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COMMON FIXED POINTS FOR δ-COMPATIBLE MAPPINGS

R.A.RASHWAN AND M.A.AHMED

ABSTRACT. In this paper, using the condition of δ-compatibility between a set-valued mapping and a single-valued mapping due to Jungck and Rhoades [6], we generalize a common fixed point theorem of Khan and Kubiaczyk [7] in a complete metric space. Their Theorem extends well known results due to Chang [1], Khan et al. [8], and Singh-Whitfield [13]. We give an example to support our extension.

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1. Basic Preliminaries.

Let (X,d) be a metric space, and B(X) be the set of all nonempty, bounded subsets of X.

As in [3,4], let $\delta(A, B)$ be the function defined by

$$\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\},$$

for all A, B in B(X).

If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$. If B also consists of a single point b, we write $\delta(A, B) = d(a, b)$.

It follows immediately from the definition that

$$\begin{aligned}
&\delta(A, B) = \delta(B, A) \geq 0,\\
&\delta(A, B) \leq \delta(A, C) + \delta(C, B),\\
&\delta(A, B) = 0 \quad iff \quad A = B = \{a\},\\
&\delta(A, A) = diamA,
\end{aligned}$$

for all A, B, C in B(X).

Department of mathematics, Faculty of science, Assiut University, Assiut, EGYPT
E-mail Address: Assiut@frcu.enu.eg

Definition 1.1 [3]. A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if

(i) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for n=1,2, ..., and a_n converges to a.

(ii) given $\epsilon > 0$ there exists a positive integer N such that $A_n \subseteq A_\epsilon$ for $n > N$ where A_ϵ is the the union of all open spheres with centers in A and radius ϵ.

Lemma 1.1 [3,4]. If $\{A_n\}$ and $\{B_n\}$ are sequences in B(X) converging to A and B in B(X) respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.

Lemma 1.2 [4]. Let $\{A_n\}$ be a sequence in B(X) and y be a point of X such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in B(X).

Definition 1.2 [4,12]. A set-valued mapping F of X into B(X) is said to be continuous at $x \in X$ if the sequence $\{Fx_n\}$ in B(X) converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X. F is said to be continuous on X if it is continuous at every point of X.

The following Lemma was proved in [4].

Lemma 1.3. Let $\{A_n\}$ be a sequence of nonempty subsets of X and let $z \in X$ such that

$$\lim_{n\to\infty} a_n = z,$$

z being independent of the particular choice of each $a_n \in A_n$. If a selfmap I of X is continuous, then Iz is the limit of the sequence $\{IA_n\}$.

Definition 1.3 [12]. The mappings $F : X \to B(X)$ and $I : X \to X$ are said to be weakly commuting on X if $IFx \in B(X)$ and

$$\delta(FIx, IFx) \leq max\{\delta(Ix, Fx), \quad diamIFx\} \tag{1.1}$$

for all x in X.

Note that if F is a single-valued mapping, then the set $\{IFx\}$ consists of a single point and therefore diam IFx=0 for all $x \in X$ and condition (1.1) therefore reduces to the condition given by Sessa [11], that is

$$d(FIx, IFx) \leq d(Ix, Fx) \tag{1.2}$$

for all x in X.

Two commuting mappings F and I clearly commute but two weakly commuting F and I do not necessarily commute as shown in [12].

In a recent paper Jungck [5] made an extension of weak commutativity in the following way

Definition 1.4.

Two single-valued mappings f and g of a metric space (X, d) into itself. are *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some z in X.

It can be seen that two *weakly commuting mappings* are *compatible* but the converse is false. Examples supporting this fact can be found in [5].

In [6], Jungck and Rhoades extended the definition 1.4 of compatibility to a set-valued setting as follows.

Definition 1.5.

The mappings $I : X \to X$ and $F : X \to B(X)$ are *weakly δ-compatible* if they commute at coincidence points, i.e., for each point u such that Fu=Iu, we have FIu=IFu. (Note that the equation Fu=Iu implies that Fu is a singleton.)

