the iteration $\{x_n\}$ $(n \geq 0)$ generated by (2) converges to x^* for each $x_0 \in E$ and any $h_0 \in E_0$ with

$$\|x_n - x^*\| \leq c^n \|x_0 - x^*\| \tag{9}$$

Moreover $x_n \in U(x^*, r) = \{x \in E \mid \|x - x^*\| \leq r\}$ $(n \geq 0)$ for $r \in [0, \|x_0 - x^*\|]$.

Proof. By hypothesie $G(x^*, h) = h$ for all $h \in E_0$, we obtain from (8) that

$$\|G(x, h) - x^*\| \leq c\|x - x^*\| \quad \text{for all} \quad x \in E,\ h \in E_0. \tag{10}$$

Let $x_0 \in E$, $h_0 \in E_0$ be arbitrary, then by (8), we get

$$\|x_{n+1} - x^*\| = \|G(x_n, h_n) - x^*\| \leq c\|x_n - x^*\| \leq \cdots \leq c^{n+1}\|x_0 - x^*\|. \tag{11}$$

But $c \in [0, 1)$ hence (11) gives $\lim_{n\to\infty} x_{n+1} = x^*$, and (9) holds for all $n \geq 0$. It also follows from (11) that $x_n \in U(x^*, r)$ for $r \in [0, \|x_0 - x^*\|]$. To show uniqueness let us assume that there exists z with $x \neq x^*$ and $G(z, h) = z$ for all $h \in E_0$. Then $\|x^* - z\| \neq 0$ and (8) gives $\|x^* - z\| = \|G(x^*, h) - G(z, h)\| \leq c\|x^* - z\|$ contradicting $c \in [0, 1)$.

That completes the proof of the theorem.

Remark 1. If E is a compact metric space then Theorem 1 remains true if we replace generalized quasicontractive by generalized quasinonexpansive. See also Edelstein's fixed point theorem [1], [3].

From now on we assume that E is a partially ordered Banach space [1], [3], [4], $D = [a, b]$, $a \leq b$ and $E_0 = E^m$.

Theorem 2. *Let $(E, \leq)$ be a partially ordered Banach space. Assume there exist:*

(a) *continuous operators $F_1: E_1 \times E_0 \to E$, $F: [a, b] \times E_0 \to E$, and $F_2: E_2 \times E_0 \to E$ with $a \leq b$ fixed in E satisfying for all $h \in E_0$*

$$x - F_1(x, h) \geq a \tag{12}$$

and

$$x - F_2(x, h) \leq b, \tag{13}$$

where

$$E_1 = \{x \in E \mid x \leq a\} \quad \textit{and} \quad E_2 = \{x \in E \mid b \leq x\};$$

(b) *a unique point $x^* \in [a, b]$ such that $F(x^*, h) = 0$ for all $h \in E_0$;*

(c) *an operator $T \in L(E^{m+2}, E)$ such that*

$$F(x, h) = T(x, x^*, h)(x - x^*) \quad \textit{for all} \quad x \in [a, b],\ h \in E_0; \tag{14}$$

and

(d) *$d \in [0, 1)$, where*

$$d = \sup_{\substack{x\in[a,b]\\ h\in E_0}} \|I - T(x, x^*, h)\|. \tag{15}$$

Then the iteration $\{x_n\}$ $(n \geq 0)$ *generated by* (2) *is well defined, remains in* $U(x^*, r_0)$ *for* $r_0 \in [0, \|x_0 - x^*\|]$ *and converges to* x^* *with* $G(x^*, h) = x^*$ *for all* $h \in E_0$ *and*

$$\|x_n - x^*\| \leq d^n \|x_0 - x^*\| \quad (n \geq 0). \tag{16}$$

Proof. We define operators $F, G\colon E \times E_0 \to E$ by

$$F(x, h) = \begin{cases} F_1(x, h) & x \leq a \\ F(x, h) & a \leq x \leq b \\ F_2(x, h) & b \leq x \end{cases} \tag{17}$$

That is, we use the same notation for the new F and

$$G(x, h) = x - F(x, h). \tag{18}$$

It follows immediately from hypothesis (b) and (18) that (7) is satisfied for $y^* = x^*$. Moreover from hypotheses (c), (d) and (14) we get

$$G(x, h) - x^* = x - x^* - F(x, h) = (I - T(x, x^*, h))(x - x^*),$$

and

$$\|G(x, h) - x^*\| \leq d\|x - x^*\| \tag{19}$$

for all $x \in [a, b]$ and $h \in E_0$. From the proof of Theorem 1 it follows that $\lim_{n\to\infty} x_n = x^*$ for each $x_0 \in [a, b]$ and all $h \in E_0$.

If some iterate $x_k \notin [a, b]$ then $x_k \in E - [a, b]$. It then follows from (12), (13) and (2) that

$$x_{k+1} = G(x_k, h_k) = x_k - F_1(x_k, h_k) \geq a \quad \text{for } x_k \leq a$$

or

$$x_{k+1} = G(x_k, h_k) = x_k - F_2(x_k, h_k) \leq b \quad \text{for } x_k \geq b.$$

Since iteration $\{x_n\}$ $(n \geq 0)$ converges to x^* and $a \leq x^* \leq b$ there exists $v \geq 0$, $v \in N$ such that

$$x_{n+1} = G(x_n, h_n) = G^{n+1}(x_0, h) \in [a, b] \quad \text{for } n \geq v.$$

That is $G^{v+1}([a, b] \times E_0) \subseteq [a, b] \times E_0$. Furthermore by (19) we have

$$\|G^n(x, h) - x^*\| \leq d^n \|x - x^*\| \quad \text{for all} \quad x \in [a, b],\ h \in E_0. \tag{20}$$

It follows that operator G^{v+1} satisfies all requirements of Theorem 1. Thus x^* is a fixed point of G^{v+1}, hence a fixed point of G and $\lim_{n\to\infty} x_n = x^*$ for each $x_0 \in [a, b]$. The rest of the conclusions of the theorem follow immediately from (20). We also note that by reordering iteration $\{x_n\}$ $(n \geq 0)$ we can take $v = 0$.

That completes the proof of the Theorem.

