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When is a Poset Isomorphic to the Poset of Connected Induced Subgraphs of a Graph?

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Abstract

Suppose G is a finite connected graph. Let $C(G)$ denote the inclusion ordering on the connected vertex-induced subgraphs of G. Resolving a problem posed by Steve Steinsaltz, we characterize posets that arise in this way. In particular, we show that, for a finite poset $\mathbf{P}$, there exists a finite graph G such that $C(G) \cong \mathbf{P}$ if and only if $\mathbf{P}$ is atom-height and properly semimodular. We present a similar characterization for infinite posets.

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1 Introduction

Suppose G is a finite connected graph. Let $C(G)$ denote the inclusion ordering on the connected vertex-induced subgraphs of G.

Partially ordered sets arising in this manner have come under study within the last ten years [7, 9, 8, 10]. In recent work with Michael Jacobson [5] the authors proved that these posets need not be Sperner. In this paper we solve a problem posed by S. Steinsaltz [9] by characterizing those posets that arise as the inclusion ordering on the connected vertex-induced subgraphs of some graph. Our characterization links commonly studied properties of posets with commonly studied properties of graphs.

In the next section we give preliminary definitions and notation. In section three, we prove that for a finite poset $\mathbf{P}$, there exists a finite graph G such that $C(G) \cong \mathbf{P}$ if and only if $\mathbf{P}$ is atom-height and properly semimodular. In section four we integrate the familiar lattice concepts of meet semidistributivity and local distributivity into this characterization. In section five we extend the characterization to infinite posets.

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2 Preliminaries

A *partially ordered set* (or *poset*) $\mathbf{P} = (P, \leq)$ is a set P of elements together with a binary relation $\leq$ on P that is reflexive, antisymmetric, and transitive. An element $x \in P$ is an *upper bound* for a subset $S \subseteq P$ if $s \leq x$ for all $s \in S$. An element $x \in P$ is a *least upper bound* for $S \subseteq P$ if x is an upper bound for S and, for any upper bound $z \in P$ for S, $x \leq z$. *Lower bounds* and *greatest lower bounds* for S are defined dually. If a subset $S \subseteq P$ has a least upper bound (greatest lower bound) then we denote it by $\bigvee S$ ($\bigwedge S$). A *lattice* is a poset in which every pair of elements has a least upper bound and greatest lower bound. If arbitrary subsets of elements have least upper bounds and greatest lower bounds, then the lattice is *complete*. A lattice is *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for all elements a, b, and c in the poset.

If the poset $\mathbf{P}$ has a unique least element, we will refer to it as 0. Given two elements a and b of a poset, if $a \leq b$ and $a \leq c \leq b$ implies $c \in \{a, b\}$, then we say that b *covers* a and write $a \prec b$ or $b \succ a$. If a poset $\mathbf{P}$ has a 0 and $0 \prec a \in P$, then a is said to be an *atom* of $\mathbf{P}$.

Two elements $a, b \in P$ are *comparable* if $a \leq b$ or $b \leq a$; otherwise they are *incomparable*. A subset of pairwise comparable elements is called a *chain*. A chain C is *saturated* if there does not exist a $z \in P - C$ such that $x \leq z \leq y$ for some $x, y \in C$. The *length* of a chain is one less than the number of elements in the chain. The *interval* between two comparable elements $a, b \in P$ is defined as $[a, b] = \{c \in P : a \leq c \leq b\}$. A poset is *graded* (or *ranked*) if, for all comparable pairs $a \leq b$, all saturated chains with least element a and greatest element b have the same length. In a finite, graded poset with 0, the *height* of the element p, denoted $\mathrm{ht}(p)$ is the length of a saturated chain from 0 to p.

Two posets $\mathbf{P}$ and $\mathbf{Q}$ are *isomorphic* if there exists an *order-preserving* bijection $\phi : P \to Q$; that is,

$$x \leq y \qquad \Leftrightarrow \qquad \phi(x) \leq \phi(y).$$

If $\mathbf{P}$ and $\mathbf{Q}$ are isomorphic, then we write $\mathbf{P} \cong \mathbf{Q}$.

A lattice $\mathbf{L} = (L, \leq)$ is *semimodular* if for any $a, b, c \in L$, $a \prec b$, $a < c$ and $b \not\leq c$ implies $c \prec b \vee c$. It is well-known that semimodular finite lattices are graded. The converse is not true.