Definition 1.6. The mappings $I : X \to X$ and $F : X \to B(X)$ are *δ-compatible* if $\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in B(X)$,

$$Fx_n \to \{t\} \quad Ix_n \to t$$

for some t in X.

Two *weakly commuting mappings* are *δ-compatible* but the converse is not true as is shown in the following example.

Example 1.1. Let $X = [0, \infty)$ with the Euclidean metric d and define

$$Ix = x^3 + 2x^2, \quad Fx = [0, x^2]$$

for all x in X. We have

$$Ix_n \to 0, \quad Fx_n \to \{0\} \quad if \quad x_n \to 0$$

and

$$\delta(FIx_n, IFx_n) \to 0 \quad as \quad x_n \to 0.$$

Also $IFX_n \in B(X)$, we see that I and F are δ-compatible. But I and F are not weakly commuting, hence are not commuting because for x=1, we have

$$\delta(FI1, IF1) = 9 > 3 = max\{\delta(I1, F1), diamIF1\},$$

and

$$FI1 = [0, 9] \neq [0, 3] = IF1.$$

The following proposition due to Jungck and Rhoades [6] is useful in the sequel.

Proposition 1.1. Let (X,d) be a complete metric space. Suppose $I : X \to X$ and $F : X \to B(X)$, and (I,F) is δ-compatible,

(1) Suppose that the sequence $\{Fx_n\}$ converges to $\{t\}$ and $\{Ix_n\}$ converges to t. If I is continuous, then $FIx_n \to \{It\}$.

(2) If $\{It\} = Ft$ for some $t \in X$, then FIt=IFt.

Throughout this paper, (X,d) stands for a complete metric space. Further Φ is the set of functions $\phi : [0, \infty)^5 \longrightarrow [0, \infty)$ which satisfies the following conditions

(i) ϕ is upper semi-continuous from the right and non-decreasing in each coordinate variable,

(ii) For each $t > 0$,

$$\Psi(t) = max\{\phi(t, t, t, t, t), \phi(t, t, t, 2t, 0), \phi(t, t, t, 0, 2t)\} < t.$$

The following Lemma is due to Matkowski [9].

Lemma 1.4. For every $t > 0$,

$$\Psi(t) < t \quad iff \quad \lim_{n\to\infty} \Psi^n(t) = 0.$$

M.S.Khan and I.Kubiaczyk [7] proved the following Theorem.

Theorem 1.1. Let $I : X \to X$ and $F, G : X \to B(X)$ such that
(I) I, F and G are continuous,

$$FX \subset IX \quad and \quad GX \subset IX, \tag{1.3}$$

$$IF = FI \quad and \quad IG = GI, \tag{1.4}$$

For all x, y in X

$$\delta(Fx, Gy) \leq \phi(d(Ix, Iy), \delta(Ix, Fx), \delta(Iy, Gy), D(Ix, Gy), D(Iy, Fx)), \tag{1.5}$$

where ϕ satisfies (i) and

$$\phi(t, t, t, at, bt) < t,$$

for each $t > 0$, and $a \geq 0$, $b \geq 0$ with $a + b \leq 2$. Then I, F and G have a unique common fixed point u such that $u = Iu \in Fu \cap Gu$.

Now there arises a natural question. Is it possible to replace the commutativity of mappings in Theorem 1.1 by the δ-compatibility of a pair of mappings (Definition 1.6) ? The answer is in the affirmative.

2.Main results.

As an improvement of Theorem 1.1 we have the following.

Theorem 2.1. Let $I, J : X \to X$ and $F, G : X \to B(X)$ be mappings such that

$$FX \subset JX \quad and \quad GX \subset IX, \tag{2.1}$$

$$\delta(Fx, Gy) \leq \phi(d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)), \tag{2.2}$$

for all $x, y \in X$ and $\phi \in \Phi$, either

(II) $\{F, I\}$ are δ -compatible, F or I is continuous and (G,J) are weakly δ -compatible, or

(III)$\{G, J\}$ are δ -compatible, G or J is continuous and $\{F, I\}$ are weakly δ -compatible.

