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SET-VALUED MAPS WITH FIXED AND COINCIDENCE POINTS

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Abstract. In this paper, we prove the existence of solutions of functional equations $f_ix \in Sx \cap Tx$ and $x = f_ix \in Sx \cap Tx$ under certain nonlinear hybrid contraction and asymptotic regularity conditions, generalize and improve a recent result due to V. Popa concerning common fixed points of multi-valued mappings weakly commuting with a single-valued mapping and satisfying a generalized contraction type condition. In the sequel, we derive several independent results.

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I. Introduction

In [16], Nadler proved a fixed point theorem for multi-valued contraction, commonly known as Nadler's contraction principle. Subsequently, a large number of generalizations of Nadler's contraction principle appeared in Čirič [1], Khan [10], Kubiak [12], [13], Kaneko [8], [9], Sessa [22]–[24], Meade-Singh [14], Smithon [27], Wegrzyk [28] and in many others.

Recently, non-linear hybrid contractions, that is, contraction types involving single-valued and multi-valued mappings have been studied by Khan et al. [11], Mukherjee [15], Naimpally et al. [17], Popa [18], Rhoades et al. [19] and Sessa et al. [24], [25].

In this note, we show the existence of solutions of functional equations $f_ix \in Sx \cap Tx$ and $x = f_ix \in Sx \cap Tx$ under certain independent non-linear hybrid contraction and asymptotic regularity conditions where $\{f_i\}$ is a family of single-valued mappings on a metric space, S and T are multi-valued mappings on a metric space. Our results are generalizations and improvement of some results of O. Hadžić, H. Kaneko, M.S. Khan et al., T. Kubiak, R.N. Mukherjee, S.A. Naimpally et al., B.E. Rhoades et al. and many others. In the sequel, we also obtain other related results by using the proximinality of sets.

II. Preliminaries

Let (X, d) be a metric space. A subset K of X is said to be proximinal if for each $x \in X$, there exists a point $y \in K$ such that $d(x, y) = D(x, K)$, where $D(x, A)$ denotes the ordinary distance between $x \in X$ and a non-empty subset A of X.

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We shall use the following notation and definitions:

$$\begin{aligned} CL &= \{A : A \text{ is a non-empty closed subset of } X\}, \\ CB(X) &= \{A : A \text{ is a non-empty closed and bounded subset of } X\}, \\ C(X) &= \{A : A \text{ is a non-empty compact subset of } X\} \text{ and} \\ P_x(X) &= \{A : A \text{ is a non-empty proximinal subset of } X\}. \end{aligned}$$

For $A, B \in CL(X)$ and $\varepsilon > 0$,
$N(\varepsilon, A) = \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}$,
$E_{A,B} = \{\varepsilon > 0 : A \subseteq N(\varepsilon, B) \text{ and } B \subseteq N(\varepsilon, A)\}$, and

$$H(A, B) = \begin{cases} inf\, E_{A,B}, & \text{if } E_{A,B} \neq \varnothing \\ +\infty & \text{if } E_{A,B} = \varnothing. \end{cases}$$

H is called the generalized Hausdorff distance function for $CL(X)$ induced by the metric d, and H defined on $CB(X)$ is said to be the Hausdorff-Pompeiu metric induced by d.

It is well known that $P_x(X) \subset CL(X)$ and $C(X) \subset P_x(X)$ [22].

Let $\{f_i\}$ be a family of single-valued mappings from X to itself and S, T be multi-valued mappings from X to the non-empty subsets of X.

Definition 2.1.
If, for $x_\circ \in X$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = f_{2n} x_{2n-1} \in S x_{2n} \text{ for every } n \in N,$$

and

$$y_{2n+1} = f_{2n+1} x_{2n} \in T x_{2n+1} \text{ for every } n \in N_\circ = N \cup \{0\},$$

then $0_{f_i}(x_\circ) = \{y_n : n = 0, 1, 2, \ldots\}$ is said to be the orbit for $(S, T : f_i)$ at $x_\circ$. Further, $0_{f_i}(x_\circ)$ is called a regular orbit for $(S, T; f)$ if

$$d(y_n, y_{n+1}) \leq \begin{cases} H(Sx_{n-1}, Tx_n), & \text{if } n \text{ is odd,} \\ H(Tx_{n-1}, Sx_n), & \text{if } n \text{ is even.} \end{cases}$$

Definition 2.2.
If, for $x_\circ \in X$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that every Cauchy sequence of the form $0_{f_i}(x_\circ)$ converges in X, then X is called $(S, T; f_i)$ -orbitally complete with respect to $x_\circ$ or simply $(S, T; f_i, x_\circ)$ -orbitally complete. If for each $i \in N$, f_i is an identity mapping on X, then $0_{f_i}(x_\circ)$ is denoted by $0(x_\circ)$ and $(S, T; f_i, x_\circ)$ -orbitally completeness by $(S, T; x_\circ)$ -orbital completeness.