Remark 2. For Newton's method (4) (i.e., $m = 0$) operators F_1 and F_2 appearing in Theorem 2 can be defined by

$$F_1(x, h) = P'(a)(x - a) + P(a) \quad \text{for all} \quad x \leq a,\ h \in E_0,$$

and

$$F_2(x, h) = P'(b)(x - b) + P(b) \quad \text{for all} \quad b \leq x,\ h \in E_0,$$

assuming that P is Fréchet-differentiable on $U(x^*, r)$, $x^* \in [a, b]$, $r \geq 0$ fixed and $P(x^*) = 0$. Convergence theorems for Newton's method have been given by various authors using the in general expansive information $(x_0, P(x_0), P'(x_0), P''(x_0))$ or the more relaxed $(x_0, P(x_0), P'(x_0))$ [1]–[4]. We compare our Theorem 2 with corresponding results involving the latter information.

Let for simplicity $E = R$, then operator T can be defined by

$$T(x, x^*, h) = \frac{P'(x^* + t(x - x^*))}{P'(x)} \qquad t \in (0,1) \text{all } x \in [a, b].$$

Moreover define $P\colon [0, 1] \to R$ by

$$P(x) = x^2 + 2x.$$

Let $a = 0$ and $b = 1$. Note that $x^* = 0$ is the only zero of equation $P(x) = 0$, on $[0, 1]$. Ostrowski's theorem on fixed points requires that $q = \max_{x\in[0,1]} |P'(0)| \in [0, 1)$ [1], [4] which is not true since $q = 2$. Newton's theorem on fixed points [3, pp. 128] (uses information on the second derivative also) or more recently Berinde's theorem [2, thm. 5, pp. 20] cannot apply because their common condition $f(a)f(b) < 0$ is violated since $f(0)f(1) = 0$. However it can easily be seen that all hypotheses of Theorem 2 are satisfied and $d = 1$ (see also Remark 1). Hence the conclusions for the scalar equation $P(x) = 0$ and Newton's method (4) follow. Finally for $x_0 = .1$ (4) gives in turn

$$x_1 = .0090909, \quad x_2 = .0000819, \quad x_3 = .000000006 \quad \text{and} \quad x_4 = 0.$$

Finally we note that Theorem 2 can be reduced to Berinde's Theorem 5 since hypotheses (a)–(d) will be true if replaced by the stronger (f_1), (f_2') and (f_3') of Berinde's and $E = R^k$, $k \in N$.

Remark 3. For the secant method (6) operators F_1 and F_2 appearing in Theorem 2 can be given by

$$F_1(x, h) = [h, a; Q](x - a) + Q(a) \quad \text{for all} \quad a \leq x,\ h \in E_0$$

and

$$F_2(x, h) = [h, b; Q](x - b) + Q(b) \quad \text{for all} \quad b \leq x,\ h \in E_0.$$

Several other choices are also possible.

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WEAK CONDITIONS FOR THE CONVERGENCE OF ITERATIONS TO SOLUTIONS OF EQUATIONS ON PARTIALLY ORDERED TOPOLOGICAL SPACES

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Abstract. We provide sufficient conditions for the monotone convergence of Newton like methods to solutions of nonlinear operator equations on partially ordered topological spaces. We assume conditions weaker than standard Lipschitz-like conditions to produce two iterations, one increasing and one decreasing, that enclose and converge to a solution of an equation. Finally examples are given to show that our results apply where previous ones fail.

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Key Words and Phrases. Partially ordered topological space, monotone convergence, fixed point, Newton-like methods.

I. INTRODUCTION

In this study we are concerned with approximating a solution of the nonlinear operator equation

$$F(x) = 0, \tag{1}$$

where F is an operator defined on a subset of partially ordered linear space $E_1 = (E_1, \leq)$ (POL-space) with values in a POL-space E_2 [1], [2], [5]–[8].

We suggest Newton-like iterations of the form

$$F(x_n) + A^1(x_{n-1}, x_n)(x_{n+1} - x_n) = 0 \tag{2}$$

$$F(y_n) + A^1(y_n, y_{n-1})(y_{n+1} - y_n) = 0 \tag{3}$$

for $[x_{-1}, y_{-1}] \subseteq D$, $x_{-1} \leq x_0, y_0 \leq y_{-1}$ and $A^1, A^2 \colon D^2 \to L(E_1, E_2)$ [2], [6]. Sufficient conditions for the monotone convergence of the sequences $\{x_n\}$, $\{y_n\}$

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($n \geq 1$) generated by (2) and (3) respectively to a solution of equation (1) have been given by various authors based mainly on the condition

$$F(y) - F(x) \leq A(w,z)(y-x) \quad \text{for } x \leq y \leq w \leq z, \tag{4}$$

$[x,z] \subseteq D$ and $A: D \to L(E_1, E_2)$ being linear operator having continuous nonsingular nonnegative left subinverse (see [2], [5], [6] and the references there). There are examples even in the scalar case (see the example after Theorem 1) where iterations $\{x_n\}$, $\{y_n\}$ converge to a solution of equation (1) but the sufficient conditions for the convergence fail to apply. That is why it is important to try to find weaker conditions under which these iterations converge to a solution. This is what we have done in this study.

Finally we note that for $A^1(x,y) = A^2(x,y) = F'(x)$ for all $x, y \in D$, where $F'(x)$ is the Fréchet-derivative of F we obtain Newton's method [3], [4], whereas for $A^1(x,y) = A^2(x,y) = [x,y;F]$ for all $x,y \in D$, where $[x,y;F]$ is the divided difference of order one of F at the points $x, y \in D$ we obtain the secant method [3], [4]. Several other choices for the operators A^1 and A^2 are also possible [2].