Then I, J, F and G have a unique common fixed point u. Moreover $Fu = Gu = \{u\}$ is the unique common fixed point of F and I and of G and J.

Proof. Let $x_0 \in X$ be an arbitrary point in X. By (2.1) we choose a point x_1 in X such that $Jx_1 \in Fx_0 = Z_0$ and for this point x_1 there exists a point x_2 in X such that $Ix_2 \in Gx_1 = Z_1$, and so on. Continuin
g in this manner we can define a sequence $\{x_n\}$ as follows:

$$Jx_{2n+1} \in Fx_{2n} = Z_{2n+1}, Ix_{2n+2} \in Gx_{2n+1} = Z_{2n+2}. \tag{2.3}$$

For simplicity, we Put $V_n = \delta(Z_n, Z_{n+1})$, for n=0, 1, 2 ... By (2.2) and (2.3), we have

$$\begin{aligned} V_{2n+1} &= \delta(Z_{2n+1}, Z_{2n+2}) = \delta(Fx_{2n}, Gx_{2n+1}) \\ &\leq \phi(d(Ix_{2n}, Jx_{2n+1}), \delta(Ix_{2n}, Fx_{2n}), \delta(Jx_{2n+1}, Gx_{2n+1}), \\ &D(Ix_{2n}, Gx_{2n+1}), D(Jx_{2n+1}, Fx_{2n})) \\ &\leq \phi(V_{2n}, V_{2n+1}, V_{2n}, V_{2n} + V_{2n+1}, 0). \end{aligned}$$

If $V_{2n+1} > V_{2n}$, then we have

$$V_{2n+1} \leq \phi(V_{2n+1}, V_{2n+1}, V_{2n+1}, 2V_{2n+1}, 0) \leq \Psi(V_{2n+1}) < V_{2n+1},$$

which ia a contradiction. Thus,

$$V_{2n+1} \leq \phi(V_{2n}, V_{2n}, V_{2n}, 2V_{2n}, 0) \leq \Psi(V_{2n}). \tag{2.4}$$

Similarly, we have

$$V_{2n+2} \leq \Psi(V_{2n+1}). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$V_n \leq \psi(V_{n-1}) \leq ... \leq \Psi^n(V_0), \tag{2.6}$$

where

$$V_0 = \delta(Z_0, Z_1) = \delta(Fx_0, Gx_1) \leq \phi(V_0, V_0, V_0, 2V_0, 0).$$

By (2.6) and Lemma 1.4 we obtain

$$\lim_{n\to\infty} V_n = 0. \tag{2.7}$$

Let z_n be an arbitrary point in Z_n for n=0, 1, 2, ... Then

$$\lim_{n\to\infty} d(z_n, z_{n+1}) \leq \lim_{n\to\infty} \delta(Z_n, Z_{n+1}) = 0.$$

In order to show that $\{z_n\}$ is a Cauchy sequence, it is sufficient to show that $\{z_{2n}\}$ is a Cauchy sequence. Suppose that $\{z_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that, for each even integer 2k, there exist even integers 2m(k) and 2n(k) such that

$$\delta(Z_{2m(k)}, Z_{2n(k)}) > \epsilon \tag{2.8}$$

for $m(k) > n(k) \geq k$.

