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GENERAL WAYS OF CONSTRUCTING ACCELERATING NEWTON-LIKE ITERATIONS ON PARTIALLY ORDERED TOPOLOGICAL SPACES

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Abstract. In this study, we examine the monotone convergence of Newton-like methods to a solution of an equation on a partially ordered topological space setting. In particular we provide sufficient conditions for the construction of accelerating sequences. This way the solution is obtained faster than in earlier results.

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I. INTRODUCTION

In this study, we are concerned with the problem of approximating a solution z of the equation

$$F_1(x) = 0 \tag{1}$$

where F_1 is a continuous operator defined on a closed convex subset D of a partially ordered topological space (POTL-space) E_1 with values in a POTL-space E_2 [3], [4], [8].

Sufficient conditions for the monotone convergence of Newton-like methods to z have been found by several authors (see [1]–[4], [7], [8] and the references there). Here we first transform (1) into a fixed point problem of the form

$$x = G_1(x) \tag{2}$$

and then consider iterations, of the form

$$x_{n+1} = G_1(x_n) \quad (n \geq 0), \tag{3}$$

$$\bar{y}_{n+1} = G_2(x_n) \quad (n \geq 0), \tag{4}$$

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and

$$y_{n+1} = G_2(y_n) \quad (n \geq 0) \tag{5}$$

for some continuous operators $G_1, G_2 \colon D \to E_2$ with $G_1(z) = G_2(z)$ whenever $F_1(z) = 0$. Under the conditions in the above-mentioned papers iteration $\{x_n\}$ $(n \geq 0)$ converges to z. We then provide sufficient conditions that show that iterations $\{\bar{y}_n\}$, $\{y_n\}$ $(n \geq 0)$ are accelerating sequences to z, in the sense that

$$y_n \leq \bar{y}_n \leq x_n \quad (n \geq 0) \quad \text{and} \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} = \lim_{n\to\infty} \bar{y}_n = z. \tag{6}$$

Moreover in the case when $E_1 = E_2 = R$ our results can be reduced to weaker versions of the corresponding ones in [5].

Finally we provide an example to justify (6).

II. MONOTONE CONVERGENCE

We state and prove the following main result on the monotone convergence of Newton-like methods on partially ordered topological spaces.

Theorem. *Let F_1, F_2 be continuous operators defined on a closed convex subset D of a partially ordered topological space $E_1 = (E_1, \leq)$ (POTL-space) with values in a POTL-space $E_2 = (E_2, \leq)$. Assume:*

(a) *there exists points a, z, b of D such that $a \leq z \leq b$, $[a, b] \subseteq D$ and z is the only point in $[a, b]$ such that*

$$F_1(z) = F_2(z) = 0. \tag{7}$$

(b) *There exists operators $A_1(x)$, $A_2(x) \in L(E_2, E_1)$ $(x \in D)$ such that the Newton-like iterations*

$$x_{n+1} = x_n - A_1(x_n)F_1(x_n) \quad (n \geq 0) \tag{8}$$

$$y_{n+1} = y_n - A_2(y_n)F_2(y_n) \quad (n \geq 0) \tag{9}$$

are decreasing and converging to z for some starting points x_0, y_0 with $x_0 = y_0 \neq z$ and $z \leq x_0 \leq b$.

(c) *For all points $x \in [a, b]$ there exists an operator $L \in L(E_1, E_2)$ depending on F_1, F_2, z and x such that*

$$0 \leq L(x - z) = (G_1 - G_2)(x) - (G_1 - G_2)(z) \tag{10}$$

where

$$G_1(x) = x - A_1(x)F_1(x) \quad \text{and} \quad G_2(x) = x - A_2(x)F_2(x). \tag{11}$$

Then

(12) (i) *the estimate* $y_n \leq x_n$ *holds for all* $n \geq 0$ *such that* $x_n \neq z$.

(ii) $x_{n-1} \neq z$ *and* $x_n = z$ *imply* $y_n = z$.

Proof: (i) We will show estimate (12) using mathematical induction on $n \in N$. For $n = 0$ estimate (12) is true as equality. Using (7)–(11) for $n = 1$ we obtain

$$0 \leq L(x_0 - z) = (G_1 - G_2)(x_0) - (G_1 - G_2)(z) = x_1 - y_1$$

which shows (12) in this case. Assume that $y_k \leq x_k$ for $k = 1, 2, \ldots, n-1$. If $x_n \neq z$ then $x_{n-1} \neq z$ and from (7)–(11) we obtain

$$\begin{aligned} 0 \leq L(x_{n-1} - z) &= (G_1 - G_2)(x_{n-1}) - (G_1 - G_2)(z) \leq G_1(x_{n-1}) - G_2(y_{n-1}) \\ &= x_n - y_n, \end{aligned}$$

(since G_2 is increasing in $[z, b]$ and $\{y_n\} \subseteq [z, b]$ $(n \geq 0)$) which shows (12) for all $n \geq 0$. Part (ii) follows easily by applying the arguments used in part (i).

That completes the proof of the Theorem.

Remark 1. It is easily seen that for $E_1 = E_2 = R$ our theorem reduces to a weaker version of Theorem 1 in [5, p. 160] (see also [6]). A similar theorem can also be immediately stated for increasing sequences to z.

Remark 2. The condition that iteration $\{x_n\}$ $(n \geq 0)$ is decreasing (similarly for iteration $\{y_n\}$ $(n \geq 0)$) can be replaced by: $A_1(x)$ is a nonnegative linear operator and $0 \leq F_1(x)$ for $x \in [z, b]$.