Theorem 1. *Consider an operator $F: D \subseteq E_1 \to E_2$ where E_1 is a regular POTL-space, E_2 is a POTL-space and D is an open convex subset of E_1. Let x_{-1}, x_0, y_{-1}, y_0 be points of D such that*

$$x_{-1} \leq x_0 \leq y_0 \leq y_{-1}, \quad [x_{-1}, y_{-1}] \subseteq D. \tag{5}$$

Assume:

(a) *there exists points $z_1, z_2 \in [x_0, y_0]$ such that*

$$z_1 \leq z_2 \quad \text{and} \quad F(z_1) = F(z_2) = 0.$$

(b) *$F(x) \leq 0$ for all $x \in [x_0, z_1]$ and $F(x) \geq 0$ for all $x \in [z_2, y_0]$.*

(c) *there exist operators $A^1, A^2: D^2 \to L(E_1, E_2)$ such that*

$$F(z_1) - F(x) \leq A^1(y,x)(z_1 - x) \quad \text{for all } y, x \text{ with } x_0 \leq y \leq x \leq z_1 \tag{6}$$

and

$$F(v) - F(z_2) \leq A^2(v,w)(v - z_2) \quad \text{for all } v, w \text{ with } z_2 \leq v \leq w \leq y_0. \tag{7}$$

(d) *The linear operators $A^1(y,x)$ and $A^2(v,w)$ have continuous nonsingular nonnegative left subinverses on $[x_{-1},\ y_{-1}]$.*

Then there exist two sequences $\{x_n\}$, $\{y_n\}$ generated by (2) and (3) respectively ($n \geq 1$) and two points x^*, y^* of E_1 such that for all $n \geq 0$

$$F(x_n) \leq 0 \leq F(y_n), \tag{8}$$

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq z_1 \leq z_2 \leq y_{n+1} \leq y_n \leq \cdots \leq y_1 \leq y_0, \tag{9}$$

$$\lim_{n\to\infty} x_n = x^*, \quad \lim_{n\to\infty} y_n = y^*, \quad x^* \le z_1 \le z_2 \le y^*, \tag{10}$$

and

$$x^* = y^* \Rightarrow x^* = z_1 = z_2 = y^*. \tag{11}$$

Moreover assume there exist operators $A^3, A^4 \colon D^2 \to L(E_1, E_2)$ *with nonnegative left superinverses satisfying*

$$F(z_1) - F(x) \ge A^3(x,y)(z_1 - x) \quad \textit{for all } x_0 \le x \le y \le z_1, \tag{12}$$

$$F(v) - F(z_2) \ge A^4(w,v)(v - z_2) \quad \textit{for all } z_2 \le w \le v \le y_0, \tag{13}$$

$F(x^*) = F(z_1)$ *and* $F(y^*) = F(z_2)$.

Then we deduce

$$x^* = z_1 \quad \textit{and} \quad y^* = z_2.$$

Furthermore if $z_1 = z_2$ *we get* $x^* = y^* = z_1 = z_2$.

Proof. Let B_0^1 be a continuous nonsingular left subinverse of $A_0^1 = A^1(x_{-1},\ x_0)$ and consider the operator $P_1 \colon [0, z_1 - x_0] \to E_1$ defined by

$$P_1(x) = x - B_0^1(F(x_0) + A_0^1(x)).$$

It is easy to see that P_1 is isotone and continuous. We also have

$$\begin{aligned} P_1(0) &= -B_0^1(F(x_0)) \ge 0 \\ P_1(z_1 - x_0) &= z_1 - x_0 - B_0^1(F(z_1)) + B_0(F(z_1) - F(x_0) - A_0^1(z, -x_0)) \\ &\le z_1 - x_0 - B_0^1(F(z_1)) \le z_1 - x_0 \end{aligned}$$

by hypotheses (a), (b) and (c).

According to Kantorovich's theorem on fixed points [2], [5], [8] the operator P_1 has a fixed point $q_1 \in [0, z_1 - x_0]$. Set $x_1 = x_0 + q_1$ then

$$F(x_0) + A_0^1(x_1 - x_0) = 0, \quad x_0 \le x_1 \le z_1.$$

By hypothesis (a) we also get

$$F(x_1) \le 0.$$

Define the operator $P_2 \colon [0, y_0 - z_2] \to E_1$ by

$$P_2(x) = x + B_0^2(F(y_0) - A_0^2(x))$$

where B_0^2 is a continuous nonsingular nonnegative left subinverse of $A_0^2 = A^2(y_0, y_{-1})$. P_2 is clearly continuous, isotone with

$$P_2(0) = B_0^2(F(y_0)) \ge 0,$$

and

$$P_2(y_0 - z_2) = y_0 - z_2 + B_0^2(F(z_2)) + B_0^2(F(y_0) - F(z_2) - A_0^2(y_0 - z_2))$$
$$\leq y_0 - z_2 + B_0^2(F(z_2)) \leq y_0 - z_2$$

by hypotheses (a), (b) and (c).

As above we define the existence of a point $q_2 \in [0, y_0 - z_2]$ such that $P_2(q_2) = q_2$. Let $y_1 = y_0 - q_2$. It follows that

$$F(y_0) + A_0^2(y_1 - y_0) = 0, \quad z_2 \leq y_1 \leq y_0.$$

By hypothesis (b) we also have

$$F(y_1) \leq 0.$$

Using induction on $n \geq 0$ we can show that there exists two sequences $\{x_n\}$, $\{y_n\}$ $(n \geq 0)$ satisfying (2), (3), (8), (9). Since E_1 is regular from (9) it follows that there exist $x^*, y^* \in E_1$ such that $x^* = \lim_{n\to\infty} x_n$ and $y^* = \lim_{n\to\infty} y_n$. We obviously also have $x^* \leq z_1 \leq z_2 \leq y^*$.

Denote by $A^5(x^*, z_1)$ a nonnegative left superinverse of $A^3(x^*, z_1)$. Then by (12) we obtain

$$0 \leq z_1 - x^* \leq A^5(x^*, z_1)A^3(x^*, z_1)(z_1 - x^*) \leq A^5(x^*, z_1)(F(z_1) - F(x^*)) = 0$$

which implies $x^* = z_1$. Similarly by (13) we obtain $y^* = z_2$. By $z_1 = z_2$ we deduce $x^* = y^*$. Finally if $x^* = y^*$, (10) gives $z_1 = z_2$.

That completes the proof of the theorem.