Definition 2.3.
A pair (S, T) is said to be asymptotically regular at $x_\circ \in X$ if for any sequence $\{x_n\}$ in X and each sequence $\{y_n\}$ in X such that

$$y_n \in Sx_{n-1} \cup Tx_{n-1}, \lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

We remark that Definitions 2.1–2.3 with $S = T$ and $f_i = f$ for each $i \in N$ reduce to Definitions 4, 6 and 7 of B.E. Rhoades et al. [15], respectively, and orbital completeness need not imply the completeness of the space. Evidently, a complete space is orbitally complete.

Definition 2.4. For each $i \in N$, the mappings f_i and S are said to be commuting at a point $x \in X$ [3] if $f_i Sx \subseteq Sf_i x$. The mappings f_i and S are said to be commuting on X if $f_i Sx \subseteq Sf_i x$ for every point $x \in X$.

In [22], S. Sessa introduced the concept of weak commutativity for single-valued mappings on a metric space, and Sessa et al. [25] extended this concept to the setting of a single-valued mapping and a multi-valued mapping on a metric space.

Definition 2.5.

For each $i \in N$, the mappings f_i and S are said to be weakly commuting at $x \in X$ if $H(f_i Sx, Sf_i x) \leq D(f_i x, Sx)$. The mappings f and S are said to be weakly commuting on X if they are weakly commuting at every $x \in X$.

It may be noted that commutativity implies weak commutativity, but the converse need not be true even in the case of single-valued mappings as shown in [20], [22]–[25] by suitable examples.

Let $\mathcal{F}$ be the family of all mappings $\varnothing$ from the set R^+ of non-negative real numbers to itself such that $\varnothing$ is upper semicontinuous, non-decreasing and $\varnothing(t) < t$ for any $t > 0$.

The existence of coincidence points of non-linear hybrid contractions have been shown by Popa [18] by generalizing the result of Hadžić [2]. In doing so, Popa used Hausdorff-Pompeiu metric as used by Hadžić. It may be remarked that the result of Popa [18] is independent of the main result of Kubiak [12].

Theorem 2.1 [18]. Let (X, d) be a complete metric space, S and T continuous mappings from X into X, A and B closed mappings from X into $CB(SX \cap TX)$ such that $ASx = SAx$, $BTx = TBx$, for every $x \in X$ and

$$\begin{aligned} H^2(Ax, By) \leq & h \max\{d(Sx, Ax).\, d(Ty, By),\ d(Sx, By).\, d(Ty, Ax), \\ & \tfrac{1}{2}[d(Sx, Ax).d(Sx, By) + d(Ty, Ax).\, d(Ty, By)],\ d^2(Sx, Ty)\} \end{aligned} \tag{2.1}$$

for every $x, y \in X$ where $h \in (0, 1)$.

Then there exists a sequence $\{x_n\}_{n \in N}$ in X such that:

1. for every $n \in N, Sx_{2n+1} \in Bx_{2n}$, $Tx_{2n} \in Ax_{2n-1}$.
2. There exists
$$z = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1}.$$
3. $Sz \in Az$, $Tz \in Bz$.

On the other hand, Rhoades et al. [19] deal with the same existence problem by using generalized Hausdorff distance function:

Theorem 2.2 [19].

Let T be a multi-valued mapping from a metric space X into $CL(X)$. If there exists a mapping f from X into itself such that $T(X) \subseteq f(X)$, for each $x, y \in X$ and $\phi \in \mathcal{F}$,

(2.2) $H(Tx, Ty) \leq \phi(\max(D(fx, Tx),\ D(fy, Ty),\ D(fx, Ty),\ D(fy, Tx),\ d(fx, fy))$

(2.3) $\phi(t) \leq qt$ for each $t > 0$ and for some $0 < q < 1$,

(2.4) there exists a point $x_o \in X$ such that T is asymptotically regular of x_o and $f(X)$ is $(T;\ f, x_o)$–orbitally complete, then f and T have a coincidence point in X.

It is interesting to note that the commuting mappings f (not the identity mapping) and T satisfying the hypothesis of Theorem 2.2 need not have a common fixed point in X.

In [19], Rhoades et al. gave the solutions of the question "What additional conditions will guarantee the existence of a common fixed point of f and T?"

The main objective of this paper is to investigate different sets of conditions under which the fixed point equation

$$x = f_i x \in Sx \cap Tx \text{ for } x \in X \text{ and each } i \in N$$

possesses a solution.

Moreover, we observe that the following conditions are independent:

(a) There exists a constant q, $0 < q < 1$ with $\phi(t) < qt$ for each $t > 0$ such that
$$H(Sx, Ty) \leq \phi(\max(D(fx, Sx),\ D(gy, Ty),\ D(fx, Ty),\ D(gy, Sx),\ d(fx, gy))).$$

(b) There exists a constant q, $0 < q < 1$ with $\phi(t) < qt$ for each $t > 0$ such that
$$\begin{aligned} H^2(Sx, Ty) \leq & \phi(\max(D(fx, Sx).\, D(gy, Ty),\ D(fx, Ty).\, D(gy, Sx), \\ & \tfrac{1}{2}[D(fx, Sx).D(fx, Ty) + D(gy, Sx).\ D(gy, Ty)], d^2(fx, gy))). \end{aligned}$$

Example 1.