Given an element p from a poset $\mathbf{P}$, the symbol $p\!\uparrow$ denotes the *principal filter* of $\mathbf{P}$ determined by p; that is, $p\!\uparrow = \{q \in P : p \leq q\}$. The set of atoms below an element $x \in P$ is denoted A_x. Hence $A_x = \{a \in P : a \leq x \text{ and } a \text{ is an atom of } \mathbf{P}\}$. A poset $\mathbf{P}$ is *atomic* if $x = \bigvee A_x$, for all $x \in P$.

We now give two definitions specific to this paper. A graded, atomic poset is *atom-height* if for all $x \in P$, $\mathrm{ht}(x) = |A_x|$. If $\mathcal{P}$ is a lattice property, then a poset is said to be *properly* $\mathcal{P}$ if every proper, principal filter is a lattice and satisfies $\mathcal{P}$.

3 A characterization of $C(G)$ posets

Suppose that $G = (V, E)$ is a (not necessarily finite) connected graph with vertex set V and edge set E. Let $C(G)$ denote the inclusion ordering on the connected vertex-induced subgraphs of G. In this section we characterize posets that arise as the inclusion ordering on the connected vertex-induced subgraphs of some finite graph.

Theorem 3.1 *Let* $\mathbf{P}$ *be a finite poset. The following are equivalent:*

(*i*) *There exists a finite, connected graph* G *such that* $\mathbf{P} \cong C(G)$,
(*ii*) $\mathbf{P}$ *is atom-height and properly semimodular.*

Our proof of Theorem 3.1 requires several lemmas.

We shall consistenly represent elements of $C(G)$ with lower case letters. Vertices of G are denoted by **bold** lower case letters. If $x \in C(G)$, the corresponding subset of V is denoted by V_x. In the special case when x is an atom of $C(G)$, $\mathbf{x}$ will denote the vertex of G corresponding to x. If $S \subset V(G)$, then the subgraph of G induced by S is denoted $G[S]$.

A few simple observations: for any $x \in C(G)$, $|V_x| = |A_x|$; and, $x \leq y$ in $C(G)$ if and only $V_x \subseteq V_y$. Suppose $0 \neq x \in C(G)$. For any y, z contained in $x\uparrow$, the induced subgraph $G[V_y \cup V_z]$ is connected because both $G[V_y]$ and $G[V_z]$ are connected, and $V_x \subseteq V_y \cap V_z$. Indeed, for any y, z contained in $x\uparrow$ (for $x > 0$), we have

$$\begin{aligned} V_{y \vee z} &= V_y \cup V_z & (1) \\ V_{y \wedge z} &= \max\{S : V_x \subseteq S \subseteq V_y \cap V_z \text{ and } G[S] \text{ is connected.}\} & (2) \end{aligned}$$

Observe that the maximum taken in (2) exists and does not depend upon the finiteness of G.

Lemma 3.2 *If* G *is a connected graph, then* $x \prec y$ *in* $C(G)$ *if and only if* $|V_y - V_x| = 1$ *and* $x \leq y$.

Proof. Suppose that $x \prec y$ in $C(G)$. Clearly $V_y \neq V_x$. If $|V_y - V_x| > 1$, then because V_y is connected and contains V_x, there must be some vertex $\mathbf{u}$ in G such that $H = V_x \cup \{\mathbf{u}\}$ induces a connected graph. No element of $C(G)$ could correspond to H because any such element $z \in C(G)$ must satisfy $x < z < y$, contradicting $x \prec y$. We conclude that $|V_y - V_x| = 1$.

Conversely, suppose that $|V_y - V_x| = 1$ and $x \leq y$. Any subset of V_y properly containing V_x must contain V_y. The result follows. □

Lemma 3.3 *If* G *is a finite, connected graph, then* $C(G)$ *is an atom-height poset.*

Proof. Suppose $u \in C(G)$. Because u is an upper bound for each element in A_u and $|V_u| = |A_u|$, it follows that $u = \bigvee A_u$. So $C(G)$ is atomic.