Example. Let $E_1 = E_2 = R$, choose any $x_{-1} \leq 0$ and $y_{-1} \geq 0$ and set $D = [x_{-1}, y_{-1}]$. Define the function F on D by

$$F(x) = \begin{cases} x & x \leq 0 \\ 2x, & x \geq 0. \end{cases} \tag{14}$$

The only solution z of equation $F(x) = 0$ is $z = 0$. Let $z_1 = z_2 = z$. Choose also $A^1 = 2$ and $A^2 = 3$. Then hypotheses (a)–(d) of Theorem 1 are satisfied and iterations (2) and (3) give respectively

$$x_{n+1} = \left(\frac{1}{2}\right)^{n+1} x_0 \tag{15}$$

and

$$y_{n+1} = \left(\frac{1}{3}\right)^{n+1} y_0 \quad (n \geq 0) \tag{16}$$

which converge to 0 as $n \to \infty$ for any $x_0 \leq 0$ and $y_0 \geq 0$. However theorems on the convergence of Newton's method fail to apply since $F'(0)$ does not exist [2], [3], [4]. We note though that Newton's method converges to 0 since $x_1 = 0$ for any x_0. Similar observations can be made for the secant method by setting $A^1(y, x) = \frac{F(x)-F(y)}{x-y}$ and $A^2(v, w) = \frac{F(w)-F(v)}{w-v}$. By (14), approximations (2) and (3) become (15) and (16) respectively in this case which converge to 0. But theorems on the convergence of the Secant method require that $F'(0)$ exists [2], [3], [4], [6], [8]. It is also simple calculus to show that condition (4) is not satisfied for both Newton's and the Secant method.

Remark 1. We note that condition (4) implies (6) and (7) but not vice versa. (See the above example.) That is (6) and (7) are weaker conditions than (4).

The regularity of the space E_1 is a rather restrictive condition. It was used to prove that sequences $\{x_n\}$, $\{y_n\}$ $(n \geq 0)$ are well defined and convergent. To avoid this regularity condition we consider the explicit version of it:

$$x_{n+1} = x_n - B_n^1(F(x_n)) \quad (n \geq 0) \tag{17}$$

$$y_{n+1} = y_n - B_n^2(F(y_n)) \quad (n \geq 0) \tag{18}$$

where B_n^1, B_n^2 are nonnegative subinverses of A_n^1 and A_n^2 respectively.

We can now state and prove the following result:

Theorem 2. *Let $F\colon D \subseteq E_1 \to E_2$ be an operator between two POL-spaces and D be an open convex subset of E_1. Let x_{-1}, x_0, y_{-1}, y_0 be points of D satisfying* (5). *Assume that hypotheses* (a)–(c) *of Theorem* 1 *are true and linear operators $A^1(y,x)$ and $A^2(v,w)$ have nonnegative subinverses on $[x_{-1}, y_{-1}]$.*

Then approximations (17) *and* (18) *where B_n^1 and B_n^2 are nonnegative subinverses of A_n^1 and A_n^2 respectively define two sequences $\{x_n\}$, $\{y_n\}$ $(n \geq 0)$ having the properties* (8) *and* (9).

Moreover if $\lim_{n\to\infty} x_n = x^* = y^* = \lim_{n\to\infty} y_n$ then the following are true

$$x^* = z_1 = z_2 = y^* \quad \textit{and} \quad F(x^*) = F(y^*) = 0.$$

Furthermore, if F is continuous at x^* and y^* and $F(x^*) = F(y^*)$ we deduce $F(x^*) = F(y^*) = 0$.

Proof. For $n = 0$ we obtain in turn from approximations (17) and (18)

$$\begin{aligned}
&x_1 - x_0 = -B_0^1(F(x_0)) \geq 0, \\
&z_1 - x_1 = z_1 - x_0 + B_0^1 F(x_0) = (z_1 - x_0) - B_0^1(F(z_1) - F(x_0)) \geq 0 \quad \text{by (6)}
\end{aligned}$$

and

$$\begin{aligned}
&F(x_1) \leq 0 \quad \text{by (b)}, \\
&y_0 - y_1 = B_0^2(F(y_0)) \geq 0 \\
&y_1 - z_2 = y_0 - B_0^2(F(y_0)) - z_2 = y_0 - z_2 - B_0^2(F(y_0) - F(z_2)) \geq 0 \quad \text{by (7)}
\end{aligned}$$

and

$$F(y_1) \geq 0 \quad \text{by (6)}.$$

Thus, we have proved $x_0 \leq x_1 \leq z_1 \leq z_2 \leq y_1 \leq y_0$. Proceeding by induction on $n \geq 0$ we obtain (8) and (9). If $x^* = \lim_{n\to\infty} x_n$, $y^* = \lim_{n\to\infty} y_n$ and $x^* = y^*$ it follows that $x^* = z_1 = z_2 = y^*$ and since $F(z_1) = F(z_2) = 0$ we get $F(x^*) = F(y^*) = 0$.

Finally if F is continuous at x^* and y^* by letting $n \to \infty$ in (8) we get $F(x^*) \leq 0 \leq F(y^*)$ and since $F(x^*) = F(y^*)$ we deduce $F(x^*) = F(y^*) = 0$.

That completes the proof of the theorem.

Remark 2. Several natural conditions (other than the ones in Theorems 1 and 2) under which x^* and y^* are solutions of equation (1) as well as other variations of approximations (2)–(3) and (17)–(18) can be found in [2, Ch. 6].

Remark 3. The conclusions of Theorems 1 and 2 remain true if estimates (6), (7), (12) and (13) are valid on $[x_0, y_0]$, since $x_n \leq z_1$ and $z_2 \leq y_n$ $(n \geq 0)$.

The following theorem is also available on fixed points.