Let $X = \{1,2,3,4\}$. Define a metric d for X and mappings f, g, S and T on X as follows:

$$\begin{aligned}
&d(1,4) = \frac{5}{2},\ d(1,2) = d(1,3) = \frac{3}{2},\ d(2,3) = \frac{4}{5},\\
&d(3,4) = \frac{7}{4},\ d(2,4) = 1,\\
&\ S(1) = \{3\},\ S(2) = S(3) = S(4) = \{4\},\\
&\ T(1) = T(2) = T(4) = \{4\},\ T(3) = \{2\},\ \text{and } f = g = id.
\end{aligned}$$

The f, g, S and T satisfy condition (a) for each q, $\frac{14}{15} \leq q < 1$.

For $x = 1$ and $y = 2$

$$\begin{aligned}
&H^2(S(1),\, T(2)) = D^2(S(1),\, T(2)) = \frac{49}{16} > \phi(\max(D(1,\, S(1).\, D(2,\, T(2)),\\
&D(1,\, T(2)).\, D(2,\, S(1)),\, \tfrac{1}{2}[D(1,\, S(1)).\, D(1,\, T(2)) + D(2,\, S(1)).\\
&D(2,\, T(2))],\, d^2(1,2)))) = \frac{91}{40}.
\end{aligned}$$

Thus (b) is not satisfied.

Example 2.

Let $X = \{1,2,3\}$. Define a metric d for X by

$$d(1,2) = 2,\ d(2,3) = \frac{3}{2},\ d(1,3) = 2.$$

Let f, g, S, and T be the mappings on X such that

$$\begin{aligned}
&S(1) = S(2) = S(3) = \{1\},\ T(1) = T(3) = \{1\},\\
&T(2) = \{3\},\ f = g = id.
\end{aligned}$$

Then f, g, S and T satisfy condition (b) for each q, $\frac{9}{16} \leq q < 1$.

For $x = 1$ and $y = 2$ by (a)

$$\begin{aligned}
&H(S(1),\, T(2)) = D(S(1),\, T(2)) = 2 \leq \phi(\max(D(1,\, S(1)), D(2,\, T(2)),\\
&D(1,\, T(2)),\, D(2,\, S(1)),\, d(1,2))) = \phi(2) \leq 2q
\end{aligned}$$

which implies $q \leq 1$. Thus (a) is not satisfied.

III. Main Results

Now we are ready to give our main theorems:

Theorem 3.1.

Let S and T be multi-valued mappings from a metric space X into $P_X(X)$ and let $\{f_i\}$ be a family of all continuous mappings from X into itself such that

(3.1) $S(X) \cup T(X) \subset f_i(X)$ for each $i \in N$,

(3.2)

$$\begin{aligned}
H^2(Sx,Ty) \leq &\phi(\max(D(f_ix,Sx).\ D(f_jy,Ty),\ \ D(f_ix,Ty).\ D(f_jy,Sx),\\
&\tfrac{1}{2}[D(f_ix,\, Sx).\ D(f_ix,\, Ty) + D(f_jy,\, Sx).\ D(f_jy,\, Ty)],\\
&d^2(f_ix,\, f_jy))),
\end{aligned}$$

for each $x, y \in X, i, j \in N,\ i \neq j$ and $\phi \in \mathcal{F}$,

(3.3) $\phi(t) \leq qt$ for each $t > 0$ and for some $q \in (0,1)$,

(3.4) there exists a point $x_o \in X$ such that the pair (S, T) is asymptotically regular at x_o, and

(3.5) for each $i \in N$, $f_i(x)$ is (S, T, f_i, x_o)–orbitally complete.

Then (1) f_i, S and T have a coincidence point in X. If z is a coincidence point of f_i, S and T and $f_i z$ is a fixed point of f_i, then we have;

(2) $f_i z$ is also a fixed point of S provided f_i is weakly commuting with S at z for every even $i \in N$ for any odd $j \in N$.

(3) $f_i z$ is also a fixed point of T provided f_i is weakly commuting with T at z.

(4) $f_i z$ is a common fixed point of S and T provided f_i is weakly commuting with each of S and T at z.

Proof.

