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A COMMON FIXED POINT THEOREM IN COMPLETE METRIC SPACES

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ABSTRACT. In this note we establish a common fixed point theorem in complete metric spaces which generalizes some results in [2].

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1. INTRODUCTION

Delbosco [1] and Skof [3]have established fixed point theorems for self - maps of complete metric spaces by altering the distances between the points with the use of a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the following properties:
(i) $\phi : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing in R^+;
(ii) $\phi(t) = 0$ iff $t = 0$;
(iii) $\phi(t) \geq Mt^u$ for every $t > 0, u > 0$ are constants.
In 1977, F. Skof [3] gave the following theorem

THEOREM A. Let T be a self-map of a complete metric space (X, d) and $\phi \in \Phi$ such that for every $x, y \in X$,

$$(I) \quad \phi(d(Tx, Ty)) \leq a\phi(d(x, y)) + b\phi(d(x, Tx)) + c\phi(d(y, Ty)),$$

where a, b and c are three nonnegative constants satisfying $a + b + c < 1$. Then T has a unique fixed point.

In [1], the author has considered functions $\phi \in \Phi$ such that $\phi(t) = t^n, n \in N$ for every $t \geq 0$. Clearly, ϕ is not a metric since ϕ has only a argument.

In 1984, M.S.Khan et al. [2] generalized Theorem (A) by using a much extensive condition than (I) and removed the hypothesis (iii) which seems superfluous. They proved the following theorem:

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THEOREM B. Let T be a self-map of a complete metric space (X, d) and ϕ satisfying (i) and (ii). Furthermore, let a, b, c be three decreasing functions from R^+ 0 into $[0, 1)$ such that

$$a(t) + 2b(t) + c(t) < 1$$

for every $t > 0$.

Suppose that T satisfies the following condition:

$$\begin{aligned}(II) \quad \phi(d(Tx, Ty)) \leq a(d(x,y))\phi(d(x,y)) + b(d(x,y))[\phi(d(x,Tx)) + \phi(d(y,Ty))] \\ + c(d(x,y)) \min\{\phi(d(x,Ty)), \phi(d(y,Tx))\},\end{aligned}$$

where $x, y \in X$ and $x \neq y$. Then T has a unique fixed point.

In this note we shall establish a common fixed point theorem for a pair of mappings S and T which satisfying the contractive defintion corresponding to (II), i.e., for all $x, y \in X$ with $x \neq y$

$$\begin{aligned}(III) \quad \phi(d(Sx, Ty)) \leq a(d(x,y))\phi(d(x,y)) + b(d(x,y))[\phi(d(x,Sx)) + \phi(d(y,Ty))] \\ + c(d(x,y)) \min\{\phi(d(x,Ty)), \phi(d(y,Sx))\},\end{aligned}$$

A COMMON FIXED POINT THEOREM

We establish the following:

THEOREM 2.1. Let S and T be self-mappings on a complete metric space (X, d) and ϕ satisfying the conditions (i) and (ii). Furthermore, let a, b, c be three decreasing functions from R^+ into $[0, 1)$ such that

$$a(t) + 2b(t) + c(t) < 1$$

for every $t > 0$.

Suppose that S and T satisfy (III). Then S and T have a unique fixed point in X.

Proof. Let x_0 be some point in X, define

$$\begin{aligned} x_{2n} &= Sx_{2n+1}, \quad n = 0, 1, 2, ... \\ x_{2n+1} &= Tx_{2n}, \quad n = 0, 1, 2, ... \end{aligned}$$

Let $\tau_n = d(x_n, x_{n+1})$. We assume that $\tau_n > 0$ for every n. Now, for an even integer n, we have

$$\begin{aligned} &\phi(\tau_n) = \phi(d(x_n, x_{n+1})) = \phi(d(Sx_{n-1}, Tx_n)) \\ &\leq A_1 + A_2 + A_3 \end{aligned}$$

where

$$A_1 = a(d(x_{n-1}, x_n))\phi(d(x_{n-1}, x_n)),$$

$$A_2 = b(d(x_{n-1}, x_n))[\phi(d(x_{n-1}, Sx_{n-1})) + \phi(d(x_n, Tx_n))],$$

$$A_3 = c(d(x_{n-1}, x_n)) \min\{\phi(d(x_{n-1}, Tx_n)), \phi(d(x_n, Sx_{n-1}))\}.$$