Theorem 3. *Let $P\colon D \subseteq E_1 \to E_1$ where E_1 is a regular POTL-space and D is a subset of E_1. Define the iterations*

$$x_{n+1} = P(x_n) \quad (n \geq 0)$$

and

$$y_{n+1} = P(y_n) \quad (n \geq 0)$$

for some $[x_0, y_0] \in D$. Assume:

(a) *there exists $z \in D$ such that*

$$x_0 \leq P(x_0) \leq z \leq P(y_0) \leq y_0;$$

(b) *$P(x) \leq z$ for $x \leq z$, $x \in D$ and $P(x) \geq z$ for $x \geq z$, $x \in D$;*

(c) *$P(x) \leq P(y)$ for $x \leq y$, $x, y \in D$ and $P(x) \geq P(y)$.*

Then there exist two points $x^*, y^* \in D$ such that

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq x^* \leq z \leq y^* \leq y_{n+1} \leq y_n \leq \cdots \leq y_1 \leq y_0, \tag{18}$$

$$\lim_{n\to\infty} x_n = x^* \quad \textit{and} \quad \lim_{n\to\infty} y_n = y^*. \tag{19}$$

Proof. By hypothesis $x_0 \leq x_1 = P(x_0) \leq z$, $y_{-1} \geq y_1 = P(y_0) \geq z$. Assume $x_{k-1} \leq x_k \leq z$ and $y_{n-1} \geq y_k \geq z$ for $k = 1, 2, \ldots, n$. Then $x_{k+1} = P(x_k) \leq z$, $y_{k+1} = P(y_k) \geq z$, $x_{k+1} - x_k = P(x_k) - P(x_{k-1}) \geq 0$ and $y_{k+1} - y_k = P(y_k) - P(y_{k-1}) \leq 0$. Moreover since E_1 is regular there exist x^*, y^* such that (18) and (19) are satisfied.

That completes the proof of the Theorem.

Remark 4. (a) If $x^* = y^*$ then $x^* = z = y^*$. Hence if z is a fixed point of P on $[x_0, y_0]$ then $P(x^*) = x^* = y^* = P(y^*)$.

(b) If z is the unique fixed point of P on $[x_0, y_0]$ and P is continuous at x^* and y^* then since $P(x^*) = x^*$ and $P(y^*) = y^*$ we get $x^* = z = y^*$.

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ON THE MONOTONE CONVERGENCE OF IMPLICIT NEWTON-LIKE METHODS

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ABSTRACT. In this study, we examine the convergence of implicit Newton-like methods to a solution of a nonlinear equation on a partially ordered topological space. We generate two sequences, one increasing and one decreasing that enclose a solution of the nonlinear equation. Finally, we provide an example that shows that our method converges whereas the standard explicit Newton-like method does not.

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Key Words and Phrases. Newton-like method, monotone convergence, partially ordered topological space.

I. INTRODUCTION

In this study, we are concerned with the problem of approximating a solution z^* of the nonlinear operator equation

$$F(x) = 0 \tag{1}$$

where F is a continuous nonlinear operator defined on a subset D of a partially ordered topological space (POTL-space) E_1 with values in a POTL-space E_2 [2], [3], [5].

We introduce implicit Newton-like methods of the form

$$F(x_{n+1}) + A^1(x_{n-1}, x_n)(x_{n+1} - x_n) = 0 \quad (n \geq 0) \tag{2}$$

$$F(y_{n+1}) + A^2(y_n, y_{n-1})(y_{n+1} - y_n) = 0 \quad (n \geq 0) \tag{3}$$

to monotonically enclose z^*. Here operators $A^1, A^2 \colon D^2 \to L(E_1, E_2)$ are satisfying certain convexity type conditions.

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So far explicit Newton-like methods of the form

$$F(x_n) + A^1(x_{n-1}, x_n)(x_{n+1} - x_n) = 0 \quad (n \geq 0) \tag{4}$$

$$F(y_n) + A^2(y_n, y_{n-1})(y_{n+1} - y_n) = 0 \quad (n \geq 0) \tag{5}$$

as well as various explicit configurations of them have been extensively examined [1]–[5] under the natural hypothesis

$$F(y) - F(x) \leq A(w, z)(y - x) \tag{6}$$

for all $x_0 \leq x \leq y \leq w \leq z \leq y_0$ where $[x_0, y_0] \subseteq D$.

One can provide examples where iterations (2) and (3) converge to a solution z^* of equation (1) whereas iterations (4) and (5) do not. (See the example at the end of this study.) On the other hand the convergence results developed so far for (4) and (5) provide no information about the behavior of iterations (2) and (3).

Sufficient convergence conditions are provided for monotonically enclosing a solution z^* of equation (1) as well as some applications. In particular we provide an example that shows that our results apply where the earlier ones fail.

II. MONOTONE CONVERGENCE

The main result of our study is contained in the following theorem:

Theorem 1. *Consider an operator $F\colon D \subseteq E_1 \to E_2$ where E_1 is a regular POTL-space and E_2 is a POTL-space. Let x_{-1}, x_0, y_{-1}, y_0 be points of D such that*

$$x_{-1} \leq x_0 \leq y_0 \leq y_{-1}, \ [x_{-1}, y_{-1}] \subseteq D \ \text{ and } \ F(x_0) \leq 0 \leq F(y_0). \tag{7}$$

Assume there exist nonnegative operators $A^1, A^2\colon D^2 \to L(E_1, E_2)$ with continuous nonsingular nonnegative left subinverses B^1 and B^2 respectively such that

$$B(v, w)(F(y) - F(x)) \leq (I - B(v, w)A(v, w))(y - x) \tag{8}$$

where (B, A) is (B^1, A^1) or (B^2, A^2) for all $v, w, x, y \in D$ with $x_0 \leq v \leq w \leq x \leq y \leq y_0$.

Then there exist two iterations $\{x_n\}$ $(n \geq 1)$, $\{y_n\}$ $(n \geq 1)$ and two points x^*, y^* of D such that for all $n \geq 0$ approximations (2) and (3) are solvable,

$$F(x_n) \leq 0 \leq F(y_n) \tag{9}$$

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq \cdots \leq y_1 \leq y_0, \tag{10}$$

$$\lim_{n\to\infty} x_n = x^*, \ \lim_{n\to\infty} y_n = y^*, \ \text{ and } \ x^* \leq y^*. \tag{11}$$

Proof: Let B_0^1 be a continuous nonsingular nonnegative left subinverse of $A_0^1 = A^1(x_{-1},\, x_0)$ and consider the operator $H\colon [x_0, y_0] \to E_1$ defined by

$$H(x) = x - B_0^1(F(x) + A_0^1(x - x_0)).$$

Clearly H is continuous and for $x \leq y$ we deduce that H is isotone. Indeed from (8) we get

$$\begin{aligned} &B_0^1(F(y) - F(x)) \leq (I - B_0^1 A_0^1)(y - x) \Rightarrow \\ &\quad x - B_0^1(F(x)) - B_0^1 A_0^1(x - x_0) \leq y - B_0^1(F(y)) - B_0^1 A_0^1(y - x_0) \\ &\Rightarrow H(x) \leq H(y). \end{aligned}$$