Remark 3. Operator L appearing in (10) can be chosen as $L = [z, x; G_1 - G_2]$. That is as a divided difference of order one on D at the points z, x for the operator $G_1 - G_2$ [3], [4], [7], [8].

Remark 4. Several sufficient conditions for the monotone convergence of iterations $\{x_n\}$, $\{y_n\}$ $(n \geq 0)$ to z can be found in [3], [4], [7], [8] and the references there. In our theorem above we did not add such conditions to avoid repetitions and because we want to keep it as uncluttered and simple as possible.

Remark 5. Another choice for the operator L can be given by $L = A_2(x)[z, x; F_2] - A_1(x)[z, x; F_1]$. Indeed we can get from the identity

$$\begin{aligned} G_1(x) - G_2(x) &= x - A_1(x)F_1(x) - x + A_2(x)F_2(x) \\ &= A_2(x)(F_2(x) - F_2(z)) - A_1(x)(F_1(x) - F_1(z)) \\ &= (A_2(x)[z, x; F_2] - A_1(x)[z, x; F_1])(x - z) \\ &= L(x - z), \end{aligned}$$

which justifies the choice for L.

Remark 6. A reasonable choice for the operator F_2 can be given by

$$F_2(x) = A_3(x)F_1(x) + F_3(x,z)$$

for some $A_3(x) \in L(E_1, E_2)$ and $F_3 : D \to E_2$ such that $F_2(z) = 0$ whenever $F_1(z) = 0$. Assume that $[z, x; F_1]$ is a divided difference of order one on D at the points $z, x \in D$ for the operator F_1. We can write

$$\begin{aligned} G_1(x) - G_2(x) &= x - A_1(x)F_1(x) - x + A_2(x)F_2(x) \\ &= A_2(x)(A_3(x)F_1(x) + F_3(x,z)) - A_1(x)F_1(x) \\ &= (A_2(x)A_3(x) - A_1(x))(F_2(x) - F_1(z)) + A_2(x)F_3(x,z) \\ &= (A_2(x)A_3(x) - A_1(x))[z, x; F_1](x - z) + A_2(x)F_3(x,z) \end{aligned}$$

and for $F_3(x,z) = F_4(x,z)(x-z)$, we get $G_1(x) - G_2(x) = L(x,z)$ where

$$L = [A_2(x)A_3(x) - A_1(x)][z, x; F_1] + A_2(x)F_4(x,z). \tag{13}$$

Natural conditions can be imposed on the operators appearing in the above approximation (13) so that (10) is satisfied. In [5] for $E_1 = E_2 = R$ operators A_3 and F_3 were chosen such that

$$A_3(x) = 1 \quad \text{and} \quad F_3(x,z) = -\frac{1}{2}F_1''(z)(x-z)^2.$$

Since z is an unknown point it was replaced by x_{n-1} and a new iterative process involving the second derivative of order three was found. With the exception of some special cases this iterative process has no practical value since they require an evaluation of the second Fréchet-derivative at each step which means a number of function evaluations proportional with the cube of the dimension of the space. In [3], [4] discretized versions of F_3 were considered on a POTL-space using divided differences of order one. Operator F_3 is now defined by

$$F_3(x_n, x_{n-1}) = -([x_{n-1}, x_n] - [x_{n-1}, x_{n-1}])(x_{n-1} - x_n) \quad (n \ge 1).$$

Several other variations of the above discretized scheme are also possible [3], [4], [7], [8].

We now complete this study by providing an example for the last remark.

Example. Let $E_1 = E_2 = R$, $[a,b] = [-\frac{1}{3}, \frac{1}{3}]$, $F_1(x) = x^2$ and $x_0 = .2$. Note that $z = 0$ is the only zero of equation $F_1(x) = 0$ on $[a,b]$. Consider iterations

$$x_{n+1} = x_n - x_n^2, \quad \bar{y}_{n+1} = x_n - 2x_n^2 \quad \text{and} \quad y_{n+1} = y_n - 2y_n^2 \quad (n \ge 0).$$

Iterations $\{\bar{y}_n\}$, $\{y_n\}$ $(n \ge 0)$ are accelerations of $\{x_n\}$ $(n \ge 0)$ with $y \le \bar{y}_n \le x_n$ $(n \ge 0)$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \bar{y}_n = \lim_{n\to\infty} x_n = z = 0$, which justifies (6).

In what follows we tabulate the first seven iterates for each iteration for comparison:

$$\begin{aligned} &x_0 = .2, \quad x_1 = .16, \quad x_2 = .1344, \quad x_3 = .1163366, \quad x_4 = .1028024, \\ &x_5 = .0922341, \quad x_6 = .083727, \quad x_7 = .0767168, \\ &\bar{y}_0 = .2, \quad \bar{y}_1 = .12, \quad \bar{y}_2 = .1088, \quad \bar{y}_3 = .0982733, \quad \bar{y}_4 = .0892682, \\ &\bar{y}_5 = .0816657, \quad \bar{y}_6 = .0752198, \quad \bar{y}_7 = .0697066, \\ &y_0 = .2, \quad y_1 = .12, \quad y_2 = .0912, \quad y_3 = .0745651, \quad y_4 = .0634452, \\ &y_5 = .0553946, \quad y_6 = .0492575, \quad y_7 = .0444049. \end{aligned}$$

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