For each even integer 2k, let 2m(k) be the least even integer exceeding 2n(k) satisfying (2.8), that is

$$\delta(Z_{2n(k)}, Z_{2m(k)-2}) \leq \epsilon \quad and \quad \delta(Z_{2n(k)}, Z_{2m(k)}) > \epsilon \tag{2.9}$$

Then for each even integer 2k,

$$\epsilon < \delta(Z_{2n(k)}, Z_{2m(k)}) \leq \delta(Z_{2n(k)}, Z_{2m(k)-2}) + V_{2m(k)-2} + V_{2m(k)-1}.$$

The foregoing inequality implies

$$\lim_{n\to\infty} \delta(Z_{2n(k)}, Z_{2m(k)}) = \epsilon. \tag{2.10}$$

Also by the triangle inequality, we have

$$|\delta(Z_{2n(k)}, Z_{2m(k)-1}) - \delta(Z_{2n(k)}, Z_{2m(k)})| \leq V_{2m(k)-1},$$

and

$$|\delta(Z_{2n(k)+1}, Z_{2m(k)-1}) - \delta(Z_{2n(k)}, Z_{2m(k)})| \leq V_{2m(k)-1} + V_{2n(k)}.$$

Using (2.10) we get

$$\delta(Z_{2n(k)}, Z_{2m(k)-1}) \to \epsilon \tag{2.11}$$

and

$$\delta(Z_{2n(k)+1}, Z_{2m(k)-1}) \to \epsilon. \tag{2.12}$$

From (2.2) and (2.3), we have

$$\begin{aligned}
&\delta(Z_{2n(k)}, Z_{2m(k)}) \leq V_{2n(k)} + \delta(Fx_{2n(k)}, Gx_{2m(k)-1})\\
&\leq V_{2n(k)} + \phi(d(Ix_{2n(k)}, Jx_{2m(k)-1}), \delta(Ix_{2n(k)}, Fx_{2n(k)}),\\
&\delta(Jx_{2m(k)-1}, Gx_{2m(k)-1}), D(Ix_{2n(k)}, Gx_{2m(k)-1}), D(Jx_{2m(k)-1}, Fx_{2n(k)}))\\
&\leq V_{2n(k)} + \phi(\delta(Z_{2n(k)}, Z_{2m(k)-1}), V_{2n(k)}, V_{2m(k)-1},\\
&\delta(Z_{2n(k)}, Z_{2m(k)}), \delta(Z_{2m(k)-1}, Z_{2n(k)+1})).
\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\epsilon \leq \phi(\epsilon, 0, 0, \epsilon, \epsilon) \leq \Psi(\epsilon) < \epsilon,$$

which is a contradiction. Therefore the sequence $\{z_n\}$ is a Cauchy sequence in X and it has a limit u in X. So the sequences $\{Ix_{2n}\}$ and $\{Jx_{2n+1}\}$ converge to u and further, the sequences of sets $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ converge to the set $\{u\}$.

Now supose that I is continuous. Then

$$I^2x_{2n} \to Iu, IFx_{2n} \to \{Iu\}$$

by Lemma 1.3. But I and F are δ-compatible. Therefore $FIx_{2n} \longrightarrow \{Iu\}$ by proposition 1.1 (1).

Using inequality (2.2), we have

$$\begin{aligned}
&\delta(FIx_{2n}, Gx_{2n+1}) \leq \phi(d(I^2x_{2n}, Jx_{2n+1}), \delta(I^2x_{2n}, FIx_{2n}),\\
&\delta(Jx_{2n+1}, Gx_{2n+1}), D(I^2x_{2n}, Gx_{2n+1}), D(Jx_{2n+1}, FIx_{2n})).
\end{aligned}$$

for $n \in N$. As $n \to \infty$ we obtain by Lemma 1.1

$$d(Iu, u) \leq \phi(d(Iu, u), 0, 0, d(Iu, u), d(Iu, u)) \leq \Psi(d(Iu, u)) < d(Iu, u)$$

which is a contradiction. Thus Iu=u.

Further

$$\delta(Fu, Gx_{2n+1}) \leq \phi(d(Iu, Jx_{2n+1}), \delta(Iu, Fu),$$
$$\delta(Jx_{2n+1}, Gx_{2n+1}), D(Iu, Gx_{2n+1}), D(Jx_{2n+1}, Fu)).$$

Letting $n \longrightarrow \infty$, it follows by Lemma 1.1 that

$$\begin{aligned}\delta(Fu, u) &\leq \phi(0, \delta(Fu, u), 0, 0, \delta(u, Fu)) \\ &\leq \Psi(\delta(u, Fu)) < \delta(u, Fu),\end{aligned}$$

which is a contradiction. So $Fu = \{u\}$.