Suppose $x, y \in C(G)$ with $x < y$. By lemma 3.2, $x \prec y$ if and only if $|V_y - V_x| = 1$. By straight-forward induction on $|V_x|$, it also follows that any saturated chain from 0 to x has length $|V_x| = |A_x|$. In other words, $C(G)$ is graded, and for $x \in C(G)$, $\mathrm{ht}(x) = |A_x|$. This proves that $C(G)$ is atom-height. □

Lemma 3.4 *Suppose that G is a connected graph. For any $0 \neq x \in C(G)$, the principal filter generated by x is a complete semimodular lattice.*

Proof. Suppose $0 \neq x \in C(G)$. For any $\{x_i\}_{i\in I} \subseteq x\uparrow$, the join $\bigvee_{i\in I} x_i$ exists because $G\left[\bigcup_{i\in I} V_{x_i}\right]$ is connected. The presence of a unique least element in $x\uparrow$ (namely x) and the existence of arbitrary joins guarantees the existence of arbitrary meets: $\bigwedge_{i\in I} x_i = \bigvee\{y \in x\uparrow : y \leq x_i \text{ for all } i \in I\}$. Hence $x\uparrow$ is a complete lattice.

To show that $x\uparrow$ is semimodular, suppose that $i, j, k \in x\uparrow$ and $i \prec j$, $i \leq k$, $j \not\leq k$. Now $V_i \subseteq V_j$ and $V_i \subseteq V_k$. Because $i \prec j$, lemma 3.2 implies $|V_j - V_i| = 1$. From (1) we know that the vertex set of $j \vee k$ is $V_j \cup V_k$. Because $|V_j \cup V_k - V_k| = 1$, lemma 3.2 guarantees that $k \prec k \vee j$ as desired. □

Observe that this last result actually does not depend upon the finiteness G. Indeed we shall use this result later in the case that G is infinite.

Consider a finite, atom-height, properly semimodular poset $\mathbf{P}$. From $\mathbf{P}$ construct a graph G_P whose vertices are the atoms of $\mathbf{P}$, and whose edges are pairs of distinct atoms $a, b \in P$ for which the join $a \vee b$ exists and has height two. In the following two lemmas, we show that there is a natural isomorphism $\mathbf{P} \cong C(G_P)$.

Lemma 3.5 *Suppose that $\mathbf{P}$ is an atom-height, properly semimodular poset. If $x \in P$, then $G_P[A_x]$ is connected.*

Proof. The proof is an induction on the $\mathrm{ht}(x)$. Recall that A_x is the set of atoms of $\mathbf{P}$ less than or equal to x. If $\mathrm{ht}(x) = 1$, then $A_x = \{x\}$ and so the subgraph of G_P induced by this set is clearly connected. Similarly, if $\mathrm{ht}(x) = 2$, then $|A_x| = 2$ because $\mathbf{P}$ is atom-height. Let $\{a, b\} = A_x$. Since $\mathbf{P}$ is atomic, $x = \bigvee A_x = a \vee b$. By definition of the graph G_P, $\mathbf{ab}$ is an edge of G_P; hence $G[\{\mathbf{a}, \mathbf{b}\}]$ is connected.

Suppose that the lemma is true for all elements of P with height at most n. Let $x \in P$ with $\mathrm{ht}(x) = n + 1 > 2$.

We claim that if $w \in P$ and $\mathrm{ht}(w) \geq 2$, then either w is the unique largest element of P or $|\{y \in P : y \prec w\}| \geq 2$. If not, a non-maximum element $w \in P$ exists with $\mathrm{ht}(w) \geq 2$ and $|\{y \in P : y \prec w\}| \leq 1$. Let y be the unique element covered by w. Now $A_w = A_y$, but because P is atomic $w = \bigvee A_w = \bigvee A_y = y$, contradicting $y \prec w$.

By the previous paragraph, we may assume that there are two distinct elements $w, y \in P$ such that $w \prec x$ and $y \prec x$. Because $\mathbf{P}$ is atom-height, $|A_w| = |A_y| = n$. By the induction hypothesis, the subgraphs of G_P induced by the vertex sets A_w and A_y are connected. Because $|A_x| = n + 1$, $A_w \cup A_y = A_x$ and $|A_w| = |A_y| = n > 2$, we conclude that $A_w \cap A_y \neq \emptyset$. Hence the subgraph of G_P induced by $A_w \cup A_y = A_x$ is connected. □

Lemma 3.6 *Suppose that $\mathbf{P}$ is an atom-height, properly semimodular poset. If $\mathcal{S}$ is a set of atoms of $\mathbf{P}$ and $G_P[\mathcal{S}]$ is connected, then there exists some $x \in P$ such that $A_x = \mathcal{S}$.*