So

$$\phi(\tau_n) =\leq a(\tau_{n-1})\phi(\tau_{n-1}) + b(\tau_{n-1})[\phi(\tau_{n-1}) + \phi(\tau_n)].$$

Hence we obtain:

$$\phi(\tau_n) \leq \frac{a(\tau_{n-1}) + b(\tau_{n-1})}{1 - b(\tau_{n-1})}\phi(\tau_{n-1}) < \phi(\tau_{n-1}). \qquad (1)$$

Since ϕ is increasing, $\{\tau_n\}$ is a decreasing sequence. Put $\lim_n \tau_n = \tau$ with $\tau > 0$. By (1), then $\tau_n \geq \tau$ implies that

$$\phi(\tau_n) \leq \frac{a(\tau) + b(\tau)}{1 - b(\tau)}\phi(\tau_{n-1}).$$

By letting $n \to \infty$, Since ϕ is continuous, we have:

$$\phi(\tau) \leq \frac{a(\tau) + b(\tau)}{1 - b(\tau)}\phi(\tau) < \phi(\tau),$$

which is a contradiction, hence $\tau = 0$.

Similarly for an odd integer n one can show that $\tau = 0$. Now, we wish to show that $\{x_n\}$ is a Cauchy sequence. Since $\tau = 0$ it suffices to prove that $\{x_{2n}\}$ is a Cauchy sequence we proceed by contradiction. Then there is an $\epsilon > 0$ such that for each even integer $2k$, $k = 0, 1, 2, ...$, there exists two sequences $\{2p(k)\}$, $\{2q(k)\}$ with $2k \leq 2q(k) < 2p(k)$ for which

$$d(x_{2p(k)}, x_{2q(k)}) > \epsilon. \qquad (2)$$

For each even integer $2k$, let $2p(k)$ be the least even integer exceeding $2q(k)$ and satisfy (2). Then

$$d(x_{2q(k)}, x_{2p(k)-2}) \leq \epsilon \quad \textit{and} \quad d(x_{2q(k)}, x_{2p(k)}) > \epsilon.$$

Put $d_{2k} = d(x_{2q(k)}, x_{2p(k)})$, then we have

$$\begin{aligned} \epsilon < d_{2k} &= d(x_{2q(k)}, x_{2p(k)}) \\ &\leq d(x_{2q(k)}, x_{2p(k)}) + d(x_{2p(k)-2}, x_{2p(k)-1}) + d(x_{2p(k)-1}, x_{2p(k)}) \\ &\leq \epsilon + \tau_{2p(k)-2} + \tau_{2p(k)-1}. \end{aligned} \qquad (3)$$

Since $\{\tau_n\}$ converges to 0, $d_{2k} \to \epsilon$. Furthermore, by triangular inequality, it follows that

$$\begin{aligned} -\tau_{2p(k)} - \tau_{2q(k)} + d_{2k} &\leq d(x_{2q(k)+1}, x_{2p(k)+1}) \\ &\leq \tau_{2p(k)} + \tau_{2q(k)} + d_{2k}, \end{aligned}$$

and therefore the sequence $\{d(x_{2q(k)+1}, x_{2p(k)+1})\}$ converges to ϵ.

Moreover, by triangle inequality we have

$$\begin{aligned} &d(x_{2p(k)+1}, x_{2q(k)}) \leq d(x_{2p(k)+1}, x_{2p(k)}) + d(x_{2p(k)}, x_{2q(k)}) \\ &\leq \tau_{2p(k)} + d_{2k} \to \epsilon, \quad \textit{as} \quad k \to \infty \end{aligned}$$

and

$$d(x_{2p(k)+2}, x_{2q(k)+1}) \leq d(x_{2p(k)+2}, x_{2p(k)+1}) + d(x_{2p(k)+1}, x_{2q(k)+1}) \leq A_4$$

So

$$A_4 = d(x_{2p(k)+2}, x_{2q(k)+1}) \leq \tau_{2p(k)+1} + d(x_{2p(k)+1, x_{2q(k)+1}})$$