(1) Let x_o be a point in X satisfying (3.4). Since $S(X) \cup T(X) \subset f_i(X)$, for each $i \in N$, we can find sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned}
y_{2n} &= f_{2n} x_{2n-1} \in S x_{2n} \text{ for every } n \in N,\\
y_{2n+1} &= f_{2n+1} x_{2n} \in T x_{2n+1} \text{ for every } n \in N_\circ,\\
d(y_{2n},\, y_{2n+1}) &\leq q^{-1/4} H(Sx_{2n},\, Tx_{2n+1}) \qquad \text{and}\\
d(y_{2n+1},\, y_{2n+2}) &\leq q^{-1/4}\ H(Tx_{2n+1},\, Sx_{2n+2}).
\end{aligned}$$

By (3.4), we have

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

Now we wish to show that $\{y_{2n}\}$ is a Cauchy sequence in $f_i(X)$. Suppose that the sequence $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists a positive number $\in$ such that, for each positive integer $2k$, there exist integers $2n(k)$ and $2m(k)$ such that

(3.6) $2k \leq 2n(k) < 2m(k)$ and

(3.7) $d[y_{2n(k)},\, y_{2m(k)}) \geq \in$.

Let $d_{i,j} = d(y_i, y_j)$ and $d_i = d(y_i, y_{i+1})$ for each $i, j \in N$. Then, for each integer $2k$, we have

(3.8)
$$\in \leq d_{2n(k),\,2m(k)} \leq d_{2n(k),\,2m(k)-2} + d_{2m(k)-2} + d_{2m(k)-1}.$$

For each integer $2k$, let $2m(k)$ denote the smallest integer satisfying (2.1) and (2.2). Then we have $d_{2m(k),\,2m(k)-2} < \in$ and it follows from (2.3) that

(3.9)
$$\lim_{k\to\infty} d_{2n(k),2m(k)} = \in .$$

Using the triangle inequality, we get

$$\begin{aligned}
&|d_{2n(k),\,2m(k)} - d_{2n(k),\,2m(k)}| \leq d_{2m(k)-1} \qquad \text{and}\\
&|d_{2n(k)+1,\,2m(k)-1} - d_{2n(k),\,2m(k)}| \leq d_{2n(k)} + d_{2m(k)-1},
\end{aligned}$$

which yield

$$\lim_{k\to\infty} d_{2n(k),\,2m(k)-1} = \lim_{k\to\infty}\ d_{2n(k)+1,\,2m(k)-1} = \in$$

in view of (3.4) and (3.9) and so by (3.2), we have

$$\begin{aligned}
d_{2n(k),\,2m(k)} \leq & d_{2n(k)} + d_{2n(k)+1,2m(k)} \\
\leq & d_{2n(k)} + q^{-1/4}\, H(Sx_{2m(k)},\, Tx_{2n(k)+1}) \\
\leq & d_{2n(k)} + q^{-1/4}\{\phi(\max(D(f_{2m(k)+1}\, x_{2m(k)},\, Sx_{2m(k)}). \\
& D(f_{2m(k)+2}\, x_{2m(k)+1},\, Tx_{2m(k)+1}),\, D(f_{2m(k)+1} \\
& x_{2m(k)},\, Tx_{2n(k)+1}).\, D(f_{2m(k)+2},\, x_{2m(k)+1},\, Sx_{2m(k)}), \\
& \tfrac{1}{2}[D(f_{2m(k)+1}\, x_{2m(k)},\, Sx_{2m(k)})).\, D(f_{2m(k)+1}\, x_{2m(k)}, \\
& Tx_{2m(k)+1} + D(f_{2m(k)+2}\, x_{2m(k)+1},\, Sx_{2m(k)}), \\
& D(f_{2m(k)+2}\, x_{2m(k)+1},\, Tx_{2m(k)+1})],\, d^2(f_{2m(k)+1}\, x_{2m(k)}, \\
& f_{2m(k)+2}\, x_{2m(k)+1})))\}^{\frac{1}{2}} \\
\leq & d_{2n(k)} + q^{-1/4}\{\phi(\max(d_{2m(k)},\, d_{2m(k)+1}, \\
& d_{2n(k)+1,\,2m(k)+1}\, d_{2m(k)+z,\,zm(k)}, \\
& \tfrac{1}{2}[d_{2m(k)}.\, d_{2m(k)+1} + d_{2m(k)+2,\,2m(k)}. \\
& d_{2m(k)+1}],\, d^2_{2m(k)+1})))\}^{\frac{1}{2}}.
\end{aligned}$$

Using the semicontinuity of ϕ and letting $k \to \infty$, this inequality yields

$$\in \leq q^{-1/4}\{\phi(\ ^2)\}^{\frac{1}{2}} \leq q^{-1/4}\,(q^2)^{\frac{1}{2}} < \in$$

Since $\in > 0$ and $q^{1/4} < 1$, which contradicts the choice of $\in$ and so the sequence $\{y_{2n}\}$ is a Cauchy sequence. Similarly, we can prove that $\{y_{2n+1}\}$ is also a Cauchy sequence in $f_i(X)$.

Since $f_i(X)$ is $(S, T,\, f_i,\, x_\circ)$-orbitally complete, the Cauchy sequence $\{y_{2n}\}$ has a limit u in $f_1(X)$ for each $i \in N$. By condition (3.4), we have

$$0 = \lim_{n\to\infty} d(y_{2n},\, y_{2n+1}) = \lim_{n\to\infty} d(u,\, y_{2n+1})$$

and so $\{y_{2n+1}\}$ also converges to u.