Proof. The proof is by induction on $|\mathcal{S}|$. If $|\mathcal{S}| = 1$, then $\mathcal{S} = \{a\}$, for some atom $a \in P$; hence $\mathcal{S} = A_a$. If $|\mathcal{S}| = 2$, then $\mathcal{S} = \{a, b\}$, for some pair of atoms $a, b \in P$ such that **ab** is an edge of G_P. Because **ab** is an edge of G_P, the element $a \vee b$ exists and has height 2. By the atom-height property, $A_{a\vee b} = \{a, b\} = \mathcal{S}$.

Suppose the lemma is true for all sets of atoms with at most $n \geq 2$ elements. Consider a set $\mathcal{S}$ of $n+1$ atoms from P such that $G_P[\mathcal{S}]$ is connected. By considering a spanning tree of $G_P[\mathcal{S}]$, we can find two leaves of $G_P[\mathcal{S}]$, and hence we can find two atoms a_1, a_2 of P such that $G_P[\mathcal{S} - \{\mathbf{a}_1\}]$, $G_P[\mathcal{S} - \{\mathbf{a}_2\}]$, and $G_P[\mathcal{S} - \{\mathbf{a}_1, \mathbf{a}_2\}]$ are all connected graphs. Note that $n \geq 2$ implies that $G_P[\mathcal{S} - \{\mathbf{a}_1, \mathbf{a}_2\}]$ contains at least one vertex. By the induction hypothesis, there exist $x, y, z \in P$ such that $\mathcal{S} - \{a_1\} = A_x$, $\mathcal{S} - \{a_2\} = A_y$, and $\mathcal{S} - \{a_1, a_2\} = A_z$. Because **P** has the atom-height property, $\text{ht}(x) = n = \text{ht}(y)$, $z \prec x$, and $z \prec y$. In particular, $z \neq 0$ so the proper semimodularity of P implies $z\!\uparrow$ is a semimodular lattice. Hence, $x \prec x \vee y$, $y \prec x \vee y$, and $\text{ht}(x \vee y) = n+1 = |A_{x\vee y}|$. Observing that $A_x \subset A_{x\vee y}$ and $A_y \subset A_{x\vee y}$, we conclude that $A_{x\vee y} = \mathcal{S}$. □

<u>Theorem 3.1</u>: *Proof.* By Lemma 3.3 and Lemma 3.4, if G is a finite connected graph, then $C(G)$ is an atom-height, properly semimodular poset.

Conversely, suppose that **P** is an atom-height, properly semimodular poset. By Lemma 3.5, the mapping $\phi : P \to C(G_P)$ defined by $\phi(x) = A_x$ is well-defined. By Lemma 3.6, the mapping ϕ is onto $C(G_P)$. Because **P** is atomic, $A_x = A_y$ implies that $x = y$ so ϕ is one-to-one. If $x \leq y$, then $A_x \subseteq A_y$. Conversely, if $A_x \subseteq A_y$, then $x \leq y$ because P is atomic. Therefore, $C(G_P) \cong \mathbf{P}$. □

4 $SD_\wedge$ and Local Distributivity

Meet semidistributivite ($SD_\wedge$) and locally distributive lattices often arise when graph theory and combinatorics is the background setting (see [4, 6, 8]). We now integrate such lattices into our description of $C(G)$ posets.

A lattice $\mathbf{L} = (L, \leq)$ is *meet semidistributive* ($SD_\wedge$) if $a \wedge b = a \wedge c$ implies $a \wedge b = a \wedge (b \vee c)$, for all elements a, b, c, d from the lattice.

Lemma 4.1 *Let G be a connected simple graph. The poset $C(G)$ is properly $SD_\wedge$.*

Proof. Suppose that $0 \neq u \in C(G)$. We may assume that u is an atom of $C(G)$. Consider $u\!\uparrow$, and let $x, y, z \in u\!\uparrow$ such that $x \wedge y = x \wedge z$. We must prove that $x \wedge (y \vee z) = x \wedge y$, or equivalently $V_{x\wedge(y\vee z)} = V_{x\wedge y}$. The vertex set $V_{x\wedge y}$ is the maximal set in

$$\{S \subset V(G) : U \subseteq S \subseteq V_x \cap V_y \text{ and } G[S] \text{ is connected.}\}.$$

as noted in (2). Because $V_{x\wedge y} = V_{x\wedge z}$, it follows that the vertex set $V_{x\wedge y}$ is contained in $V_x \cap V_y \cap V_z$. Because of (1), the vertex set $V_{y\vee z}$ is $V_y \cup V_z$. This means that $V_{x\wedge(y\vee z)}$ is a subset of $V_x \cap (V_y \cup V_z) = (V_x \cap V_y) \cup (V_x \cap V_z)$.