Since $F(X) \subset J(X)$, there exists a point $w \in X$ such that $\{u\} = Fu = \{Jw\}$.

We shall now show that $Gw = \{Jw\}$. From (2.2),

$$\delta(Fu, Gw) \leq \phi(d(Iu, Jw), \delta(Iu, Fu), \delta(Jw, Gw), D(Iu, Gw), D(Jw, Fu)),$$

or

$$\delta(Jw, Gw) \leq \phi(\delta(Jw, Gw)),$$

which implies that $Gw = \{Jw\}$. Thus Gw is a singleton and w is a coincidence point for G and J.

Since they are weakly δ -compatible, GJw=JGw.

It then follows that $\{Ju\} = JGw = GJw = Gu$. Using (2.2),

$$\delta(Fu, Gu) \leq \phi(d(Iu, Ju), \delta(Iu, Fu), \delta(Ju, Gu), D(Iu, Gu), D(Ju, Fu))$$

or

$$\delta(u, Ju) \leq \phi(\delta(u, Ju)),$$

which implies that u=Ju. Since $\{Ju\} = Gu$, u is a common fixed point of F,G, I and J.

The proof assuming the continuity of J is similar, as is the proof of (III).

Now suppose that F and I have a second common fixed point v such that $Fv = \{v\} = \{Iv\}$. Then, using inequality (2.2), we obtain

$$d(v, u) = \delta(Fv, Gu) \leq \phi(d(v, u), 0, 0, d(v, u), d(v, u)) \leq \Psi(d(v, u)) < d(v, u),$$

which is a contradiction. So v=u, and u is the unique common fixed point of F and I such that $Fu = \{u\}$.

Similarly, it can be shown that u is the unique common fixed point of G and J such that $Gu = \{u\}$.

Remark. In Theorem 2.1, if F and G are single-valued mappings, we obtain Theorem 2.2 of [10].

Now , we give an example to support our extension.

Example. Let $X=[0,\infty)$ with the Euclidean metric d. Define $F,G:X\to B(X)$ and $I,J:X\to X$ by

$$Fx=[0,\frac{x^4}{2}],Gx=[0,\frac{x^2}{2}]$$
$$Ix=x^4+2x^2,Jx=x^8+x^4+x^2$$

for all x, y in X. Now:

$$F(X)\subset I(X),G(X)\subset J(X),$$

and F, G, I and J are continuous. Moreover, for any sequence $\{x_n\}$ in X, we have

$$Ix_n\to 0,Fx_n\to\{0\}\quad as\quad x_n\longrightarrow 0,$$

$IFx_n\in B(X)$,

$$\delta(FIx_n,IFx_n)=max\{\frac{(x_n^4+2x_n^2)^4}{2},(\frac{x_n^4}{2})^2+2(\frac{x_n^4}{2})^2\}\to 0\quad as\quad x_n\to 0,$$

thus F and I are δ -compatible . Similarly, one can show that G and J are δ -compatible.

Assuming $\phi\in\Phi$ as

$$\phi(t_1,t_2,t_3,t_4,t_5)=\frac{1}{2}max\{t_2,t_3\}$$

for every $t_i\in R^+$, i=1,2,3,4,5.

We have for any $x,y\in X$

$$\begin{aligned}\delta(Fx,Gy)&=max\{\frac{x^4}{2},\frac{y^2}{2}\}\\&=\frac{1}{2}max\{x^4,y^2\}\\&\le\frac{1}{2}max\{x^4+2x^2,y^8+y^4+y^2\}\\&=\frac{1}{2}max\{\delta(Ix,Fx),\delta(Jy,Gy)\}\\&=\phi(d(Ix,Jy),\delta(Ix,Fx),\delta(Jy,Gy),\delta(Ix,Gy),\delta(Jy,Fx)).\end{aligned}$$

Since ϕ obviously satisfies the conditions (i) and (ii), all the assumptions of Theorem 2.1 are verified and 0 is the unique common fixed point of F, G, I and J.