Hence there exists a point $z \in X$ such that $f_i z = u$. By condition (3.2), for any even $i \in N$,

$$\begin{aligned}
D(f_i z,\, Sz) \leq & d(f_i z,\, y_{2n+1}) = D(y_{2n+1},\, Sz) \\
\leq & d(f_i z,\, y_{2n+1} + H(Sz,\, Tx_{2n+1}) \\
\leq & d(f_i z, y_{2n+1}) + \{\phi(\max(D(f_i z,\, Sz).\, D(f_{2n+2}\, x_{2n+1}, \\
& Tx_{2n+1}), D(f_i z,\, Tx_{2n+1}).\, D(f_{2n+2}\, x_{2n+1},\, Sz), \\
& \tfrac{1}{2}[D(f_i z,\, Sz).\, D(f_i z,\, Tx_{2n+1}) + D(f_{2n+2}\, x_{2n+1},\, Sz). \\
& D\,(f_{2n+2}\, x_{2n+1},\, Tx_{2n+1})],\, d^2(f_i z,\, f_{2n+2}\, x_{2n+1})))\}^{\frac{1}{2}} \\
\leq & d(f_i z,\, y_{2n+1}) + \{\phi(\max(D(f_i z,\, Sz).\, D(y_{2n+2},\, y_{2n+1}), \\
& D(f_i z,\, y_{2n+1}).\, D(y_{2n+2},\, Sz),\, \tfrac{1}{2}[D(f_i z,\, Sz). \\
& D(f_i z,\, y_{2n+1}) + D(y_{2n+2},\, Sz).\, D(y_{2n+2},\, y_{2n+1})], \\
& d^2(f_i z,\, y_{2n+2})))\}.
\end{aligned}$$

Letting $n \to \infty$, this inequality yields

$$D(f_i z, Sz) \leq \{\phi(\max(0, 0, 0, 0))\}^{\frac{1}{2}} = 0.$$

If $f_i z \in Sz$ for any even $i \in N$, then $D(f_i z,\, Sz) > 0$ and so the above inequality yields a contradiction.

Hence $f_i z \in Sz$ for every $i \in N$ since every proximinal set is closed. Similarly, we have $f_i z \in Tz$ for any odd $i \in N$. Therefore, z is a coincidence point of f_i, S and T for each $i \in N$.

(2) If for any even $i \in N$, $u = f_i z$ is a fixed point of f_i, then $u = f_i u = f_i f_i z \in f_i Sz$. If f_i is weakly commuting with S at z, then $f_i Sz = Sf_i z$ since $f_i z \in Sz$. Therefore, we have $u \in f_i Sz = Sf_i z = Su$; that is, u is a fixed point of S.

(3) It follows by the same techniques in (2) and by (2) and (3), we have (4).

Remark.

Theorem 3.1 is true even if we replace $P_X(X)$ by $CL(X)$.

By replacing the condition (3.1) of Theorem 3.1 by the orbital regularity, we have the following:

Theorem 3.2.

Let S and T be multi-valued mappings from a metric space X into $P_X(X)$ and let $\{f_i\}$ be the family of all continuous mappings from X into itself such that the conditions (3.2) holds,

(3.10) $\phi(t) < t$ for each $t > 0$ and some $\phi \in \mathcal{F}$.

(3.11) for some $x_\circ \in X$, and each $i \in N$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that the orbit $0_{f_i}(x_\circ)$ is regular, the pair (S, T) is asymptotically regular at $x_\circ$ and $f(X)$ is $(S, T, f_i, x_\circ)$–orbitally complete. Then we have:

(1) for each $i \in N$, f_i, S and T have a coincidence point in X,

(2) if the limit of $0_{f_i}(x_\circ)$ is a fixed point of f_i, then all the conclusions of (2) and (3) of Theorem 3.1 are also true.

It is well known that if S is a multi-valued mapping from X into $C(X)$, then for every $x, y \in X$ and $u \in Sx$, there exists a point $v \in Sy$ such that

$$d(u, v) \leq H(Sx, Sy).$$

It follows that if S and T are multi-valued mappings from X into $C(X)$, then the orbital regularity in Theorem 3.2 can be dropped. Thus we have the following:

Theorem 3.3. Let S and T be multi-valued mappings from a metric space X into $C(X)$ and let $\{f_i\}$ be a family of continuous mappings from X into itself such that the conditions (3.1), (3.2), (3.4), (3.5) and (3.6) hold. Then all the conclusions of Theorem 3.2 are also true.