Let $\{\mathbf{u}\} = V_u$, and let $\mathbf{v} \in V_{x\wedge(y\vee z)}$. By symmetry, we may assume that $\mathbf{v} \in V_x \cap V_y$. Because $\mathbf{u}$ and $\mathbf{v}$ are in $V_{x\wedge(y\vee z)}$, there must be a path from $\mathbf{u}$ to $\mathbf{v}$ using only vertices in this set. Let

$$\mathbf{u} = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_n = \mathbf{v}$$

be the sequence of vertices in this path, where $\mathbf{v}_i \in (V_x \cup V_y) \cup (V_x \cap V_z)$ and $\mathbf{v}_i\mathbf{v}_{i+1} \in E(G)$ for each i. Note that both $\mathbf{u} = \mathbf{v}_1$ and $\mathbf{v} = \mathbf{v}_n$ are vertices of $V_x \cap V_y$.

The subgraph of $G[V_u \cup \{\mathbf{v}_2\}]$ is connected and contains $\mathbf{u}$. Suppose that w is the element of $u\!\uparrow$ in $C(G)$ corresponding to this induced subgraph. If $\mathbf{v}_2 \in V_x \cap V_y$, then it follows that $w \leq x \wedge y = x \wedge z$. In particular, $\{\mathbf{u}, \mathbf{v}_2\} \subseteq V_x \cap V_z$. We have shown that $\mathbf{v}_2 \in V_x \cap V_y$ implies that $\mathbf{v}_2 \in V_x \cap V_z$. By induction, it follows that $\mathbf{v} \in V_x \cap V_y \cap V_z$. That is, $V_{x\wedge(y\vee z)}$ is contained in $V_x \cap V_y \cap V_z$ and contains $\mathbf{u}$. Maximal among connected vertex subsets containing $\mathbf{u}$ and contained in $V_x \cap V_y \cap V_z$ is $V_{x\wedge y\wedge z} = V_{x\wedge y} = V_{x\wedge z}$. Hence, $x \wedge (y \vee z) \leq x \wedge y = x \wedge z$. Consequently, $x \wedge (y \vee z) = x \wedge y$, as desired. □

Let $\mathbf{L} = (L, \leq)$ be a lattice. For $a \in L$, define $C_a = \{b \in L : a \prec b\}$. Let $a^+ = \bigvee C_a$. If for all $a \in L$, the interval $[a, a^+]$ is distributive, then L is said to be *locally distributive.* (In the literature, locally distributive lattices are sometims called *upper locally distributive.* See the paper by Monjardet [6] for more information.) The following result is well-known.

Proposition 4.2 *If* $\mathbf{L}$ *is a finite lattice, then* $\mathbf{L}$ *is locally distributive if and only if* $\mathbf{L}$ *is both semimodular and* $SD_\wedge$.

Here is an extension of Theorem 3.3.

Theorem 4.3 *For a finite poset* $\mathbf{P}$ *the following are equivalent:*

- (i) *There exists a connected graph* G *such that* $\mathbf{P} \cong C(G)$*,*
- (ii) $\mathbf{P}$ *is an atom-height, properly semimodular poset,*
- (iii) $\mathbf{P}$ *is an atom-height, properly semimodular, and properly* $SD_\wedge$ *poset,*
- (iv) $\mathbf{P}$ *is an atom-height, and properly locally distributive poset.*

Proof. By Theorem 3.3, (i) and (ii) are equivalent. By Lemma 4.1 and Theorem 3.3, (i) implies (iii). Proposition 4.2 and Theorem 3.3 shows that (iv) implies (i). □

Using (1) and Lemma 3.2, one can show directly that, for any graph $C(G)$ is locally distributive. Infinte locally distributive lattices are not in general semimodular or $SD_\wedge$; hence, for infinite graphs, Lemmas 3.4 and 4.1 are not direct consequences of the local distributivity of $C(G)$ as they are in the finite case via Proposition 4.2.