Note that Theorem 1.1 is not applicable even if F=G and I=J because F and I are not weakly commuting mappings at x=2, for

$$\begin{aligned}&\delta(FI1,IF1)=\frac{81}{2}>3=max\{\delta(F1,I1),diamIF1\},\\&\delta(GJ1,JG1)=\frac{9}{2}>3=max\{\delta(G1,J1),diamJG1\},\\&FI1\neq IF1,GJ1\neq JG1.\end{aligned}$$

3. A common fixed point theorem in compact metric space.

Mativated by the paper [2], we have the following.

Theorem 3.1. Let (X,d) be a compact metric space and let I, J be functions from X into X, $F, G : X \to B(X)$ be two set-valued functions with $FX \subset JX$ and $GX \subset IX$ such that for all x, y in X and $\phi \in \Phi$, the inequality

$$\delta(Fx, Gy) < \phi(d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)),$$

holds whenever the right hand side of (3.1) is positive. If the pairs (F,I) and (G,J) are δ - compatible, and if the functions F and I are continuous, then there is a unique fixed point u in X such that

$$Fu = Gu = \{u\} = \{Iu\} = \{Ju\}.$$

Proof. Let $\eta = \inf_{x\in X} \delta(Ix, Fx)$. Since X is compact metric space, there is a convergent sequence $\{x_n\}$ with limit x_0 in X such that

$$\delta(Ix_n, Fx_n) \to \eta \quad as \quad n \to \infty.$$

Since

$$\delta(Ix_0, Fx_0) \leq d(Ix_0, Ix_n) + \delta(Ix_n, Fx_n) + \delta(Fx_n, Fx_0),$$

then by the continuity of F and I and $\lim_{n\to\infty} x_n = x_0$ we get $\delta(Ix_0, Fx_0) \leq \eta$ and thus $\delta(Ix_0, Fx_0) = \eta$.

Since $FX \subset JX$, then there exists a point y_0 in X with $Jy_0 \in Fx_0$ and $d(Ix_0, Jy_0) \leq \eta$.

If $\eta > 0$ then

$$\begin{aligned}\delta(Jy_0, Gy_0) &\leq \delta(Fx_0, Gy_0)\\ &< \phi(d(Ix_0, Jy_0), \delta(Ix_0, Fx_0),\\ &\delta(Jy_0, Gy_0), D(Ix_0, Gy_0), D(Jy_0, Fx_0))\\ &\leq \phi(\eta, \eta, \delta(Jy_0, Gy_0), \eta + \delta(Jy_0, Gy_0), 0).\end{aligned}$$

If $\delta(Jy_0, Gy_0) > \eta$ in the above inequality, then we have

$$\begin{aligned}&\delta(Jy_0, Gy_0)\\ &< \phi(\delta(Jy_0, Gy_0), \delta(Jy_0, Gy_0), \delta(Jy_0, Gy_0), 2\delta(Jy_0, Gy_0), 0)\\ &\leq \Psi(\delta(Jy_0, Gy_0)) < \delta(Jy_0, Gy_0),\end{aligned}$$

which is a contradiction. Thus $\delta(Jy_0, Gy_0) \leq \eta$.