In Theorem 3.1, if we replace (3.2) by the following condition

$$\begin{aligned} H^2(Sx, Ty) \leq &\phi(\max(D(f_i x, Sx).\, D(f_i y, Ty),\, D(f_i x, Ty).\, D(f_j y, Sx), \\ &\tfrac{1}{2}[D(f_i x, Sx).\, D(f_i x, Ty) + D(f_j y, Sx).\, D(f_i y, Ty)], \\ &d^2(f_i x, F_j y))) \end{aligned} \tag{3.8}$$

for each $x, y \in X$, $i, j \in N$, $i \neq j$, and $\phi \in \mathcal{F}$, then the asymptotic regularity of (S, T), (3.4) is not needed. In fact, we have the following:

Theorem 3.4. Let S and T be multi-valued mappings from a metric space X into $P_x(X)$ and let $\{f_i\}$ be a family of continuous mappings from X into itself such that the conditions (3.1), (3.5) and (3.8) hold. Then all the conclusions of Theorem 3.1 are also true.

In Theorem 3.1 and 3.3, taking $f = f_i$ on X for each $i \in N$ and defining $\phi(t) = qt$, $o < q < 1$, we have the following:

Corollary 3.5. Let S and T be multi-valued mappings from a metric space into $P_x(X)$ and let f be a continuous mapping from X into itself such that the condition (3.4) holds:

$$S(X) \cup T(X) \subset f(X) \tag{3.9}$$

$$\begin{aligned} H^2(Sx, Ty) \leq &q \max(D(fx, Sx).\, D(fy, Ty),\, D(fx, Ty).\, D(fy, Sx), \\ &\tfrac{1}{2}[D(fx, Sx).\, D(fx, Ty) + D(fy, Sx).\, D(fy, Ty)],\, d^2(fx, fy)) \end{aligned} \tag{3.10}$$

for each $x, y \in X$ and for some $0 < q < 1$,

(3.11) $f(X)$ is $(S, T;\, f, x_\circ)$–orbitally complete, and

(3.12) f is weakly commuting with each of S and T.

Then f, S, and T have a common fixed point in X.

Corollary 3.6. Let S and T be multi-valued mappings from a metric space X into $P_X(X)$ and let f be a continuous mapping from X into itself such that the condition (3.9) and (3.12) hold, and

$$\begin{aligned} H^2(Sx, Ty) \leq & q \max(D(fx, Sx).\, D(fy, Ty),\ \tfrac{1}{2}[D^2(fx, Ty) + D^2(fy, Sx)], \\ & \tfrac{1}{2}[D(fx, Sx).\, D(fx, Ty) + D(fy, Sx).\, D(fy, Ty)],\ d^2(fx, fy))) \end{aligned} \tag{3.13}$$

For each $x, y \in X$ and for some $0 < q < 1$. Then f, S and T have a common fixed point in X.

By using the technique of Kubiak [13], we also have the following theorem.

Theorem 3.7. Let S, T be multi-valued mappings from a metric space X into $P_X(f(X) \cap g(X))$ and f, g be continuous mappings from X into itself such that the condition (3.12) and (3.13) hold.

Then f, g, S and T have a common fixed point in X.

Remark. Our results differ from the corresponding theorems of Hadžić [3] and Sessa et al. [24] to the version hybrid contraction mappings.

IV. Some More Related Results

Let X be a reflexive Banach space and $WC(X)$ denote the family of non-empty weakly compact subset of X. Recall that every non-empty weakly compact subset of a reflexive Banach space is proximinal [26] and hence closed.

Our main theorem in this section is prefaced with the following lemma:

Lemma 4.1 [14].

Let $f \in \mathcal{F}$ and $t_\circ > 0$. If $t_{n+1} \leq f(t_n)$ for all $n \in N$, then the sequence $\{t_n\}$ converges to 0.

The following theorem is a variant of results due to Kaneko [9] and Khan et al. [11].

Theorem 4.2.

Let X be a reflexive Banach space and let $\{S_n\}$, $\{T_n\}$ be sequences of mappings from X into $WC(X)$ such that

$$\begin{aligned} H^2(S_m x,\, T_n y) \leq & \phi(\max(D(x,\, S_m x).\, D(y,\, T_n y),\ D(x, T_n y).\, D(y,\, S_m x), \\ & \tfrac{1}{2} D(x,\, S_m x).\, D(x,\, T_n y),\ \tfrac{1}{2} D(y,\, S_m x).\, D(y,\, T_n y),\ d^2(x, y)) \end{aligned} \tag{4.1}$$

for each $x, y \in X$ and for some $\phi \in \mathcal{F}$. Then there exists a point $z \in X$ such that $z \in S_m z \cap T_n z$ for each $m, n \in N$.

Proof.

Firstly, we assume that $h = 0$. Let $x_\circ \in X$ be an arbitrary point. Then there exists a point $x_1 \in S_1 x_\circ$ such that $d(x_\circ, x_1) = D(x_\circ, S x_\circ)$. This is guaranteed due to the fact that $S_1 x_\circ$ is a non-empty weakly compact subset of a reflexive Banach space and so it is proximinal. Now, for all $n \in N$, we have

$$D(x_1,\, T_n x_1) \leq H(S_1 x_\circ,\, T_n x_1) = 0$$

which implies that $x_1 \in T_n x_1$. Similarly, for all $n \in N$,

$$D(x_1,\, S_1 x_1) \leq H(T_n x_1,\, S_1 x_1) = 0$$

yields $x_1 \in S_n x_1$. Thus x_1 is a common fixed point of the two sequences $\{S_n\}$ and $\{T_n\}$.