We also have

$$H(x_0) = x_0 - B_0^1(F(x_0) + A_0^1(x_0 - x_0)) = x_0 - B_0^1(F(x_0)) \geq x_0$$

by (7),

$$H(y_0) = y_0 - B_0^1(F(y_0) + A_0^1(y_0 - x_0)) \leq y_0$$

by (7) and the fact that A_0^1 and B_0^1 are nonnegative operators on D^2. It follows from the Kantorovich theorem on fixed points that H has a fixed point $x_1 \in [x_0, y_0]$. That is $H(x_1) = x_1$ which shows that x_1 solves (2) for $n = 0$. It also follows from (2) for $n = 0$ that $F(x_1) \leq 0$.

Let B_0^2 be a continuous nonsingular nonnegative left subinverse of $A_0^2 = A^2(y_0,\, y_{-1})$ and define the operator $G\colon [x_1, y_0] \to E_1$ by

$$G(x) = x - B_0^2(F(x) - A_0^2(y_0 - x)).$$

Operator G is clearly continuous and exactly as above using (8) we can show that it is isotone.

Moreover we have

$$G(x_1) = x_1 - B_0^2(F(x_1) - A_0^2(y_0 - x_1)) \geq x_1$$

(since $F(x_1) \leq 0$ and A_0^2 is nonnegative),

$$G(y_0) = y_0 - B_0^2(F(y_0) - A_0^2(y_0 - x_0)) \leq y_0$$

(since $F(y_0) \geq 0$). Hence there exists $y_1 \in [x_1, y_0]$ with $G(y_1) = y_1$. That is y_1 solves approximation (3) for $n = 0$ and $F(y_1) \geq 0$.

Proceeding by induction on $n \geq 0$ we can show that there exist two iterations $\{x_n\}$ $(n \geq 1)$, $\{y_n\}$ $(n \geq 1)$ satisfying (2), (3), (9) and (10). Moreover since the space E_1 is regular it follows that there exist $x^*, y^* \in E_1$ satisfying (11).

That completes the proof of the Theorem.

In what follows we shall give some natural conditions under which the points x^* and y^* are solutions of the equation $F(x) = 0$.

Theorem 2. *Under the hypotheses of Theorem 1 suppose that F is continuous at x^* and y^*. If one of the following conditions is satisfied:*

(a) $x^* = y^*$;

(b) *operators A_n^1, A_n^2 $(n \geq 0)$ are equicontinuous;*

(c) *E_2 is normal and there exists an operator $P: E_1 \to E_2$ $(P(0) = 0)$ continuous on the origin and such that $A_n^1 \leq S$, $A_n^2 \leq S$ for sufficiently large n;*

(d) *E_1 is normal and there exists an operator $Q: E_1 \to E_2$ $(Q(0) = 0)$ which has an isotone inverse continuous at the origin and such that $A_n^1 \leq Q$, $A_n^2 \leq Q$ for sufficiently large n.*

Then we have

$$F(x^*) = F(y^*) = 0.$$

Proof. Using the continuity of F and (9) we get $F(x^*) \leq 0 \leq F(y^*)$ by letting $n \to \infty$. That is $F(x^*) = F(y^*) = 0$.

(b) By the equicontinuity of the operators A_n^1, A_n^2 it follows that $\lim_{n\to\infty} A_n^1(x_n - x_{n+1}) = \lim_{n\to\infty} A_n^2(y_n - y_{n+1}) = 0$, and by (2) and (3), by letting $n \to \infty$ and using the continuity of F at x^* and y^* we get $F(x^*) = F(y^*) = 0$.

(c) For sufficiently large n we get

$$0 \geq F(x_{n+1}) \geq P(x_n - x_{n+1}), \quad 0 \leq F(y_{n+1}) \leq P(y_n - y_{n+1}).$$

By the normality of E_2 and the continuity of F and P we get $F(x^*) = F(y^*) = 0$.

(d) By (2), (3), (9) and (10) we get

$$0 \geq F(x_{n+1}) = A_n^1(x_n - x_{n+1}) \geq Q(x_n - x_{n+1}),$$
$$0 \leq F(y_{n+1}) = A_n^2(y_n - y_{n+1}) \leq Q(y_n - y_{n+1}).$$

Hence, we get

$$\lim_{n\to\infty} Q^{-1}(F(x_{n+1})) = \lim_{n\to\infty} Q^{-1}(F(y_{n+1})) = 0,$$

since E_1 is normal and $\lim_{n\to\infty}(x_n - x_{n+1}) = \lim_{n\to\infty}(y_n - y_{n+1}) = 0$. That is by the continuity of F we finally get $F(x^*) = F(y^*) = 0$.

That completes the proof of the Theorem.

Example. Let $D = E_1 = E_2 = \mathbf{R}$ and $F(x) = \frac{1}{2}x$. Choose $A^1 = A^2 = \frac{1}{4}$, $B^1 = B^2 = 1$, $x_{-1} = -2$, $x_0 = -1$, $y_0 = 1$ and $y_{-1} = 2$. Then it can easily be seen that all hypotheses of Theorem 1 are satisfied and (2) becomes $x_{n+1} = \frac{1}{3}x_n = \left(\frac{1}{3}\right)^{n+1} x_0$ which converges to the unique solution $z^* = 0$ of equation $F(x) = 0$. However condition (6) is not satisfied and iteration (4) becomes $x_{n+1} = -x_n \not\to 0$ for any $x_0 \neq 0$.