Since $GX \subset IX$, then there is a point z_0 in X such that $Iz_0 \in Gy_0$ and $d(Iz_0, Jy_0) \leq \eta$. Hence we have

$$\begin{aligned}&\eta \leq \delta(Iz_0, Fz_0) \leq \delta(Fz_0, Gy_0)\\ &< \phi(d(Iz_0, Jy_0), \delta(Iz_0, Fz_0), \delta(Jy_0, Gy_0), D(Iz_0, Gy_0), D(Jy_0, Fz_0))\\ &\leq \phi(\eta, \delta(Iz_0, Fz_0), \eta, 0, \eta + \delta(Iz_0, Fz_0)). \qquad (3.1)\end{aligned}$$

If $\delta(Iz_0, Fz_0) > \eta$ in the last inequality, then we get

$$\begin{aligned}
&\eta \le \delta(Iz_0, Fz_0) \\
&< \phi(\delta(Iz_0, Fz_0), \delta(Iz_0, Fz_0), \delta(Iz_0, Fz_0), 0, 2\delta(Iz_0, Fz_0)) \\
&\le \Psi(\delta(Iz_0, Fz_0)) < \delta(Iz_0, Fz_0))
\end{aligned}$$

a contradiction. Thus $\delta(Iz_0, Fz_0) \le \eta$.

Substituting into (3.1) yields

$$\eta \le \phi(\eta, \eta, \eta, 0, 2\eta) \le \Psi(\eta) < \eta,$$

a contradiction. Therefore $\eta = 0$ and hence

$$Ix_0 = Jy_0 = Iz_0 = \{Gy_0\} = \{Fz_0\}.$$

If $\delta(Jy_0, Gy_0) > 0$, then

$$\begin{aligned}
&\delta(Jy_0, Gy_0) \le \delta(Fx_0, Gy_0) \\
&< \phi(d(Ix_0, Jy_0), \delta(Ix_0, Fx_0), \delta(Jy_0, Gy_0), D(Ix_0, Gy_0), D(Jy_0, Fx_0)) \\
&= \phi(0, 0, \delta(Jy_0, Gy_0), 0, 0) \le \Psi(\delta(Jy_0, Gy_0)) < \delta(Jy_0, Gy_0),
\end{aligned}$$

a contradiction. Therefore

$$\{Gy_0\} = \{Jy_0\} = \{Fx_0\} = \{Ix_0\}.$$

Since F and I are δ -compatible and $\delta(Ix_0, Fx_0) = 0$, we get $\delta(FIx_0, IFx_0) = 0$, and we get

$$\delta(F^2x_0, I^2x_0) = \delta(FIx_0, I^2x_0) = \delta(IFx_0, I^2x_0) = \delta(I^2x_0, I^2x_0) = 0.$$

We have

$$F^2x_0 = FIx_0 = IFx_0 = \{I^2x_0\}.$$

If $I^2x_0 \ne Ix_0$ then we have

$$\begin{aligned}
&d(I^2x_0, Ix_0) = \delta(F^2x_0, I^2x_0) \\
&< \phi(\delta(IFx_0, Jy_0), \delta(IFx_0, F^2x_0), \delta(Jy_0, Gy_0), D(IFx_0, Gy_0), D(Jy_0, F^2x_0) \\
&= \phi(d(I^2x_0, Ix_0), 0, 0, d(I^2x_0, Ix_0), d(I^2x_0, Ix_0) \\
&\le \Psi(d(I^2x_0, Ix_0) < d(I^2x_0, Ix_0),
\end{aligned}$$

which is a contradiction, and so $I^2x_0 = Ix_0$. Hence

$$FIx_0 = \{Ix_0\} = \{I^2x_0\}.$$

Similarly, we have

$$GJy_0 = \{Jy_0\} = \{J^2y_0\}.$$

Let $u = Ix_0 = Jy_0$. Then

$$Fu = \{u\} = \{Iu\} = \{Ju\} = Gu.$$

Suppose that the point y in X is a common fixed point of F,G, I and J with $y \ne u$

$$\begin{aligned}
&d(u, y) = \delta(u, y) = \delta(Fu, Gy) \\
&< \phi(d(Iu, Jy), \delta(Iu, Fu), \delta(Jy, Gy), D(Iu, Gy), D(Jy, Fu)) \\
&= \phi(d(u, y), 0, 0, d(u, y), d(u, y)) \le \Psi(d(u, y)) < d(u, y),
\end{aligned}$$

whence u is the unique common fixed point of F, G, I and J.

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