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A GENERALIZATION OF EDELSTEIN'S THEOREM ON FIXED POINTS AND APPLICATIONS

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Abstract. In this study, we approximate fixed points of nonlinear equations on topological vector spaces using implicit methods. A new Edelstein type theorem on fixed points is proved. The advantage of these methods is that they can be used to solve equations that usual explicit methods cannot.

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I. INTRODUCTION

In this study, we are concerned with the problem of approximating a fixed point z of the equation

$$x = F(x, x, \dots, x), \tag{1}$$

where F is an operator defined on a subset D^{m+1} of a topological vector space E^{m+1} with values in D, D is a compact set, and m is a nonnegative integer.

Explicit multistep methods of the form

$$x_{n+1} = F_n(x_n, x_{n-1}, \dots, x_{n-m}) \quad (m \geq m) \tag{2}$$

have been used extensively to approximate a fixed point z of equation (1). See for example [1], [2], [4]–[6], and the references there. Picard, Newton-like methods can be considered approximations of the form (2). With the exception of the work in [3] (for $m = 1$) no attention has been given to implicit multistep methods of the form

$$x_n = F(x_n, y_n^m) \tag{3}$$

$$y_n^m = (y_{n-1}, y_{n-2}, \dots, y_{n-m}), \tag{4}$$

$$y_{n-i} = g_n^i(x_n, x_{n-1}, \dots, x_{n-m}), \tag{5}$$

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where $g_n^i: D^{m+1} \to D$ for $n \geq m$, $i = 1, 2, \ldots, m$.

We provide sufficient conditions for the convergence of the sequence $\{x_n\}$ ($n \geq m$) generated by (3)–(5) to a fixed point z of equation (1) for any starting points $x_0, x_1, \ldots, x_{m-1} \in D$. Moreover we provide a generalization of Edelstein's theorem on fixed points [4], [2].

Using examples we finally show that the advantage of these methods is that they can be used to solve equations that explicit methods cannot.

II. CONVERGENCE ANALYSIS

The main result of our study is contained in the following convergence result for the method (3)–(5).

Theorem 1. *Let m be a nonnegative integer, E a topological vector space, D a compact subset of E, F continuous operators from D^{m+1} into D, $S = \{x \in D: F(x, x, \ldots, x) = x\}$, and an operator $f: D \times S \to [0, +\infty)$. Assume:*

(a) *$S \neq \emptyset$ and $f = 0$, $x \in D$, $z \in S \Leftrightarrow x = z \in S$;*

(b) *$f(F(z_n, z_{n-1}, \ldots, z_{n-m}), z) < \max\{f(z_n, z), f(z_{n-1}, z), \ldots, f(z_{n-m}, z)\}$, for all $n \geq m$, $z_n, z_{n-1}, \ldots, z_{n-m} \in D$, $z \in S$ such that there are no relations such as $z_n = z_{n-1} = \cdots = z_{n-m} = z$;*

(c) *if there exist operators $h_n^i: D^{m+1} \to D$, $i = 1, 2, \ldots, m$, $n \geq m$ such that $z_{n-i} = h_n^i(v_n, v_{n-1}, \ldots, x_{n-m})$ for some $v_n, v_{n-1}, \ldots, v_{n-m} \in D$ then*

$$f(F(z_n, z_{n-1}, \ldots, z_{n-m}), z) < f(z_{n-i}, z) < f(v_{n-1}, z)$$

for all $z_n, v_{n-1}, z_{n-i} \in D$, $n \geq m$, $i = 1, 2, \ldots, m$;

(d) *operator $f(x, z)$ is continuous in x;*

(e) *the system (3)–(5) is solvable for arbitrary $x_0, x_1, x_2, \ldots, x_m \in D$ for all $n \geq m$.*

Then the sequences $\{x_n\}$, $\{y_{n-i}\}$ ($n \geq m$), $i = 1, 2, \ldots, m$ generated by (3)–(5) converge in the topology of E to the unique fixed point z of equation (1).

Proof: Using (3)–(5) and (c) we get

$$\begin{aligned} f(x_n, z) &= f(F(x_n, y_{nm}), z) < f(y_{n-i}, z) \\ &< f(x_{n-1}, z), \quad n \geq m, \ i = 1, 2, \ldots, m, \end{aligned}$$

which implies that

$$\lim_{n \to \infty} f(x_n, z) = \lim_{n \to \infty} f(y_{n-i}, z) \quad \text{for all } i = 1, 2, \ldots, m.$$

By the compactness of D we can find subsequences $\{x_{n_j}\} \subseteq \{x_n\}$, $\{y_{n_j-i}\} \subseteq \{y_{n-i}\}$ and points $x, y^1, y^2, \ldots, y^m \in E$ such that

$$\lim_{j \to \infty} x_{n_j} = x, \ \lim_{j \to \infty} y_{n_j-i} = y^i.$$

Since f is continuous in x we get $f(x, z) = f(y^i, z)$ for $i = 1, 2, \ldots, m$. Moreover by letting $j \to \infty$ we obtain

$$\lim_{j \to \infty} F(x_{n_j}, y_{n_j-1}, \ldots, y_{n_j-m}) = F(x, y^1, y^2, \ldots, y^m)$$

and by (b)

$$f(x,z) = f(F(x,y^1,y^2,\dots,y^m),z) < \max\{f(x,z), f(y^1,z),\dots,f(y^m,z)\},$$

which is a contradiction. That is we deduce

$$x = y^1 = y^2 = \cdots = y^m = z.$$

Furthermore we note that S is a singleton set. If $a \neq b$, $a,b \in S$, then by (b) we obtain

$$f(a,b) = f(F(a,b,\dots,b),b) < f(a,b)$$

which is a contradiction. Hence we get $a = b$.