Next, we assume that $h \neq 0$. Let $x_\circ \in X$ be arbitrary but fixed. Then we construct a sequence $\{x_n\}$ in X such that

$$x_{2n-1} \in S_n x_{2n-2}, x_{2n} \in T_{2n} x_{2n-1}, d(x_{2n-2}, x_{2n-1}) = D(x_{2n-2},$$
$$S_n x_{2n-2}), \text{ and } d(x_{2n-1}, x_{2n}) = D(x_{2n-1}, T_n x_{2n}).$$

The existence of such a sequence in X is guaranteed by the proximinality of $S_m(x)$ and $T_n(x)$ for each $m, n \in N$.

Now, suppose that $x_n = x_{n+1}$ for some $n \in N$. If $n \in N$ is even, then we have $x_{2n} \in S_{n+1}\, x_{2n}$. Further, for each $m \in N$, we have

$$\begin{aligned} D^2(x_{2n},\, T_m x_{2n}) \leq & H^2(S_{n+1} x_{2n},\, T_m x_{2n}) \\ \leq & \phi(\max(D(x_{2n},\, S_{n+1} x_{2n}).\, D(x_{2n},\, T_m x_{2n}),\, D(x_{2n},\, T_m x_{2n}). \\ & D(x_{2n},\, S_{n+1} x_{2n}), \tfrac{1}{2} D(x_{2n},\, S_{2n+1}\, x_{2n}).\, D(x_{2n},\, T_m x_{2n}), \\ & \tfrac{1}{2} D(x_{2n},\, S_{2n+1} x_{2n}).\, D(x_{2n},\, T_m x_{2n})\, d^2(x_2 n, x_{2n}))) \\ = & \phi(\circ), \end{aligned}$$

which yields $x_{2n} \in T_m x_{2n}$ for each $m \in N$ similarly, $D^2(x_{2n},\, S_m x_{2n}) \leq H^2(S_m x_{2n},\, T_n x_{2n}) \leq \phi(\circ)$ gives $x_{2n} \in S_m x_{2n}$ for each $m \in N$.

In case, when an odd integer $n \in N$, we have the same results.

Therefore, in each case, we have a common fixed point for the sequences $\{S_m\}$ and $\{T_n\}$.

Now suppose that $x_n \neq x_{n+1}$ for all $n \in N$. We shall show that $\{x_n\}$ is a Cauchy sequence in X. We observe that

$$\begin{aligned} d^2(x_{2n},\, x_{2n+1}) = & D^2(x_{2n}, S_{n+1} x_{2n}) \\ \leq & H^2(T_n\, x_{2n-1},\, S_{n+1} x_{2n}) \\ \leq & \phi(\max(D(x_{2n},\, S_{n+1} x_{2n}).\, D(x_{2n-1},\, T_n x_{2n-1}), \\ & D(x_{2n-1},\, T_n x_{2n-1}).\, D(x_{2n},\, S_{2n+1} x_{2n}), \\ & \tfrac{1}{2} D(x_{2n-1},\, S_{n+1} x_{2n}).\, D(x_{2n},\, S_{n+1} x_{2n}), \\ & \tfrac{1}{2} D(x_{2n-1},\, S_{n+1} x_{2n}).\, D(x_{2n-1},\, T_n x_{2n-1}), \\ & d^2(x_{2n-1},\, x_{2n}))) \\ = & \phi(\max(d(x_{2n}\ x_{2n+1}).\, d(x_{2n-1},\, x_{2n}),\, d(x_{2n-1}, x_{2n}). \\ & d(x_{2n},\, x_{2n+1}), \tfrac{1}{2} d(x_{2n-1},\, x_{2n+1}).\, d(x_{2n},\, x_{2n+1}), \\ & \tfrac{1}{2} d(x_{2n-1},\, x_{2n+1}).\, d(x_{2n-1},\, x_{2n}),\, d^2(x_{2n-1},\, x_{2n}))) \\ \leq & \phi(\max(d(x_{2n-1},\, x_{2n}).\, d(x_{2n},\, x_{2n+1}),\, \tfrac{1}{2}\{d(x_{2n-1},\, x_{2n}) \\ & + d(x_{2n},\, x_{2n+1})\}\, d(x_{2n},\, x_{2n+1}),\, \tfrac{1}{2}\{d(x_{2n-1}, x_{2n}) + \\ & d(x_{2n},\, x_{2n+1})\}\, d(x_{2n-1},\, x_m),\, d^2(x_{2n-1},\, x_{2n}))). \end{aligned}$$

If $d(x_{2n-1}, x_{2n}) < d(x_{2n}, x_{2n+1})$, then the above inequality yields

$$d^2(x_{2n},\, x_{2n+1}) \leq \phi(d^2(x_{2n},\, x_{2n+1})) < d^2(x_{2n},\, x_{2n+1}),$$

a contradiction. Therefore, we get

$$d^2(x_{2n}, x_{2n+1}) \leq \phi(d^2(x_{2n-1}, x_{2n})).$$

.