Hence

$$A_4 \to \epsilon \quad as \quad k \to \infty$$

From (III), we also deduce:

$$\begin{aligned} \phi(d(x_{2p(k)+2}, x_{2q(k)+1})) &= \phi(d(Sx_{2p(k)+1}, Tx_{2q(k)})) \\ &\leq L_1 + L_2 + L_3 \leq L_4 + L_5 + L_6 \end{aligned}$$

where

$$L_1 = a(d(x_{2p(k)+1}, x_{2q(k)}))\phi(d(x_{2p(k)+1}, x_{2q(k)})),$$

$$L_2 = b(d(x_{2p(k)+1}, x_{2q(k)}))[\phi(d(x_{2p(k)+1}, Sx_{2p(k)+1})) + \phi(d(x_{2q(k)}, Tx_{2q(k)}))],$$

$$L_3 = c(d(x_{2p(k)+1}, x_{2q(k)})) \min\{\phi(d(x_{2p(k)+1}, Tx_{2q(k)})), \phi(d(x_{2q(k)}, Sx_{2p(k)+1}))\},$$

$$L_4 = a(\tau_{2p(k)} + d_{2k})\phi(\tau_{2p(k)} + d_{2k}),$$

$$L_5 = b(\tau_{2p(k)} + d_{2k})[\phi(\tau_{2p(k)+1}) + \phi(\tau_{2q(k)})],$$

$$\begin{aligned} L_6 = c(\tau_{2p(k)} + d_{2k}) \min\{&\phi(d(x_{2p(k)+1}, x_{2q(k)}) + \tau_{2q(k)}), \\ &\phi(\tau_{2q(k)} + d(x_{2q(k)+1}, x_{2p(k)+2}))\}. \end{aligned}$$

As $k \to \infty$, we have

$$\begin{aligned} \phi(\epsilon) &\leq a(\epsilon)\phi(\epsilon) + c(\epsilon)\phi(\epsilon) \\ &= \{a(\epsilon) + b(\epsilon)\}\phi(\epsilon) < \phi(\epsilon). \end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in a complete metric space X, then there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now, we shall show that $Sz = z = Tz$.

Hence using (III), we obtain:

$$\begin{aligned} &\phi(d(x_{2p(k)+2}, Tz)) = \phi(d(Sx_{2p(k)+1}, Tz)) \\ &\leq a(d(x_{2p(k)+1}, z))\phi(d(x_{2p(k)+1}, z)) \\ &+ b(d(x_{2p(k)+1}, z))[\phi(d(x_{2p(k)+1}, x_{2p(k)+2})) + \phi(d(z, Tz))] \\ &+ c(d(x_{2p(k)+1}, z)) \min\{\phi(d(x_{2p(k)+1}, Tz)), \phi(d(z, x_{2p(k)+2}))\}. \end{aligned} \tag{4}$$

Letting $k \to \infty$, and use $b < \frac{1}{2}$ in (4), we have

$$\phi(d(z, Tz)) < \frac{1}{2}\phi(d(z, Tz)).$$

That is, $\phi(d(z,Tz)) = 0$ and therefore $d(z,Tz) = 0$, i.e., $z = Tz$. Similarly, it can be shown that $Sz = z$. Hence S and T have a common fixed point $z \in X$. We claim that z is the unique common fixed point of S and T. For this, we suppose that $w(w \neq z)$ is another fixed point of S and T. Then

$$\begin{aligned}
&\phi(d(z,w)) = \phi(d(Sz,Tw)) \\
&\leq a(d(z,w))\phi(d(z,w)) + b(d(z,w))[\phi(d(z,Sz)) + \phi(d(w,Tw))] \\
&+ c(d(z,w)) \min\{\phi(d(z,Tw)), \phi(d(w,Sz))\} \\
&\leq [a(d(z,w)) + c(d(z,w))]\phi(d(z,Sz)) \\
&< \phi(d(z,Tw))
\end{aligned}$$

a contradiction. This completes the proof of the theorem.

Finally, we give an example to discuss the validity of the hypothesis of Theorem 2.1.

EXAMPLE 2.1

Let $X = \{1,2,3\}$ be with the Euclidean metric induced by d which is defined by

$$d(1,2) = d(2,3) = 2, d(1,3) = 1.$$

Suppose S and T are maps on X defined by

$$S1 = S2 = S3 = 1, \quad T1 = 1, \quad T2 = 3, \quad T3 = 1.$$

Then it is clear that X is complete with respect to d and S, T satisfy all the conditions of Theorem 2.1. Now setting

$$\begin{aligned}
&x = 1, \quad y = 2, \quad a(t) = \frac{2}{5}, \quad b(t) = \frac{1}{5}, \quad c(t) = \frac{1}{6}, \\
&\phi(t) = t^2, \quad t \in R^+.
\end{aligned}$$

We observe that S and T satisfy condition (III), and 1 is the unique common fixed point of S and T.

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