That completes the proof of the Theorem.

Remark 1. Let $m = 1$, $g_n^1 = g$, $g(x_n, x_{n-1}) = F_k(x_n, x_{n-1})$, $n \geq 1$, $k \geq 0$, where $F_0(v,w) = w$, $F_1(v,w) = F(v,w)$, $F_i(v,w) = F(v, F_{i-1}(v,w))$, for $v, w \in D$, $i = 2,3,\dots,k$, $y_n^1 = g(x_n, x_{n-1})$ $(n \geq 1)$, then Theorem 1 reduces to Theorem 1 in [3]. Theorem 1 cannot be used for the important class of multistep Newton-like methods where $m > 1$ [1], [2].

The following theorem generalizes Edelstein's Theorem on fixed points [4], [2].

Theorem 2. *Let m be a nonnegative integer, D a closed bounded subset of a normed space E, and F an operator from D^{m+1} into D.*

Assume:

(a)

$$\|F(v_1,v_2,\dots,v_{m+1}) - F(w_1,w_2,\dots,w_{m+1}\|$$
$$\begin{cases} < \max\{\|v_i - w_i\|\}, i = 1,2,\dots,m+1 \\ \textit{if } (v_1,v_2,\dots,v_{m+1}) \neq (w_1,w_2,\dots,w_{m+1}) \\ \text{and } v_i \neq v_j;\ \text{or } w_i \neq w_j \text{ for } i \neq j \\ \\ \leq \|v_i - w_i\|,\ i = 1,2,\dots,m+1 \\ \text{if } v_1 = v_2 = \cdots = v_{m+1} \text{ and } w_1 = w_2 = \cdots = w_{m+1}. \end{cases}$$

(b) *Either D is compact or F maps $D \times D$ into a compact subset of D.*

Then the following are true:

(i) *Equation*

$$F(x,x,\dots,x) = x$$

has a unique z solution in D.

(ii) *Every equation*

$$x_n = F(x_n, x_{n-1},\dots,x_{n-m}) \quad (n \geq m) \tag{6}$$

has also a unique solution x_n and $\lim_{n\to\infty} x_n = z$ for any $x_0 \in D$.

Proof: It follows immediately by using the operator $F_y(x) = F(x,y)$ for all $x \in D$, $y \in D^m$ and Edelstein's Theorem on fixed points [4], [2].

Remark 2. Theorem 2 generalizes Edelstein's Theorem on fixed points [4], [2], and reduces to Theorem 6 in [3] for $m = 1$. Obviously earlier results [1], [2], [4]–[6] cannot be used to assert existence or the convergence of approximate solutions for equation (1) if instead of (6) Picard's iteration for $G(x) = F(x, x, \ldots, x)$ is used. Moreover the results in [3] cannot be used for multistep Newton-like methods where $m > 1$ [1], [2].

In the following example we provide an application for Theorem 2.

Example 1. Let $E = R$, $D = [a, b]$ with $a \leq 0 \leq b$, $\|x - y\| = |x - y|$ and $F(x, y) = -\frac{1}{6}x + \frac{2}{3}y$. Then we have

$$\begin{aligned} \|F(x,y) - F(v,w)\| &= \left\| -\frac{1}{6}(x - v) + \frac{2}{3}(y - w) \right\| \\ &\leq \frac{1}{6}|x - v| + \frac{2}{3}|y - w| \end{aligned} \tag{7}$$

for all $x, y, v, w \in D$. It now follows from (7) that hypothesis (a) of Theorem 2 is satisfied. Moreover (6) (for $m = 1$) becomes

$$x_n = \frac{4}{7}x_{n-1} = \left(\frac{4}{7}\right)^n x_0 \to 0 = z \quad \text{as } n \to \infty$$

for any $x_0 \in R$.

In the example that follows we provide an explicit sequence that does not converge and an implicit sequence that converges to a fixed point of an operator.

Example 2. Let E, D be as in Example 1 and consider the problem of approximating the unique fixed point $z = 0$ of the operator F on D given by $F(x, y) = cx$ for all $x, y \in D$ with $c \neq 0$. We use the approximations

$$F(a_n) + d(a_{n+1} - a_n) = 0 \quad (n \geq 0), \tag{8}$$

$$F(b_{n+1}) + d(b_{n+1} - b_n) = 0 \quad (n \geq 0) \tag{9}$$

for $a_0, b_0 \in D$. Approximation (8) is explicit whereas (9) is an implicit method. With the above choices (8) and (9) become respectively

$$a_{n+1} = \frac{d - c}{d} a_n \quad (n \geq 0)$$

and

$$b_{n+1} = \frac{d}{c + d} b_n \quad (n \geq 0).$$

For $c = 3$ and $d = 1$ we obtain

$$a_{n+1} = -2a_n = (-2)^{n+1}a_0 \not\to 0 = z \quad \text{for any } a_0 \neq 0$$

whereas

$$b_{n+1} = \frac{1}{4}b_n = \left(\frac{1}{4}\right)^{n+1} b_0 \to 0 = z \quad \text{for any } b_0.$$

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