Similarly, we can show that

$$d^2(x_{2n+1},\, x_{2n+2}) \le \phi(d^2(x_{2n}, x_{2n+1})).$$

It follows from the above relation that

$$d^2(x_n, x_{n+1}) \le \phi(d^2(x_{2n}, x_{2n+1}))$$

for all $n \in N$. By Lemma 4.1, we have

$$\lim_{n\to\infty} d^2(x_n,\, x_{n+1}) = 0,$$

which in turn implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is a Banach space, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z.$$

Then we have

$$\begin{aligned}
D^2(x_{2n-1}, T_m z) \le& H^2(S_n x_{2n-2},\, T_m z)\\
\le& \phi(\max(D(x_{2n-2},\, S_n x_{2n-2}).\, D(z,\, T_m z),\, D(x_{2n-2},\, T_m z).\\
& D(z,\, S_n x_{2n-2}),\, \tfrac{1}{2} D(x_{2n-2},\, S_n x_{2n-2}).\, D(x_{2n-2},\, T_m z),\\
& \tfrac{1}{2} D(z,\, S_n x_{2n-2}).\, D(z,\, T_m z),\, d^2(x_{2n-2},\, z)))\\
\le& \phi(\max(d(x_{2n-2},\, x_{2n-1}).\, D(z,\, T_m z),\, D(x_{2n-2},\, T_m z),\\
& d(z, x_{2n-1}),\, \tfrac{1}{2} d(x_{2n-2},\, x_{2n-1}).\, D(x_{2n-2},\, T_m z),\\
& \tfrac{1}{2} d(z,\, x_{2n-1}).\, D(z,\, T_m z),\, d^2(x_{2n-2} z))).
\end{aligned}$$

Letting $n \to \infty$, the above inequality yields

$$D^2(z,\, T_m z) = 0$$

Since $T_m z$ is closed, $z \in T_m z$ for all $m \in N$. Similarly, we have also $z \in S_m z$ for all $m \in M$.
Therefore, z is a common fixed point of the sequences $\{S_n\}$ and $\{T_n\}$. This completes the proof.
In the last two decades, a large number of fixed point theorems appeared for commuting and weakly commuting mappings (see, for instance, [5]–[7] and [19]–[24]). By using the technique of Khan et al. [11], we have the following results.

Corollary 4.3.
Let f, g, S and T be single-valued mappings from a complete metric space X into itself such that

$$\begin{aligned}
d^2(fx, gy) \le& \phi(\max(d(fx, Sx),\, d(gy, Ty), d(fx, Ty).\, d(gy, Sx),\\
& \tfrac{1}{2}[d(fx, Sx).\, d(fx, Ty) + d(gy, Sx).\, d(gy, Ty)]\, d^2(Sx, Ty))),
\end{aligned} \tag{4.2}$$

for each $x, y \in X$ and for some $\phi \in \mathcal{F}$, and

$$\{f, S\} \text{ and } \{g, T\} \text{ are weakly commuting pairs.} \tag{4.3}$$

Then f, g, S and T have a unique common fixed point in X.

Corollary 4.4.

Let S and T be two continuous mappings from a complete metric space X into itself. S and T have a common fixed point in X if and only if there are two self-mappings A and B on X such that

$$A(X) \cup B(X) \subset S(X) \cap T(X) \tag{4.4}$$

$$\begin{aligned} d^2(Ax, Bx) \leq & \phi(\max(d(Sx, Ax), d(Ty, By), d(Sx, By).\, d(Ty, Ax), \\ & \tfrac{1}{2}[d(Sx, Ax).\, d(Ty, Ax) + d(Sx, By).\, d(Ty, By)],\ d^2(Sx, Ty))), \end{aligned} \tag{4.5}$$

for each $x, y \in X$ and for some $\phi \in \mathcal{F}$, and

$$\{A, S\} \text{ and } \{B, T\} \text{ are weakly commuting pairs.} \tag{4.6}$$

Further, z is the unique common fixed point of A, B, S and T.

Theorem 4.5.

Let X be a compact metric space and let S, T be multi-valued mappings from X into $P_X(X)$ such that either S or T is continuous,

$$\begin{aligned} H^2(Sx, Ty) \leq & \phi(\max(d(x, Sx)\, d(y, Ty),\ d(y, Sx)\, d(x, Ty),\ \tfrac{1}{2}d(x, Sx) \\ & d(x, Ty),\ \tfrac{1}{2}d(y, Sx)\, d(y, Ty),\ d^2(x, y))) \end{aligned} \tag{4.7}$$

for each $x, y \in X$, $x \neq y$, and for some $\phi \in \mathcal{F}$. Then either S or T has a fixed point in X.

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