5 The Infinite Case

In this section we characterize those infinite partial orders that arise as the inclusion ordering on the connected vertex-induced subgraphs of an infinite graph. It is natural to try to reduce the infinite case to the finite case via compactness. Here, compactness arises in the context of lattices, and we are able to describe $C(G)$ posets (G infinite), using Theorem 4.3 (the finite case) along with a compactness condition In this way we avoid having to deal with infinite cardinals.

By a connected infinite graph G we mean an infinite graph in which, for any pair of vertices u and v of G, there exists a finite connected subgraph of G containing u and v. A poset is *graphic* if it is isomorphic the inclusion ordering on the connected vertex-induced subgraphs of a graph.

It makes sense here to define the *height* of an element in a poset with a 0 to be the length of the *shortest* saturated chain from 0 to that element (Note that the usual definition of height of an element is the length of the *longest* saturated chain from 0 to that element). With this definition of height, we say that an infinite poset $\mathbf{P}$ with 0 is *finitely graphic* if for any element $a \in P$ with finite height, the interval $[0, a]$ is graphic.

Recall that a lattice $\mathbf{L} = (L, \leq)$ is *complete* if $\bigvee S$ and $\bigwedge S$ exists for all $S \subseteq L$. Suppose that $\mathbf{L}$ is a complete lattice. An element $k \in L$ is *compact* if, for every $S \subseteq L$, $k \leq \bigvee S$ implies that $k \leq \bigvee T$, for some finite subset $T \subset S$. A lattice $\mathbf{L}$ is *algebraic* if, for each $a \in L$,

$$a = \bigvee \{k \in L | k \text{ is compact and } k \leq a\}.$$

Now we are ready to state the characterization in the infinite case.

Theorem 5.1 *Suppose that* $\mathbf{P}$ *is a poset with* 0*. The following are equivalent:*

(i) $\mathbf{P} \cong C(G)$*, for some connected graph* G

(ii) $\mathbf{P}$ *satisfies the following three conditions:*

- (a) $\mathbf{P}$ *is finitely graphic*
- (b) $\mathbf{P}$ *is atomic*
- (c) *for* $0 < x \in P$*,* $x\!\uparrow$ *is an algebraic semimodular lattice whose compact elements are the elements of finite height.*

Proof. "(i) $\Rightarrow$ (ii)" Suppose that $\mathbf{P}$ is a poset with 0, and G is a connected graph such that $\mathbf{P} \cong C(G)$. For an element $x \in P$ with finite height, the interval $[0, x]$ in P is isomorphic to $C(X)$, where X is the subgraph of G corresponding to x. Hence P is finitely graphic.

For an arbitrary element $0 < x \in P$, the connected subgraph of G corresponding to x has a vertex set that is simply the union of its vertices. Therefore P is atomic. The completeness and semimodularity of $x\!\uparrow$ follows from Lemma 3.4. Because every element of $x\!\uparrow$ corresponds to the vertex set of a connected subgraph of G containing x, and such vertex sets are unions of finite subgraphs containing x, if follows that the compact elements of $x\!\uparrow$ are precisely the elements of finite height. The filter $x\!\uparrow$ is algebraic for the same reasons.

"$(ii) \Rightarrow (i)$" Suppose that P is a poset with 0 satisfying conditions $(a), (b)$, and (c). Recall that, given a poset $\mathbf{P}$, we define the graph G_P whose vertices are the atoms of $\mathbf{P}$, and whose edges are pairs of distinct atoms $a, b \in P$ for which the join $a \vee b$ exists and has height two. We prove (ii) implies (i) by showing $\mathbf{P} \cong G_P$ in two steps. The first step is an analog of Lemma 3.5.

Step 1: For all $w \in P$, $G_P[A_w]$ is connected.

If w has finite height, then condition (a) implies that $G_P[A_w]$ is connected. Assume w does not have finite height. By condition (b), w is the join of atoms. In particular, w is in the filter of some atom a, so condition (c) guarantees that $w = \bigvee_{i\in I} p_i$, where the p_i's have finite height in $a\uparrow$. Consequently, the p_i's have finite height in $\mathbf{P}$. Let A_i be the set of atoms below p_i, for $i \in I$. Condition (a) guarantees that each $G_P[A_i]$ is connected. Hence,

$$G_P[A_w] = G_P[\bigcup_{i\in I} A_i]$$

The last subgraph is connected because it is the union of connected graphs containing the vertex $\mathbf{a}$.