The uniqueness of the solution z^* of the equation $F(x) = 0$ in $[x_0, y_0]$ can be proved by assuming a condition complementary to (8). In particular we can show the following:

Theorem 3. *Let E_1 and E_2 be two POL-spaces, $F\colon D \subseteq E_1 \to E_2$ be a nonlinear operator and $x_0, y_0 \in D$ with $x_0 \le y_0$ and $[x_0, y_0] \subseteq D$. Assume:*

(a) *there exists an operator $M\colon D^2 \to L(E_1, E_2)$ such that $M(x,y)$ has a nonnegative left superinverse on D^2 and*

$$F(y) - F(x) \ge M(x,y)(y-x) \tag{12}$$

for all $(x,y) \in D^2$.

(b) *$(x^*, y^*) \in D$ and $F(x^*) = F(y^*)$.*

Then we deduce

$$x^* = y^*.$$

Proof: Let $M_1(x^*, y^*)$ be a nonnegative left superinverse of $M(x^*, y^*)$. Then by (12) we get

$$\begin{aligned} 0 \le y^* - x^* &\le M_1(x^*, y^*) M(x^*, y^*)(y^* - x^*) \\ &\le M_1(x^*, y^*)(F(x^*) - F(y^*)) = 0. \end{aligned}$$

That is we deduce $x^* = y^*$.

That completes the proof of the Theorem.

Remark. Several configurations other than (2) and (3) can also be considered. See [1], [2], [4] for other configurations for explicit Newton-like methods. For example our results can apply to prove convergence for the implicit Newton-like method given by

$$\begin{aligned} F(x_{n+1}) + A(y_n, y_{n-1})(x_{n+1} - x_n) = 0 \quad (n \ge 0) \\ F(y_{n+1}) + A(y_n, y_{n-1})(y_{n+1} - y_n) = 0 \quad (n \ge 0) \end{aligned}$$

where $A\colon D^2 \to L(E_1, E_2)$.

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A GENERALIZATION OF EDELSTEIN'S THEOREM ON FIXED POINTS AND APPLICATIONS

Ioannis K. Argyros

Abstract. In this study, we approximate fixed points of nonlinear equations on topological vector spaces using implicit methods. A new Edelstein type theorem on fixed points is proved. The advantage of these methods is that they can be used to solve equations that usual explicit methods cannot.

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I. INTRODUCTION

In this study, we are concerned with the problem of approximating a fixed point z of the equation

$$x = F(x, x, \dots, x), \tag{1}$$

where F is an operator defined on a subset D^{m+1} of a topological vector space E^{m+1} with values in D, D is a compact set, and m is a nonnegative integer.

Explicit multistep methods of the form

$$x_{n+1} = F_n(x_n, x_{n-1}, \dots, x_{n-m}) \quad (m \geq m) \tag{2}$$

have been used extensively to approximate a fixed point z of equation (1). See for example [1], [2], [4]–[6], and the references there. Picard, Newton-like methods can be considered approximations of the form (2). With the exception of the work in [3] (for $m = 1$) no attention has been given to implicit multistep methods of the form

$$x_n = F(x_n, y_n^m) \tag{3}$$

$$y_n^m = (y_{n-1}, y_{n-2}, \dots, y_{n-m}), \tag{4}$$

$$y_{n-i} = g_n^i(x_n, x_{n-1}, \dots, x_{n-m}), \tag{5}$$

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where $g_n^i: D^{m+1} \to D$ for $n \geq m$, $i = 1, 2, \ldots, m$.

We provide sufficient conditions for the convergence of the sequence $\{x_n\}$ ($n \geq m$) generated by (3)–(5) to a fixed point z of equation (1) for any starting points $x_0, x_1, \ldots, x_{m-1} \in D$. Moreover we provide a generalization of Edelstein's theorem on fixed points [4], [2].

Using examples we finally show that the advantage of these methods is that they can be used to solve equations that explicit methods cannot.

II. CONVERGENCE ANALYSIS

The main result of our study is contained in the following convergence result for the method (3)–(5).

Theorem 1. *Let m be a nonnegative integer, E a topological vector space, D a compact subset of E, F continuous operators from D^{m+1} into D, $S = \{x \in D: F(x, x, \ldots, x) = x\}$, and an operator $f: D \times S \to [0, +\infty)$. Assume:*

(a) *$S \neq \emptyset$ and $f = 0$, $x \in D$, $z \in S \Leftrightarrow x = z \in S$;*

(b) *$f(F(z_n, z_{n-1}, \ldots, z_{n-m}), z) < \max\{f(z_n, z), f(z_{n-1}, z), \ldots, f(z_{n-m}, z)\}$, for all $n \geq m$, $z_n, z_{n-1}, \ldots, z_{n-m} \in D$, $z \in S$ such that there are no relations such as $z_n = z_{n-1} = \cdots = z_{n-m} = z$;*

(c) *if there exist operators $h_n^i: D^{m+1} \to D$, $i = 1, 2, \ldots, m$, $n \geq m$ such that $z_{n-i} = h_n^i(v_n, v_{n-1}, \ldots, x_{n-m})$ for some $v_n, v_{n-1}, \ldots, v_{n-m} \in D$ then*

$$f(F(z_n, z_{n-1}, \ldots, z_{n-m}), z) < f(z_{n-i}, z) < f(v_{n-1}, z)$$

for all $z_n, v_{n-1}, z_{n-i} \in D$, $n \geq m$, $i = 1, 2, \ldots, m$;

(d) *operator $f(x, z)$ is continuous in x;*

(e) *the system (3)–(5) is solvable for arbitrary $x_0, x_1, x_2, \ldots, x_m \in D$ for all $n \geq m$.*

Then the sequences $\{x_n\}$, $\{y_{n-i}\}$ ($n \geq m$), $i = 1, 2, \ldots, m$ generated by (3)–(5) converge in the topology of E to the unique fixed point z of equation (1).