The next step is an analog of Lemma 3.6.

Step 2: Suppose that C is a set of atoms in $\mathbf{P}$. If $G_P[C]$ is connected, then some $x \in P$ exists such that $A_x = C$.

Suppose first that $|C|$ is finite. We prove the result by induction on $|C|$. The basis cases are $|C| = 1, 2$. If $|C| = 1$, then $C = A_a$ where $C = \{a\}$. If $|C| = 2$, then $C = \{a, b\}$ for atoms $a \neq b$ of $\mathbf{P}$ such that $a \vee b$ exists and $a \vee b$ has height two. Condition (a) implies that $[0, a \vee b]$ is graphic and, consequently $C = A_{a\vee b}$. If $|C| \geq 3$, then $G_P[C]$ contains a vertex a such that C is the disjoint union of two sets C_1 and C_2 with $C_1 \cap C_2 = \{a\}$ and $G_P[C_i]$ is connected for $i = 1, 2$ (consider a spanning tree T of $G_P[C]$, and let b be a leaf T with neighbor a. Let $C_1 = \{a, b\}$ and let $C_2 = C - \{b\}$.) By the inductive hypothesis, $C_1 = A_w$ and $C_2 = A_z$, where $w, z \geq a$ and both w and z have finite height. Because $a\uparrow$ is semimodular, it follows that $w \vee z$ has finite height. Therefore $A_{w\vee z}$ contains C. Moreover, $G_P[C]$ is a connected subgraph of $G_P[A_{w\vee z}]$ so $C = A_{w\vee z}$.

Suppose now that $|C|$ is infinite. Let $a \in C$ and consider $a\uparrow$. Define

$$\mathcal{C} = \{S \subset C : |S| \text{ is finite, } G_P[S] \text{ is connected, } a \in S\}.$$

By the previous paragraph, for all $S \in \mathcal{C}$, an element $s \in P$ exists such that $S = A_s$ and $s \geq a$. Because $a\uparrow$ is complete, the join $\bigvee\{s \in P | A_s \in \mathcal{C} \text{ and } s \geq a\}$ exists; let $w \in P$ equal this join. We want to show $A_w = C$. Clearly $C \subseteq A_w$. It suffices to show $A_w \subseteq C$. Step 1 guarantees that $G_P[A_w]$ is connected. Furthermore, $G_P[A_w]$ contains C. Suppose, for a contradiction, that there exists some element $b \in A_w - C$. We may assume that $\mathbf{b}$ has a neighbor $\mathbf{x}$ in $G_P[A_w]$ such that $x \in C$. Consider the set $\{s \in P | A_s \in \mathcal{C} \text{ and } s \geq a \text{ and } s \geq x\} = \{w_i\}_{i\in I}$. Note that $\bigvee_{i\in I} w_i = w$ because every $s \in P$ such that $A_s \in \mathcal{C}$ and $s \geq a$ is below some w_i. By assumption, $b \in A_w$. Hence $b \leq w = \bigvee_{i\in I} w_i$. Consequently,

$$\begin{aligned} x \vee b &\leq x \vee \left(\bigvee_{i\in I} w_i\right) \\ &\leq \bigvee_{i\in I} (x \vee w_i) \end{aligned}$$

$$
\begin{aligned}
&\leq \bigvee_{i\in I} w_i && \text{(because } x \leq w_i, \text{ for all } i \in I) \\
&\leq \bigvee_{i\in J\subset I, |J|<\infty} w_i && \text{(because } x{\uparrow} \text{ is compact)} \\
&\leq v
\end{aligned}
$$

where $v = \bigvee\{w_i : i \in J\}$. The semimodularity of $x{\uparrow}$ ensures that v has finite height. Because v has finite height, it is graphic. Hence $A_v \in \mathcal{C}$, and $b \in C$. This shows that $A_w = C$.

The theorem now follows from steps 1 and 2 by the same reasoning as the proof of Theorem 2.1 (define a mapping $\phi : P \to C(G_P)$ via $\phi(x) = A_x$ and show it is an order-preserving bijection). $\square$

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