Proof: Using (3)–(5) and (c) we get

$$\begin{aligned} f(x_n, z) &= f(F(x_n, y_{nm}), z) < f(y_{n-i}, z) \\ &< f(x_{n-1}, z), \quad n \geq m, \ i = 1, 2, \ldots, m, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} f(x_n, z) = \lim_{n\to\infty} f(y_{n-i}, z) \quad \text{for all } i = 1, 2, \ldots, m.$$

By the compactness of D we can find subsequences $\{x_{n_j}\} \subseteq \{x_n\}$, $\{y_{n_j-i}\} \subseteq \{y_{n-i}\}$ and points $x, y^1, y^2, \ldots, y^m \in E$ such that

$$\lim_{j\to\infty} x_{n_j} = x, \ \lim_{j\to\infty} y_{n_j-i} = y^i.$$

Since f is continuous in x we get $f(x, z) = f(y^i, z)$ for $i = 1, 2, \ldots, m$. Moreover by letting $j \to \infty$ we obtain

$$\lim_{j\to\infty} F(x_{n_j}, y_{n_j-1}, \ldots, y_{n_j-m}) = F(x, y^1, y^2, \ldots, y^m)$$

and by (b)

$$f(x,z) = f(F(x,y^1,y^2,\dots,y^m),z) < \max\{f(x,z), f(y^1,z),\dots,f(y^m,z)\},$$

which is a contradiction. That is we deduce

$$x = y^1 = y^2 = \cdots = y^m = z.$$

Furthermore we note that S is a singleton set. If $a \neq b$, $a, b \in S$, then by (b) we obtain

$$f(a,b) = f(F(a,b,\dots,b),b) < f(a,b)$$

which is a contradiction. Hence we get $a = b$.

That completes the proof of the Theorem.

Remark 1. Let $m = 1$, $g_n^1 = g$, $g(x_n, x_{n-1}) = F_k(x_n, x_{n-1})$, $n \geq 1$, $k \geq 0$, where $F_0(v,w) = w$, $F_1(v,w) = F(v,w)$, $F_i(v,w) = F(v, F_{i-1}(v,w))$, for $v, w \in D$, $i = 2, 3, \dots, k$, $y_n^1 = g(x_n, x_{n-1})$ $(n \geq 1)$, then Theorem 1 reduces to Theorem 1 in [3]. Theorem 1 cannot be used for the important class of multistep Newton-like methods where $m > 1$ [1], [2].

The following theorem generalizes Edelstein's Theorem on fixed points [4], [2].

Theorem 2. *Let m be a nonnegative integer, D a closed bounded subset of a normed space E, and F an operator from D^{m+1} into D.*

Assume:

(a)

$$\|F(v_1,v_2,\dots,v_{m+1}) - F(w_1,w_2,\dots,w_{m+1}\|$$
$$\begin{cases} < \max\{\|v_i - w_i\|\}, i = 1,2,\dots,m+1 \\ \textit{if } (v_1,v_2,\dots,v_{m+1}) \neq (w_1,w_2,\dots,w_{m+1}) \\ \text{and } v_i \neq v_j;\ \text{or } w_i \neq w_j \text{ for } i \neq j \\ \\ \leq \|v_i - w_i\|,\ i = 1,2,\dots,m+1 \\ \text{if } v_1 = v_2 = \cdots = v_{m+1} \text{ and } w_1 = w_2 = \cdots = w_{m+1}. \end{cases}$$

(b) *Either D is compact or F maps $D \times D$ into a compact subset of D.*

Then the following are true:

(i) *Equation*

$$F(x,x,\dots,x) = x$$

has a unique z solution in D.

(ii) *Every equation*

$$x_n = F(x_n, x_{n-1},\dots,x_{n-m}) \quad (n \geq m) \tag{6}$$

has also a unique solution x_n and $\lim_{n\to\infty} x_n = z$ for any $x_0 \in D$.

Proof: It follows immediately by using the operator $F_y(x) = F(x,y)$ for all $x \in D$, $y \in D^m$ and Edelstein's Theorem on fixed points [4], [2].

Remark 2. Theorem 2 generalizes Edelstein's Theorem on fixed points [4], [2], and reduces to Theorem 6 in [3] for $m = 1$. Obviously earlier results [1], [2], [4]–[6] cannot be used to assert existence or the convergence of approximate solutions for equation (1) if instead of (6) Picard's iteration for $G(x) = F(x, x, \dots, x)$ is used. Moreover the results in [3] cannot be used for multistep Newton-like methods where $m > 1$ [1], [2].

In the following example we provide an application for Theorem 2.

Example 1. Let $E = R$, $D = [a, b]$ with $a \leq 0 \leq b$, $\|x - y\| = |x - y|$ and $F(x, y) = -\frac{1}{6}x + \frac{2}{3}y$. Then we have

$$\begin{aligned} \|F(x,y) - F(v,w)\| &= \left\| -\frac{1}{6}(x - v) + \frac{2}{3}(y - w) \right\| \\ &\leq \frac{1}{6}|x - v| + \frac{2}{3}|y - w| \end{aligned} \tag{7}$$

for all $x, y, v, w \in D$. It now follows from (7) that hypothesis (a) of Theorem 2 is satisfied. Moreover (6) (for $m = 1$) becomes

$$x_n = \frac{4}{7}x_{n-1} = \left(\frac{4}{7}\right)^n x_0 \to 0 = z \quad \text{as } n \to \infty$$

for any $x_0 \in R$.

In the example that follows we provide an explicit sequence that does not converge and an implicit sequence that converges to a fixed point of an operator.

Example 2. Let E, D be as in Example 1 and consider the problem of approximating the unique fixed point $z = 0$ of the operator F on D given by $F(x, y) = cx$ for all $x, y \in D$ with $c \neq 0$. We use the approximations

$$F(a_n) + d(a_{n+1} - a_n) = 0 \quad (n \geq 0), \tag{8}$$

$$F(b_{n+1}) + d(b_{n+1} - b_n) = 0 \quad (n \geq 0) \tag{9}$$

for $a_0, b_0 \in D$. Approximation (8) is explicit whereas (9) is an implicit method. With the above choices (8) and (9) become respectively

$$a_{n+1} = \frac{d - c}{d} a_n \quad (n \geq 0)$$

and

$$b_{n+1} = \frac{d}{c + d} b_n \quad (n \geq 0).$$

For $c = 3$ and $d = 1$ we obtain

$$a_{n+1} = -2a_n = (-2)^{n+1} a_0 \not\to 0 = z \quad \text{for any } a_0 \neq 0$$

whereas

$$b_{n+1} = \frac{1}{4} b_n = \left(\frac{1}{4}\right)^{n+1} b_0 \to 0 = z \quad \text{for any } b_0.$$

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